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CATEGORY OF SOME SUBALGEBRAS OF THE TOEPLITZ ALGEBRA

K. H. Hovsepyan

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We perform the structural analysis of C^*-subalgebras of the Toeplitz algebra that are generated by inverse subsemigroups of a bicyclic semigroup. We construct a category of the sets of natural numbers of length $k < m$ and associate each set with a certain C^*-algebra. As a result, we obtain a category of C^*-algebras. The existence of a functor between these categories is proved. In particular, we establish the conditions under which the category of C^*-algebras turns into a bundle of C^*-algebras.

1. Introduction

One of the most well-known algebraic objects in the contemporary mathematical physics is the Toeplitz algebra $\mathcal{T}$. Both this algebra and various its modifications have been investigated in numerous works [1–10]. The present paper is also devoted to the study of one of possible generalizations of the Toeplitz algebra encountered in the investigation of C^*-algebras generated by the inverse subsemigroups of a bicyclic semigroup. In [1], Barnes proved that a bicyclic semigroup possesses, to within a unitary equivalence, one infinite-dimensional exact irreducible representation and that a series of one-dimensional representations is parametrized by a unit neighborhood. By the Coburn theorem [2], all C^*-algebras generated by nonunitary isometric representations of a semigroup of nonnegative integers $\mathbb{Z}_+$ are canonically isomorphic. This theorem was generalized by Douglas [4] for semigroups with Archimedean ordering and by Murphy [6] for semigroups with total ordering. In [5], Aukhadiev and Tepoyan proved the statement converse to the Murphy theorem [6], namely, that all C^*-algebras generated by the exact isometric nonunitary representations of a semigroup are canonically isomorphic only if the semigroup has a total order. Thus, the C^*-algebra generated by the exact infinite-dimensional representations of a bicyclic semigroup is isomorphic to the Toeplitz matrix.

Earlier, we originated our investigation of C^*-subalgebras of the Toeplitz algebra $\mathcal{T}$ generated by monomials whose index is a multiple of the number m. This C^*-algebra was denoted by $\mathcal{T}_m$. It was shown that this C^*-algebra is fixed under a finite subgroup of the group S^1 of order m. All irreducible infinite-dimensional representations of this C^*-algebra were described in [11–13].

In [14], the C^*-algebra $\mathcal{T}_m$ was studied from a somewhat different point of view. The complete description of all invariant ideals of the algebra $\mathcal{T}_m$ was obtained. It was shown that their number is finite and equal to 2^m and that each of them is generated by one or several differences of projectors of the form

$$T^iT^{*i} - T^jT^{*j}, \quad 0 \le i < j \le m.$$

It was also proved that if J is an invariant ideal of the C^*-algebra $\mathcal{T}_m$ and $J \neq \mathcal{K}_m$, then it can be represented in the form of direct sum $\mathcal{T}_m \cong \mathcal{T}_n \oplus J$ for some $n < m$.

In [15], it was shown that the algebra $\mathcal{T}_m$ can be represented in the form of the cross product:

$$\mathcal{T}_m = \varphi(\mathcal{A}) \times_{\delta_m} \mathbb{Z},$$

Ijevan Branch of Yerevan State University, Ijevan, Armenia; e-mail: karen.hovsep@gmail.com.

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where

$$\mathcal{A} = C_0(\mathbb{Z}_+) \oplus \mathbb{C}I,$$

i.e., the algebra of continuous functions on $\mathbb{Z}_+$ with finite limits at infinity. In addition, the complete description of automorphisms of the C^*-algebra $\mathcal{T}_m$ and its subalgebras $\mathcal{T}(m)$ and $\mathcal{K}_m$ can be found in [16].

2. Subalgebras of the Toeplitz Algebra

Consider a bicyclic semigroup S with generating element a. It is clear that each element of the bicyclic semigroup has the form $a^m a^{*n}$, where m and n are nonnegative integers. An element of the form $a^m a^{*n}$ is called monomial.

For the monomial $b = a^m a^{*n}$ from S, we say that the number $m - n$ is the *index* of this monomial and denote it by $\operatorname{ind}(b)$. Note that

$$\operatorname{ind}(b \cdot c) = \operatorname{ind}(b) + \operatorname{ind}(c)$$

for any elements $b, c \in S$ (see [11]). We fix an integer $m \in \mathbb{N}$ and denote

$$S_m = \{b \in S \colon\ \operatorname{ind}(b) = k \cdot m,\ k \in \mathbb{Z}\}.$$

Let $S(m) \subset S$ be a subsemigroup generated by an element a^m. It is clear that $S(m)$ and S_m are inverse subsemigroups of the bicyclic semigroup S. The relationship between these semigroups was presented in [11]. It is known that a bicyclic semigroup possesses, to within unitary equivalence, a single exact infinite-dimensional irreducible representation (see [1])

$$\pi \colon S \to B(l^2(\mathbb{Z}_+)), \qquad \pi(a^n a^{*m}) = T^n T^{*m},$$

where T is the operator of shift on $l^2(\mathbb{Z}_+)$, i.e., on the basis $\{e_k\}_{k \in \mathbb{Z}_+}$, this operator acts as follows:

$$Te_k = e_{k+1}$$

and this representation generates a Toeplitz algebra. The following question naturally arises: Is it possible to generalize a Toeplitz algebra such that the number of irreducible infinite-dimensional unitarily nonequivalent representations generated by the inverse subsemigroups of a bicyclic semigroup is finite? It turns out that these C^*-algebras are generated by all inverse subsemigroups $S_m,\ m \in \mathbb{N}$, of the bicyclic semigroup S. By $\mathcal{T}_m$ we denote a C^*-subalgebra of the Toeplitz algebra $\mathcal{T}$ generated by the inverse subsemigroup $\pi(S_m)$. In other words, $\mathcal{T}_m$ is generated by all monomials of the form $T^k T^{*l}$, where

$$\operatorname{ind}(T^k T^{*l})/m = k - l/m \in \mathbb{Z}.$$

Let $\mathcal{T}(m)$ be a C^*-subalgebra of the Toeplitz algebra generated by $\pi(S(m))$. It is clear that $\mathcal{T}(m) \subset \mathcal{T}_m$. We now present some results obtained for the analyzed algebras in [14] and frequently used in what follows.

We now consider a representation of the Hilbert space $l^2(\mathbb{Z}_+)$ in the form of direct sum

$$l^2(\mathbb{Z}_+) = H_1 \oplus H_2 \oplus \ldots \oplus H_m, \tag{1}$$

where the basis of the subspace H_i is formed by the vectors $\{e_{i-1+km}\}_{k \in \mathbb{Z}_+},\ 1 \le i \le m$. Then the subspaces H_i, $1 \le i \le m$, are invariant under the algebra $\mathcal{T}_m$.

In view of (1), any element $A \in \mathcal{T}_m$ can be uniquely represented in the form

$$A = A|_{H_1} \oplus \ldots \oplus A|_{H_m}. \tag{2}$$

Let $\mathcal{K}$ be the C^*-subalgebra of all compact operators of the Toeplitz algebra $\mathcal{T}$ and let $\mathcal{K}_m$ be the C^*-subalgebra of all compact operators of the algebra $\mathcal{T}_m$.

Lemma 1. *The identity*

$$\mathcal{K}_m = \mathcal{K}(H_1) \oplus \ldots \oplus \mathcal{K}(H_m)$$

is true.

Lemma 2. *The algebra $\mathcal{T}_m$ is a C^*-algebra generated by the operators T^m and T^{*m} and the projectors $P_1, \ldots, P_m$, where $P_l = T^l T^{*l}$, $0 \le l \le m-1$.*

Theorem 1. *Any element $A \in \mathcal{T}_m$ has the form $A = C + D$, where $C \in \mathcal{T}(m)$ and $D \in \mathcal{K}_m$, i.e.,*

$$\mathcal{T}_m = \mathcal{T}(m) + \mathcal{K}_m.$$

We now define operators $\alpha\colon \mathcal{T} \to \mathcal{T}$ and $\beta\colon \mathcal{T} \to \mathcal{T}$:

$$\alpha(A) = TAT^*, \qquad \beta(A) = T^*AT, \quad A \in \mathcal{T}. \tag{3}$$

It is clear that the operator α is an endomorphism of the algebra $\mathcal{T}$.

Lemma 3. *The operators α and β given by relations (3) satisfy the relations:*

(i) $\beta \circ \alpha = \mathrm{id}$;

(ii) $\beta(\mathcal{T}_m) = \mathcal{T}_m$;

(iii) $\beta \circ \alpha^k(\mathcal{T}_m) = \alpha^{k-1}(\mathcal{T}_m)$.

Proof. The first relation is obvious. It suffices to perform the proof for monomials. We prove the second relation. Let $V = T^k T^{*l} \in \mathcal{T}_m$. Then

$$\beta(V) = T^* T^k T^{*l} T = T^{k-1} T^{*l-1}.$$

It is easy to see that $\mathrm{ind}(\beta(V)) = \mathrm{ind}(V) = kl$. This directly yields the second relation. The third relation follows from the first relation.

3. Structures of the Subalgebras $\mathcal{T}_m$ and $\mathcal{T}(m)$ and Relationship between Them

By $\mathcal{T}(m)^+$ ($\mathcal{T}(m)^-$) we denote a subalgebra of the algebra $\mathcal{T}(m)$ such that the index of each its element is positive (negative), i.e.,

$$\mathcal{T}(m)^+ = \{A \in \mathcal{T}(m)\colon\ \mathrm{ind}(A) \ge 0\}, \qquad \mathcal{T}(m)^- = \{A \in \mathcal{T}(m)\colon\ \mathrm{ind}(A) < 0\}.$$

Similarly, denote $\mathcal{T}_m^+$ and $\mathcal{T}_m^-$. It is clear that these algebras are not C^*-algebras.

Theorem 2. *The Banach algebra $\mathcal{T}_m^+$ can be represented in the form of the direct sum of spaces as follows:*

$$\mathcal{T}_m^+ = (T(m)^+) \oplus \alpha(\mathcal{T}(m)^+) \oplus \ldots \oplus \alpha^{m-1}(\mathcal{T}(m)^+).$$

Proof. We now show that, for any monomial $V \in \mathcal{T}_m^+$ $V \in \alpha^k(\mathcal{T}(m)^+)$ and some $0 \leq k \leq n-1$, the sum on the right-hand side is also the direct sum. Let V be a monomial from $\mathcal{T}_m^+$ of the form $V = T^{mk+l}T^{*mr+l}$, where $0 < l < m, k > r$. Then

$$V = T^l T^{mk} T^{*mr} T^{*l} \in \alpha^l(\mathcal{T}(m)^+).$$

We now show that $\alpha^k(\mathcal{T}(m)^+) \cap \alpha^j(\mathcal{T}(m)^+) = 0$ for $k \neq j$. Assume that $V \in \alpha^k(\mathcal{T}(m)^+) \cap \alpha^j(\mathcal{T}(m)^+)$. Then

$$V = \alpha^k(T^{mn}T^{*ml}) = \alpha^j(T^{mi}T^{*ms}).$$

Hence,

$$V = T^{mn+k}T^{*ml+k} = T^{mr+j}T^{*ms+j},$$

i.e., $mn + k = mr + j$ and $ml + k = ms + j$. Since $0 \leq k \leq m-1$ and $0 \leq j \leq m-1$, these equalities are possible only for $k = j, n = r, l = s$. The assertion of the theorem follows from the fact that the monomials are dense in $\mathcal{T}_m^+$.

Corollary 1. *The Banach algebra $\mathcal{T}_m^-$ can be represented in the form of direct sum of spaces as follows:*

$$\mathcal{T}_m^- = (T(m)^-) \oplus \alpha(\mathcal{T}(m)^-) \oplus \ldots \oplus \alpha^{m-1}(\mathcal{T}(m)^-).$$

Corollary 2. *The C^*-algebra $\mathcal{T}_m$ can be represented in the form of direct sum of spaces as follows:*

$$\mathcal{T}_m = \mathcal{T}_m^- \oplus \mathcal{T}_m^+.$$

Lemma 4. *The following inclusion is true:*

$$\alpha^k(\mathcal{T}(m)^+)\alpha^j(\mathcal{T}(m)^-) \subset \alpha^k(\mathcal{T}(m)^+) \oplus \alpha^j(\mathcal{T}(m)^-),$$

where $0 \leq k, j \leq m-1$.

Proof. For the sake of definiteness, we assume that $k > j$. Let $V_1 \in \alpha^k\mathcal{T}(m)^+$ be a monomial of the form $V_1 = T^{nm+k}T^{*ml+k}$, $n > l$, and let $V_2 \in \alpha^j\mathcal{T}(m)^-$ be a monomial of the form $V_2 = T^{cm+j}T^{*am+j}$, where $c < a$. In this case, if $c > l$, then

$$\begin{aligned} V_1V_2 &= T^{nm+k}T^{*ml+k}T^{cm+j}T^{*am+j} \\ &= T^{nm+k+cm+j-(ml+k)}T^{*am+j} \\ &= T^jT^{m(n+c-l)}T^{*am}T^{*j} \in \alpha^j(\mathcal{T}(m)^+) \end{aligned} \tag{4}$$

and if $c < l$, then

$$V_1 V_2 = T^{nm+k} T^{*(ml+k)} T^{cm+j} T^{*(am+j)}$$

$$= T^{nm+k} T^{*(-cm)-j+(ml+k)} T^{*(am+j)}$$

$$= T^k T^{mn} T^{*m(l+a-c)} T^{*k} \in \alpha^k(\mathcal{T}(m)^-). \tag{5}$$

Thus, the assertion of the lemma is true for monomials. To complete the proof, we note that the element of subalgebra $\alpha^k(\mathcal{T}(m)^+) \cdot \alpha^j(\mathcal{T}(m)^-)$ has the form AB, where A is a linear combination of monomials of the form V_1 and B is a linear combination of monomials of the form V_2. Thus, AB is a linear combination of monomials of the form (4) or (5). Therefore,

$$AB \in \alpha^k(\mathcal{T}(m)^+)\alpha^j(\mathcal{T}(m)^-).$$

Consider a family $\{\mathcal{T}_m\}_{m=1}^{\infty}$ of C^*-algebras. The following theorem is true:

Theorem 3. *Suppose that* $m \in \mathbb{N}$. *Then, for any* $n \in \mathbb{N}$,

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \ldots \oplus \alpha^{i_n}(\mathcal{T}(m)) \cong \mathcal{T}(n) \oplus \alpha(\mathcal{T}(n)) \oplus \ldots \oplus \alpha^{n-1}(\mathcal{T}(n)) = \mathcal{T}_n,$$

where $1 \leq i_1 < i_2 < \ldots < i_n \leq m$.

Proof. We prove the theorem by induction. If $n = 1$, then $\alpha^{i_1}(\mathcal{T}(m)) \cong \mathcal{T}(m)$ and $\mathcal{T}(m) \cong \mathcal{T}$. This implies that $\alpha^{i_1}(\mathcal{T}(m)) \cong \mathcal{T}$.

We now show that the theorem is true for $n = 2$, i.e.,

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \alpha^{i_2}(\mathcal{T}(m)) \cong \mathcal{T}(2) \oplus \alpha(\mathcal{T}(2)).$$

Since $i_1 < i_2$, we get

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \alpha^{i_2}(\mathcal{T}(m)) = \alpha^{i_1}(\mathcal{T}(m)) \oplus \alpha^{i_2-i_1}(\mathcal{T}(m)) \cong \mathcal{T}(m) \oplus \alpha^{i_2-i_1}(\mathcal{T}(m)).$$

We denote $i_0 = i_2 - i_1 \geq 1$ and prove that

$$\mathcal{T}(m) \oplus \alpha^{i_0}(\mathcal{T}(m)) \cong \mathcal{T}(2) \oplus \alpha(\mathcal{T}(2)).$$

Consider the isomorphisms

$$\varphi_1 \colon \mathcal{T}(m) \to \mathcal{T}(2) \colon \varphi_1(T^m) = T^2$$

and

$$\varphi_2 \colon \alpha^{i_0}(\mathcal{T}(m)) \to \alpha(\mathcal{T}(2)) \colon \varphi_1(T^{m+i_0} T^{*i_0}) = \alpha(T^2).$$

Denote

$$\varphi = \varphi_1 \oplus \varphi_2 \colon \mathcal{T}(m) \oplus \alpha^{i_0}(\mathcal{T}(m)) \to \mathcal{T}(2) \oplus \alpha(\mathcal{T}(2)).$$

We now show that φ is an isomorphism.

Let $V_1 = T^{am}T^{*bm} \in \mathcal{T}(m)$ and $V_2 = T^{dm+i_0}T^{*cm+i_0} \in \alpha^{i_0}(\mathcal{T}(m))$. According to Lemma 2 in [14], either

(i) $V_1 \cdot V_2 \in \mathcal{T}(m)$

or

(ii) $V_1 \cdot V_2 \in \alpha^{i_0}(\mathcal{T}(m))$.

In order to prove that φ is an isomorphism, it suffices to show that φ is a homomorphism, i.e.,

$$\varphi_1(V_1) \cdot \varphi_2(V_2) \in \mathcal{T}(2) \quad \text{in case (i)}$$

and

$$\varphi_1(V_1) \cdot \varphi_2(V_2) \in \alpha(\mathcal{T}(2)) \quad \text{in case (ii).}$$

Consider the first case, i.e., $V_1 \cdot V_2 \in \mathcal{T}(m)$. In view of the fact that

$$V_1 \cdot V_2 = T^{am}T^{*bm}T^{dm+i_0}T^{*cm+i_0} \in \mathcal{T}(m),$$

we obtain $bm > dm + i_0$. Since $\varphi_1(V_1) = T^{2a}T^{*2b}$ and $\varphi_2(V_2) = T^{2d+1}T^{*(2c+1)}$, we get

$$\varphi_1(V_1) \cdot \varphi_2(V_2) = T^{2a}T^{*2b}T^{2d+1}T^{*(2c+1)} \qquad \text{and} \qquad \varphi_1(V_1) \cdot \varphi_2(V_2) \in \mathcal{T}(2)$$

if and only if $2b > 2d + 1$. We now show that the inequality $bm > dm + i_0$ implies that $2b > 2d + 1$. Indeed,

$$bm > dm + i_0 \Rightarrow b \geq d + \frac{i_0}{m} \Rightarrow 2b > 2d \Rightarrow 2b \geq 2d + 1, \quad b, d \in \mathbb{N}.$$

In the second case, the proof is similar. Thus,

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \alpha^{i_2}(\mathcal{T}(m)) \cong \mathcal{T}(2) \oplus \alpha(\mathcal{T}(2)).$$

Assume that the theorem is true for $n = k$, i.e.,

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \ldots \oplus \alpha^{i_k}(\mathcal{T}(m)) \cong \mathcal{T}(k) \oplus \alpha(\mathcal{T}(k)) \oplus \ldots \oplus \alpha^{i_{k-1}}(\mathcal{T}(k)) = \mathcal{T}_k.$$

This enables us to prove it for $n = k + 1$. It is necessary to show that

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \ldots \oplus \alpha^{i_k+1}(\mathcal{T}(m)) \cong \mathcal{T}(k+1) \oplus \alpha(\mathcal{T}(k+1)) \oplus \ldots \oplus \alpha^{i_k}(\mathcal{T}(k+1)) = \mathcal{T}_{k+1}.$$

By the inductive assumption, we have

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \ldots \oplus \alpha^{i_k}(\mathcal{T}(m)) \cong \mathcal{T}(k) \oplus \alpha(\mathcal{T}(k)) \oplus \ldots \oplus \alpha^{i_{k-1}}(\mathcal{T}(k)) = \mathcal{T}_k.$$

However, on the other hand,

$$\mathcal{T}_k \cong \mathcal{T}(k+1) \oplus \alpha(\mathcal{T}(k+1)) \oplus \ldots \oplus \alpha^{i_{k-1}}(\mathcal{T}(k+1)).$$

By using the above-mentioned equalities and the step of induction for $n = 2$, we get

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \ldots \oplus \alpha^{i_k}(\mathcal{T}(m)) \oplus \alpha^{i_{k+1}}(\mathcal{T}(m))$$

$$\cong \mathcal{T}(k+1) \oplus \alpha(\mathcal{T}(k+1)) \oplus \ldots \oplus \alpha^{i_{k-1}}(\mathcal{T}(k+1)) \oplus \alpha^{i_k}(\mathcal{T}(k+1)) = \mathcal{T}_{k+1}.$$

Corollary 3. *There exists a chain of embedded algebras*

$$\mathcal{T}_2 \hookrightarrow \mathcal{T}_3 \hookrightarrow \ldots \hookrightarrow \mathcal{T}_n \hookrightarrow \ldots,$$

where $\hookrightarrow$ *stands for the embedding of the linear spaces of the corresponding algebras but not of the algebras themselves, i.e.,* $\hookrightarrow$ *does not preserve the structure of an algebra.*

4. Category of C^*-Algebras

We define a unitary operator $u_j : H_j \to l^2(\mathbb{Z}_+),\ 0 \le j \le m-1$, by setting $u_j(e_{j+km}) = e_k$ on the basis elements. Since H_j are invariant spaces for the C^*-algebra $\mathcal{T}_m$, the unitary operator

$$u = u_0 \oplus \ldots \oplus u_{m-1} : H_0 \oplus H_1 \oplus \ldots \oplus H_{m-1} \to \bigoplus_{j=0}^{m-1} l^2(\mathbb{Z}_+)$$

generates the embedding

$$\sigma : \mathcal{T}_m \to \bigoplus_{j=0}^{m-1} B(l^2(\mathbb{Z}_+)),$$

$$\sigma(A) = uAu^*, \quad \text{where} \quad A \in \mathcal{T}_m.$$

Since $T^m e_{i+km} = e_{i+(k+1)m}$, we obtain that

$$\sigma(T^m) = T \oplus \ldots \oplus T$$

is the mth copy of the operator of shift T. The algebra $\mathcal{T}(m)$ is generated by the operators T^m and T^{*m}. Thus, for any $A \in \mathcal{T}(m)$, there exists an operator $B \in \mathcal{T}$ such that

$$\sigma(A) = B \oplus \ldots \oplus B.$$

It is clear that the converse assertion is also true: For any $B \in \mathcal{T}$, there exists an operator $A \in \mathcal{T}(m)$ such that $\sigma(A) = B \oplus \ldots \oplus B$. Therefore, the algebra $\mathcal{T}(m)$ is identified with the algebra $\sigma(\mathcal{T}(m))$:

$$\mathcal{T}(m) \approx \sigma(\mathcal{T}(m)) = m\mathcal{T} = \{A : A = B \oplus B \oplus \ldots \oplus B,\ B \in \mathcal{T}\} \hookrightarrow \bigoplus^{m} \mathcal{T}, \tag{6}$$

where $\bigoplus^{m} \mathcal{T}$ denotes the direct sum of m copies of the Toeplitz algebra $\mathcal{T}$.

As shown in [14],

$$P_j|_{H_i} = \begin{cases} I, & i-1 \geq j, \\ T^m T^{*m}, & i-1 < j, \end{cases} \qquad 0 \leq i \leq m-1.$$

This implies that

$$\sigma(P_i) = TT^* \oplus \ldots \oplus TT^* \oplus I \oplus \ldots \oplus I.$$

In what follows, we always identify projectors $P_i,\ 0 \leq i \leq m-1$, with the projectors $\sigma(P_i)\colon\ P_i \approx \sigma(P_i)$, $0 \leq i \leq m-1$. In particular, by using these results and Lemma 1, we conclude that the subalgebra of compact operators $\mathcal{K}_m$ in $\mathcal{T}_m$ can be identified with the algebra $\sigma(\mathcal{K}_m)$:

$$\mathcal{K}_m \approx \sigma(\mathcal{K}_m) = \bigoplus^m \mathcal{K}. \tag{7}$$

By virtue of Theorem 1 and relations (6) and (7), the algebra $\mathcal{T}_m$ can be identified with the algebra $\sigma(\mathcal{T}_m)$:

$$\mathcal{T}_m \approx \sigma(\mathcal{T}_m) = \Big\{A\colon\ A = (B + K_1) \oplus \ldots \oplus (B + K_m),\ B \in \mathcal{T},\ K_1, \ldots, K_m \in \mathcal{K}\Big\}.$$

Consider a set of numbers $M = \{1, \ldots, m\}$ and denote by

$$N = \big\{(i_1, \ldots, i_k),\ \text{ where }\ i_k \in M,\ k = 1, \ldots, m\big\}$$

the set of all possible collections of numbers of length smaller than m. We introduce the order on N as follows:

$$(i_1, \ldots, i_j) \leq (i_1, \ldots, i_l) \quad \text{for} \quad j \leq l, \qquad \text{and} \qquad i_t \in \{i_1, \ldots, i_l\}, \quad t = 1, \ldots, j.$$

It is clear that the relation on N defined in this way is a partial ordering.

Definition 1. *The category $\mathcal{C}$ consists of a class of objects $Ob_{\mathcal{C}}$ and, for each pair of objects A and B, the set of morphisms (or arrows) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is given. Moreover, each morphism is associated with unique A and B. For a pair of morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, the composition $g \circ f \in \mathrm{Hom}(A, C)$ is given. For each object A, the identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ is given. In addition, the following two axioms hold:*

(a) the operation of composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f\colon A \to B,\ g\colon B \to C,\ h\colon C \to D$;

(b) the action of the identity morphism is trivial: $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$ for all $f\colon A \to B$.

It is clear that $(N, \leq)$ is a category in which the classes of morphisms form partial ordering and objects are all possible collections from N.

We associate each collection $(i_1, i_2, \ldots, i_k) \in N$ with the C^*-algebra

$$\mathcal{T}_m(i_1, i_2, \ldots, i_k) = \mathcal{T}(m) + \mathcal{K}_{i_1, i_2, \ldots, i_k},$$

where

$$\mathcal{K}_{i_1,i_2,\ldots,i_k} = \big\{K_0 \oplus K_0 \oplus \ldots \oplus K_{i_1} \oplus K_{i_2} \oplus \ldots \oplus K_{i_k} \oplus K_0 \oplus \ldots \oplus K_0,\ \text{ where }\ K_0, K_{i_1}, \ldots, K_{i_k} \in \mathcal{K}\big\}$$

$$= \big\{i_1 K_0 \oplus K_{i_1} \oplus K_{i_2} \oplus \ldots \oplus K_{i_k} \oplus (m-k)K_0,\ \text{ where }\ K_0, K_{i_1}, \ldots, K_{i_k} \in \mathcal{K}\big\}.$$

Lemma 5. *For any collection $(i_1, i_2, \ldots, i_k) \in N$, the C^*-algebra $\mathcal{T}_m(i_1, i_2, \ldots, i_k)$ is isomorphic to the C^*-algebra $\mathcal{T}_{k+1}$:*

$$\mathcal{T}_m(i_1, i_2, \ldots, i_k) \cong \mathcal{T}_{k+1}.$$

Proof. We prove the lemma for $k = 2$, i.e., we show that $\mathcal{T}_m(i_1, i_2) \cong \mathcal{T}_3$. Consider decomposition (1). By using the structural analysis of the algebras $\mathcal{T}(m)$ and $\mathcal{K}_{i_1,i_2}$, we conclude that any element $A \in \mathcal{T}_m(i_1, i_2)$ can be represented in the form

$$A = (T \oplus \ldots \oplus T) + (K_0 \oplus K_0 \oplus \ldots \oplus K_{i_1} \oplus K_0 \oplus \ldots \oplus K_{i_2} \oplus K_0 \oplus \ldots \oplus K_0)$$

$$= (T + K_0) \oplus \ldots \oplus (T + K_{i_1}) \oplus \ldots \oplus (T + K_{i_2}) \oplus \ldots \oplus (T + K_0). \tag{8}$$

We define the mapping $\psi\colon \mathcal{T}_m(i_1, i_2) \to \mathcal{T}_3$ as follows:

$$\psi\big((T + K_0) \oplus \ldots \oplus (T + K_{i_1}) \oplus \ldots \oplus (T + K_{i_2}) \oplus \ldots \oplus (T + K_0)\big) = (T + K_0) \oplus (T + K_{i_1}) \oplus (T + K_{i_2}).$$

It is necessary to show that this mapping is a homomorphism. Indeed, let B and D be any elements from $\mathcal{T}_m(i_1, i_2)$. We now prove that, in this case, $\psi(B \cdot D) = \psi(B) \cdot \psi(D)$. It is clear that B and D have the form (8). It follows from the definition of the algebra $\mathcal{T}_m(i_1, i_2)$ that $\mathcal{T}_m(i_1, i_2) \subsetneq \mathcal{T}_m$. On the other hand, since the algebras $\mathcal{T}(m)$ and $\mathcal{K}_{i_1,i_2}$ are invariant under H_i, by using decomposition (1), we establish the invariance of the algebra $\mathcal{T}_m(i_1, i_2)$ under H_i. Thus, decomposition (2) remains true for elements of the algebra $\mathcal{T}_m(i_1, i_2)$. Hence,

$$\psi(B \cdot D) = \psi\Big(\big((T + K_0) \oplus \ldots \oplus (T + K_{i_1}) \oplus \ldots \oplus (T + K_{i_2}) \oplus \ldots \oplus (T + K_0)\big) \times \big((T + K_1) \oplus \ldots \oplus (T + K'_{i_1}) \oplus \ldots \oplus (T + K'_{i_2}) \oplus \ldots \oplus (T + K_1)\big)\Big)$$

$$= \psi\big((T + K_0)(T + K_1) \oplus \ldots \oplus (T \oplus K_{i_1})(T + K'_{i_1}) \oplus \ldots \oplus (T \oplus K_{i_2})(T + K'_{i_2}) \oplus \ldots \oplus (T + K_0)(T + K_1)\big)$$

$$= (T + K_0)(T + K_1) \oplus (T \oplus K_{i_1})(T + K'_{i_1}) \oplus (T \oplus K_{i_2})(TK'_{i_2}). \tag{9}$$

Here, we have used the fact that

$$B \cdot D = (B|_{H_1} \oplus \ldots \oplus B|_{H_m})(D|_{H_1} \oplus \ldots \oplus D|_{H_m}) = B|_{H_1} \cdot D|_{H_1} \oplus \ldots \oplus B|_{H_m} \cdot D|_{H_m}.$$

On the other hand,

$$\begin{aligned}\psi(B)\cdot\psi(D) &= \psi\big((T+K_0)\oplus\ldots\oplus(T+K_{i_1})\oplus\ldots\oplus(T+K_{i_2})\oplus\ldots\oplus(T+K_0)\big)\\ &\qquad\times\psi\big((T+K_1)\oplus\ldots\oplus(T+K'_{i_1})\oplus\ldots\oplus(T+K'_{i_2})\oplus\ldots\oplus(T+K_1)\big)\\ &= \big((T+K_0)\oplus(T+K_{i_1})\oplus(T+K_{i_2})\big)\cdot\big((T+K_1)\oplus(T+K'_{i_1})\oplus(T+K'_{i_2})\big)\\ &= (T+K_0)(T+K_1)\oplus(T\oplus K_{i_1})(T+K'_{i_1})\oplus(T\oplus K_{i_2})(TK'_{i_2}). \qquad (10)\end{aligned}$$

It follows from (9) and (10) that ψ is a homomorphism. By using the structural analysis of the above-mentioned algebras, we can directly establish the injectivity and surjectivity of ψ.

The proof of the theorem for $k > 2$ is the same as for $k = 2$.

We now consider a family $\{\mathcal{T}_m(i_1,\ldots,i_k)\}_{(i_1,\ldots,i_k)\in N}$ of C^*-algebras and define an embedding of the algebra $\mathcal{T}_m(i_1,\ldots,i_l)$ into the algebra $\mathcal{T}_m(i_1,\ldots,i_k)$ (here and in what follows, an embedding is regarded as an embedding of linear spaces of the corresponding algebras). It turns out that this embedding can be determined if and only if $l < k$. Indeed, if $l < k$, then, by Lemma 5, $\mathcal{T}_m(i_1,\ldots,i_l)\cong\mathcal{T}_l$ and $\mathcal{T}_m(i_1,\ldots,i_k)\cong\mathcal{T}_k$. On the other hand, according to Corollary 3 and the structural analysis of the investigated algebras, the embedding $\mathcal{T}_k\hookrightarrow\mathcal{T}_l$ takes place. This means that $\mathcal{T}_m(i_1,\ldots,i_l)\hookrightarrow\mathcal{T}_m(i_1,\ldots,i_k)$. Thus, there exists a natural embedding of the algebra $\mathcal{T}_m(i_1,\ldots,i_l)$ into the algebra $\mathcal{T}_m(i_1,\ldots,i_k)$ for $l < k$.

It is clear that the set of C^*-algebras generated by all possible collections $(i_1,\ldots,i_k)\in N$ forms a category whose morphisms are the above-mentioned embeddings:

$$\big(\{\mathcal{T}_m(i_1,\ldots,i_k)\}_{(i_1,\ldots,i_k)\in N},\hookrightarrow\big).$$

Definition 2. *The functor $\mathcal{F}\colon\mathcal{C}\to\mathcal{D}$ from a category $\mathcal{C}$ into a category $\mathcal{D}$ is a mapping that associates each object $X\in\mathcal{C}$ with an object $\mathcal{F}(X)\in\mathcal{D}$ and each morphism $f\colon X\to Y$ in the category $\mathcal{C}$ with a morphism $\mathcal{F}(f)\colon\mathcal{F}(X)\to\mathcal{F}(Y)$ in the category $\mathcal{D}$. This association must have the following properties: $\mathcal{F}(\mathrm{id}_A)=\mathrm{id}_{\mathcal{F}(A)}$ and $\mathcal{F}(g\circ f)=\mathcal{F}(g)\circ\mathcal{F}(f)$.*

Lemma 6. *There exists a functor between the following categories:*

$$F_m\colon (N,\le)\to\big(\{\mathcal{T}_m(i_1,\ldots,i_k)\}_{(i_1,\ldots,i_k)\in N},\hookrightarrow\big).$$

Proof. Since the algebra $\mathcal{T}_m(i_1,\ldots,i_k)$ is defined for all collections $(i_1,\ldots,i_k)$ from N, we conclude that F_m maps objects and morphisms from $(N,\le)$ into objects and morphisms from

$$\big(\{\mathcal{T}_m(i_1,\ldots,i_k)\}_{(i_1,\ldots,i_k)\in N},\hookrightarrow\big),$$

i.e., $F_m((i_1,\ldots,i_k))=\mathcal{T}_m(i_1,\ldots,i_k)$ and $F_m(\le)=\hookrightarrow$. In addition, if $(i_1,\ldots,i_k)\le(i_1,\ldots,i_j)$, then

$$F_m((i_1,\ldots,i_k))=\mathcal{T}_m(i_1,\ldots,i_k)\hookrightarrow F_m((i_1,\ldots,i_j))=\mathcal{T}_m(i_1,\ldots,i_j).$$

This means that F_m is a functor.

The number k is called the length of the C^*-algebra $\mathcal{T}_m(i_1,\ldots,i_k)$.

Remark. If, in the category of C^*-algebras $\big(\{\mathcal{T}_m(i_1, \ldots, i_k)\}_{(i_1,\ldots,i_k)\in N}, \hookrightarrow\big)$, the role of objects is played solely by the C^*-algebras with identical lengths, then the category turns into a bundle of C^*-algebras:

$$\big(\{\mathcal{T}_m(i_1, \ldots, i_k)\}_{(i_1,\ldots,i_k)\in N}, \cong\big),$$

where, by virtue of Lemma 5, the role of morphisms is played by the isomorphisms of C^*-algebras .

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