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ON THE CONSTRUCTIVE DESCRIPTION OF GIBBS MEASURES FOR THE POTTS MODEL ON A CAYLEY TREE

M. M. Rahmatullaev,[1,2] F. K. Rafikov,[3] and Sh. Kh. Azamov[4] UDC 517.9

We consider the Potts model on a Cayley tree and prove the existence of Gibbs measures constructed by the method proposed in [H. Akin, U. A. Rozikov, and S. Temir, *J. Stat. Phys.,* **142**, 314 (2011)]. In addition, we prove that there exist (k_0)-translation invariant Gibbs measures for the Potts model on a Cayley tree and compute the free energy of these Gibbs measures.

1. Introduction

The notion of Gibbs measure for the Potts model on a Cayley tree is introduced in a standard way (see [1–4]). In [5], the ferromagnetic Potts model with three states on a Cayley tree of order two was studied and it was shown that there exists a critical temperature T_c such that, for $T < T_c$, there exist three translation-invariant Gibbs measures and uncountably many Gibbs measures that are not translation invariant. The results obtained in [6] were generalized in [5] for the Potts model with finitely many states on a Cayley tree of any (finite) order.

In [7], it was proved that the translation-invariant Gibbs measure for the antiferromagnetic Potts model on a Cayley tree with external field is unique. The work [8] was devoted to the investigation of the Potts model on a Cayley tree with countably many states in the presence of a nonzero external field. It was proved that this model possesses a unique translation-invariant Gibbs measure.

All translation-invariant Gibbs measures of the Potts model with q $(q \geq 3)$ states (spins) were found in [9]. In particular, it was shown that, for sufficiently low temperatures, their number is equal to $2^q - 1$. It was proved that there exist $[q/2]$ critical temperatures and the exact number of translation-invariant Gibbs measures was indicated for each intermediate temperature.

In [10], a weakly periodic Gibbs measure was introduced and some these measures were found for the Ising model. In [11], weakly periodic ground states and weakly periodic Gibbs measures were studied for the Potts model. In [16, 17], weakly periodic Gibbs measures were investigated for the Potts model with external field. In [15, 18], the free energy was studied for the known Gibbs measures in the Ising and Potts models.

In [12], some other Gibbs measures (in what follows, they are called Gibbs measures obtained by using the ART-structure) were constructed for the Ising model on Cayley tree. In [13, 14], a new Gibbs measure on a Cayley tree of order k was constructed for the Ising model with the help of a translation-invariant Gibbs measure on a Cayley tree of order k_0 $(k_0 < k)$. This measure was called a (k_0)-translation-invariant Gibbs measure.

The aim of the present paper is to construct a Gibbs measure obtained by using the ART-structure and a (k_0)-translation-invariant Gibbs measure for the Potts model. The paper is organized as follows: Main definitions

[1] V. Romanovskii Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan; e-mail: mrahmatullaev@rambler.ru.
[2] Corresponding author.
[3] Kokand State Pedagogic University, Kokand, Uzbekistan; e-mail: rafiqovfaxriddin@gmail.com.
[4] Kokand State Pedagogic University, Kokand, Uzbekistan; e-mail: azamovsherzodjon@gmail.com.

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and known results are presented in Sec. 2. The results for Gibbs measures obtained by using the ART-structure are presented in Sec. 3. The results obtained for (k_0)-translation-invariant Gibbs measures are given in Sec. 4. The free energies for Gibbs measures obtained by using the ART-structure and one (2)-translation-invariant Gibbs measure are computed in Sec. 5.

2. Definitions and Known Facts

A Cayley tree T^k of order $k \geq 1$ is an infinite tree, i.e., a graph without loops with exactly $k+1$ edges leaving each vertex. Let $T^k = (V, L, i)$, where V is the set of vertices of T^k, L is the set of its edges, and i is the incidence function that associates each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called the *nearest neighbors of the vertex* and denoted by $l = \langle x, y\rangle$.

The distance $d(x, y),\ x, y \in V$, on a Cayley tree is given by the formula

$$d(x,y) = \min\bigl\{d \mid \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \quad \text{such that } \langle x_0, x_1\rangle, \dots, \langle x_{d-1}, x_d\rangle\bigr\}.$$

For fixed $x^0 \in V$, we denote

$$W_n = \{x \in V \mid d(x, x^0) = n\},$$

$$V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

$$L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

Further, for $x \in W_n$, we set

$$S(x) = \{y \in W_{n+1} \colon d(x,y) = 1\}.$$

It is known that there exists a one-to-one correspondence between the set V of vertices of a Cayley tree of order $k \geq 1$ and a group G_k obtained as the free product of $k+1$ cyclic groups of the second order with generatrices $a_1, a_2, \dots, a_{k+1}$, respectively, (see [4]).

We consider a model in which spin variables take values from the set $\Phi = \{1, 2, \dots, q\},\ q \geq 2$, and are located at the vertices of the tree. Then the *configuration* σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$, and the set of all configurations coincides with $\Omega = \Phi^V$. Let $\Omega_n = \Phi^{V_n}$ be the space of configurations defined on V_n.

The Hamiltonian of the Potts model is introduced as follows:

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J \in \mathbb{R}$, $\langle x, y\rangle$ are the nearest neighbors and δ_{ij} is the Kronecker symbol

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

We consider a set $\Phi' = \{\sigma_1, \dots, \sigma_q\}$, where $\sigma_i \in \mathbb{R}^{q-1}$, and introduce a scalar product $\sigma_i \sigma_j$ as follows:

$$\sigma_i \sigma_j = \begin{cases} -\dfrac{1}{q-1} & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j. \end{cases}$$

This yields

$$\delta_{\sigma(x)\sigma(y)} = \frac{q-1}{q}\left(\sigma(x)\sigma(y) + \frac{1}{q-1}\right).$$

By using this formula, we can reduce the Hamiltonian of the Potts model to the Hamiltonian of the Ising model with q values of spin, namely,

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y).$$

We fix a basis $\{e_1, \dots, e_{q-1}\}$ in $\mathbb{R}^{q-1}$ such that $e_i = \sigma_i,\ i = 1, 2, \dots, q-1$. It is clear that

$$\sum_{i=1}^{q} \sigma_i = 0.$$

Note that if $h = (h_1, \dots, h_{q-1})$, then

$$h\sigma_i = \begin{cases} \dfrac{q}{q-1} h_i - \dfrac{1}{q-1} \sum_{j=1}^{q-1} h_j & \text{for} \quad i = 1, \dots, q-1, \\ -\dfrac{1}{q-1} \sum_{j=1}^{q-1} h_j & \text{for} \quad i = q. \end{cases}$$

We define a finite-dimensional distribution of the probability measure μ in the volume V_n as follows:

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\right\}, \tag{2}$$

where $\sigma_n \in \Omega_n,\ \beta = 1/T,\ T > 0$ is temperature, $h_x \in \mathbb{R}^{q-1}$,

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y),$$

and Z_n^{-1} is the normalization factor,

$$Z_n = Z_n(\beta, h) = \sum_{\sigma_n \in \Omega_n} \exp\left(-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\right).$$

The collection of vectors $h = \{h_x \in \mathbb{R}^{q-1}, x \in V\}$ specifies the (generalized) boundary condition.

Definition 1. *The free energy corresponding to the boundary condition h is defined as the following limit (if it exists):*

$$E(\beta, h) = -\lim_{n\to\infty} \frac{1}{\beta|V_n|} \ln Z_n(\beta, h).$$

It is said that the probability distributions (2) are consistent if, for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$,

$$\sum_{\sigma^{(n)}\in\Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{3}$$

where $\sigma_{n-1} \vee \sigma^{(n)}$ is the union of configurations.

In this case, there exists a unique measure μ on Φ^V such that

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n)$$

for all n and $\sigma_n \in \Phi^{V_n}$. This measure is called the *limit Gibbs measure* corresponding to Hamiltonian (1) and the vector-valued function h_x, $x \in V$.

The following statement describes the condition imposed on the function h_x to guarantee the consistency of measures $\mu_n(\sigma_n)$:

Theorem 1 [15]. *Measures (2) satisfy condition (3) if and only if, for all $x \in V \setminus \{x^0\}$, the equation*

$$h_x = \sum_{y\in S(x)} F(h_y, \theta) \tag{4}$$

is true. Here, the function $F\colon\, h = (h_1, \ldots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta) = (F_1, \ldots, F_{q-1}) \in \mathbb{R}^{q-1}$ is given by the formula

$$F_i = \ln\left(\frac{(\theta-1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}}\right), \qquad \theta = \exp(J\beta).$$

Each solution h_x of the functional equation (4) is associated with a single Gibbs measure, and vise versa.

3. ART-Structure

In [12], some Gibbs measures were constructed for the Ising model on a Cayley tree. In this section, we construct a similar measure for the Potts model.

Let μ be a Gibbs measure on a Cayley tree of order $k_0 \leq k$, let $h_x(\mu) \in \mathbb{R}^{q-1}$ be a collection of vectors corresponding to the measure μ, and let $q \geq 2$.

For μ, we now construct a Gibbs measure $\nu = \nu(\mu)$ on a Cayley tree of order $k \geq k_0$. Let V^k be the set of all vertices of T^k and let V^{k_0} be the set of all vertices of T^{k_0}. We construct a collection of vectors $\tilde{h}_x = \tilde{h}_x(\nu) \in \mathbb{R}^{q-1}$ on T^K corresponding to the measure $\nu(\mu)$ as follows:

$$\tilde{h}_x = \begin{cases} h_x(\mu), & x \in V^{k_0}, \\ \overline{0}, & x \in V^k \setminus V^{k_0}, \end{cases} \tag{5}$$

where $\overline{0} = (0, 0, \ldots, 0) \in \mathbb{R}^{q-1}$.

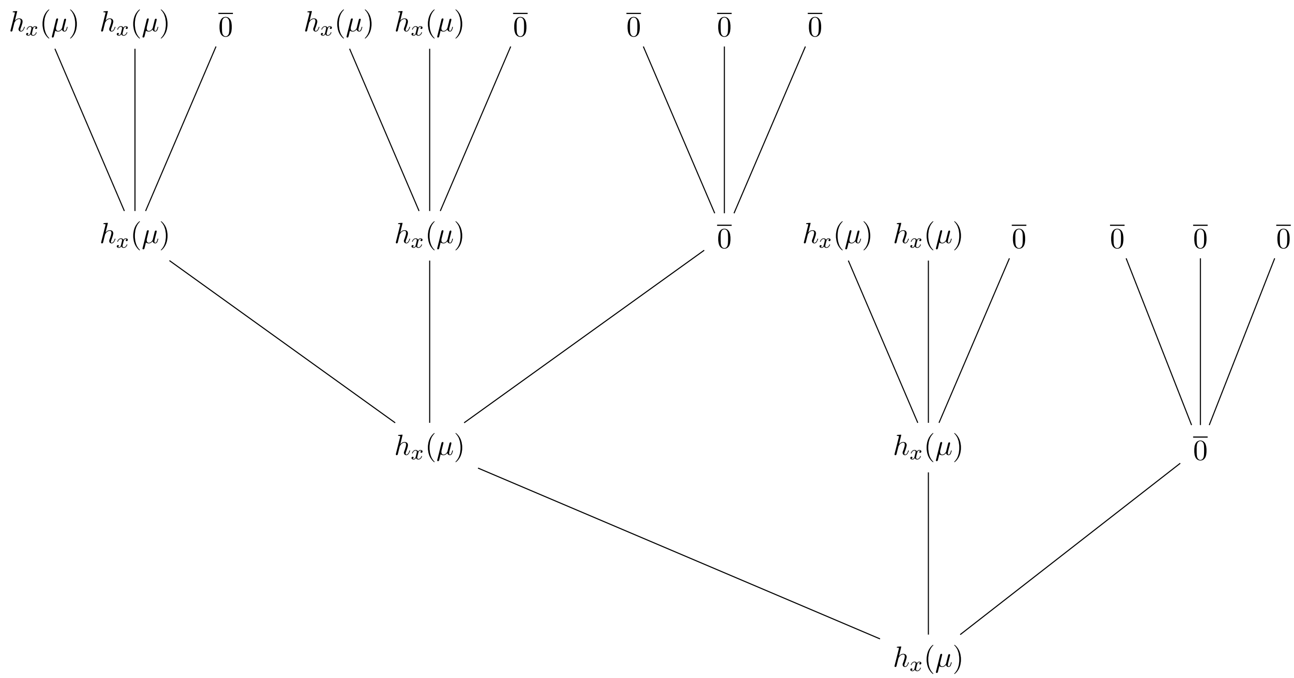

Fig. 1

For a Cayley tree of order $k = 3$, this function is shown in Fig. 1.

For $x \in V^k$, by $S_{k_0}(x)$ we denote arbitrary k_0, $1 \le k_0 \le k$, elements of $S(x)$. Note that $S(x) = S_k(x)$. We now verify whether (5) satisfies (4) on the Cayley tree T^k.

Let $x \in V^{k_0} \subset V^k$. Then the following equalities are true:

$$\tilde{h}_x = \sum_{y \in S_k(x)} F(\tilde{h}_y, \theta)$$

$$= \sum_{y \in S_k(x) \cap V^{k_0}} F(h_x(\mu), \theta) + \sum_{y \in S_k(x) \cap (V^k \setminus V^{k_0})} F(\overline{0}, \theta)$$

$$= \sum_{y \in S_{k_0}(x)} F(h_x(\mu), \theta) = h_x(\mu).$$

Here, we have used the equality $F(\overline{0}, \theta) = 0$, i.e.,

$$\sum_{y \in S(x) \cap \left(V^k \setminus V^{k_0}\right)} F\left(\overline{0}, \theta\right) = 0.$$

It is easy to see that if $x \in V^k \setminus V^{k_0}$, then $S(x) \subset V^k \setminus V^{k_0}$. Thus, we get

$$\tilde{h}_x = \sum_{y \in S(x)} F(\tilde{h}_y, \theta) = \sum_{y \in S(x)} F(\overline{0}, \theta) = 0.$$

Hence, the function $\tilde{h}_x$ defined by (5) satisfies the functional equation (4). By $\nu = \nu(\mu)$ we denote the measure of the corresponding collection of vectors $\tilde{h}_x$. This measure is called a Gibbs measure obtained by using the ART-structure.

Remark 1.

1. On a Cayley tree of order 2, for $\theta > \theta_{cr} = 1 + 2\sqrt{q-1}$, there exist translation-invariant Gibbs measures different from the measure μ_0 corresponding to the vector $h_x = \overline{0}\ \forall x \in V^2$ (see [9]). We construct a Gibbs measure obtained by using the ART-structure with the help of these measures. Hence, the condition $\theta > \theta_{cr} = 1 + 2\sqrt{q-1}$ must be satisfied. The Gibbs measures thus constructed differ from the known measures (see [9, 19, 20]).

2. In the case where $k > 2$, other Gibbs measures may exist for the other values of θ, (see [11, 16, 17]). By using these measures, one can construct a Gibbs measure obtained by using the ART-structure.

As a result, we have proved the following theorem:

Theorem 2. *Suppose that $k \geq 3$. If $\theta > \theta_{cr} = 1 + 2\sqrt{q-1}$, then, for the Potts model on a Cayley tree, there exists an uncountable set of Gibbs measures obtained by using the ART-structure.*

4. (k_0)-Translation-Invariant Gibbs Measure

For any k and q, translation-invariant Gibbs measures for the Potts model were studied in [9].

In the case where $k = 2$ and $q = 3$, for the collection of translation-invariant vectors, we obtain the following system of equations from (4):

$$\begin{aligned} h_1 &= \sum_{y \in S(x)} \ln \frac{\theta e^{h_1} + e^{h_2} + 1}{\theta + e^{h_1} + e^{h_2}}, \\ h_2 &= \sum_{y \in S(x)} \ln \frac{\theta e^{h_2} + e^{h_1} + 1}{\theta + e^{h_1} + e^{h_2}}. \end{aligned} \tag{6}$$

Since $k = 2$, we get the following system of equations:

$$h_1 = 2 \ln \frac{\theta e^{h_1} + e^{h_2} + 1}{\theta + e^{h_1} + e^{h_2}},$$

$$h_2 = 2 \ln \frac{\theta e^{h_2} + e^{h_1} + 1}{\theta + e^{h_1} + e^{h_2}}.$$

This system possesses the solutions

$$\big(h_1^{(i)}, 0\big), \quad \big(0, h_1^{(i)}\big), \quad \big(-h_1^{(i)}, -h_1^{(i)}\big), \quad (0,0), \quad i = 1, 2,$$

where

$$h_1^{(i)} = 2 \ln x_i, \qquad x_1 = \frac{\theta - 1 - \sqrt{(\theta-1)^2 - 8}}{2}, \qquad x_2 = \frac{\theta - 1 + \sqrt{(\theta-1)^2 - 8}}{2}. \tag{7}$$

In [13, 14], for the Ising model, with the help of translation-invariant Gibbs measures on a Cayley tree of order k_0, a new Gibbs measure on a Cayley tree of order k, $k_0 < k$, was constructed and called a (k_0)-translation-invariant Gibbs measure. In this section, for the Potts model, with the help of a translation-invariant Gibbs measure on a Cayley tree of order 2 $(k_0 = 2)$, by analogy with the structure proposed in [13, 14], we prove the existence of new Gibbs measures on a Cayley tree of the fifth order. These measures are also called (k_0)-translation-invariant.

The following theorem is true:

Theorem 3. *For the Potts model on a Cayley tree of the fifth order with $q = 3$ and $\theta = \dfrac{11}{2}$, there exist at least six (2)-translation-invariant Gibbs measures .*

Proof. Consider a Cayley tree of the fifth order. Recall that, for $x \in V^k$, any k_0, $1 \le k_0 \le k$, elements of $S(x)$ are denoted by $S_{k_0}(x)$. First, by using $\big(h_1^{(1)}, 0\big)$ and $\big(h_1^{(2)}, 0\big)$, we construct a collection of vectors h_x on V^5 satisfying the functional equation (4). We specify this collection of vectors as follows:

(l_1) If, at a vertex $x \in V^5$, we have $h_x = \big(h_1^{(1)}, 0\big)$, then we associate the vertices of $S_4(x)$ with the vector $h_x = \big(h_1^{(1)}, 0\big)$ and the other vertices of $S_1(x)$ with the vector $h_x = \big(h_1^{(2)}, 0\big)$. If, at a vertex $x \in V^5$, we have $h_x = \big(h_1^{(2)}, 0\big)$, then we associate the vertices of $S_3(x)$ with the vector $h_x = \big(h_1^{(2)}, 0\big)$ and the remaining vertices of $S_2(x)$ with the vector $h_x = \big(h_1^{(1)}, 0\big)$. As a result, we arrive at the following system of equations from (4):

$$h_1^{(1)} = 4 \ln \frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + \ln \frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}},$$
$$h_1^{(2)} = 2 \ln \frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + 3 \ln \frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}}. \tag{8}$$

In view of the fact that

$$h_1^{(i)} = 2 \ln \frac{\theta e^{h_1^{(i)}} + 2}{\theta + 1 + e^{h_1^{(i)}}}, \quad i = 1, 2, \tag{9}$$

it follows from Eqs. (8) that

$$2 \ln \frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + \ln \frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}} = 0. \tag{10}$$

Note that

$$h_1^{(i)} = h_1^{(i)}(\theta), \quad i = 1, 2.$$

Hence, the left-hand side of (10) depends only on θ. For the values of θ satisfying (10) and $\theta > \theta_{cr} = 1 + 2\sqrt{2}$, the collection of vectors h_x on V^5 constructed according to the rules (l_1) satisfies the functional equation (4).

By using (10) and (9), we get

$$h_1^{(1)} + \frac{h_1^{(2)}}{2} = 0. \tag{11}$$

Hence, in view of (11) and (7), we arrive at the equation

$$\left(\frac{\theta-1-\sqrt{(\theta-1)^2-8}}{2}\right)^2=\frac{2}{\theta-1+\sqrt{(\theta-1)^2-8}}.$$

This equation possesses the solution $\theta=\frac{11}{2}$, i.e., for $\theta=\frac{11}{2}$, the collection of vectors constructed according to the rules (l_1) satisfies the functional equation (4). Following [13, 14], for the Potts model, we say that the measure corresponding to the collection of vectors and constructed by the rules (l_1) is a (2)-translation-invariant Gibbs measure. Similarly, for the vectors $h_x=\big(0,h_1^{(i)}\big)$, $i=1,2$, we prove the existence of one more (2)-translation-invariant Gibbs measure for $\theta=\frac{11}{2}$.

Further, by using $\big(h_1^{(1)},0\big)$, $\big(h_1^{(2)},0\big)$, and $\big(-h_1^{(1)},-h_1^{(1)}\big)$, we construct a collection of vectors h_x on V^5 satisfying the functional equation (4). We specify this collection of vectors h_x as follows:

(l_2) If, at the vertex $x\in V^5$, we have $h_x=\big(-h_1^{(1)},-h_1^{(1)}\big)$, then we associate the vertices of $S_2(x)$ with the vector $h_x=\big(-h_1^{(1)},-h_1^{(1)}\big)$, the vertices of $S_2(x)$ with the vector $h_x=\big(h_1^{(1)},0\big)$, and the remaining vertices of $S_1(x)$ with the vector $h_x=\big(h_1^{(2)},0\big)$. If, at the vertex $x\in V^5$, we have $\big(h_1^{(1)},0\big)$ or $h_x=\big(h_1^{(2)},0\big)$, then we associate the vertices of $S(x)$ with the vectors $\big(h_1^{(1)},0\big)$ and $h_x=\big(h_1^{(2)},0\big)$ by the rules (l_1). As a result, we derive the following system of equations from (4):

$$\begin{aligned}
-h_1^{(1)}&=2\ln\frac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}}+2\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}}+\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}},\\
-h_1^{(2)}&=2\ln\frac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}},\\
h_1^{(1)}&=4\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}}+\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}},\\
h_1^{(2)}&=2\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}}+3\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}.
\end{aligned}\tag{12}$$

In view of (9), we get Eq. (10) from (12). Equation (10) possesses the solution $\theta=\frac{11}{2}$, i.e., for $\theta=\frac{11}{2}$, the collection of vectors constructed according to the rules (l_2) satisfies the functional equation (4). Similarly, for the set of vectors

$$\big\{\big(0,h_1^{(1)}\big),\big(0,h_1^{(2)}\big),\ \big(-h_1^{(1)},-h_1^{(1)}\big)\big\},\quad \big\{\big(0,h_1^{(1)}\big),\ \big(0,h_1^{(2)}\big),\ \big(-h_1^{(2)},-h_1^{(2)}\big)\big\},$$

$$\big\{\big(h_1^{(1)},0\big),\ \big(h_1^{(2)},0\big),\ \big(-h_1^{(2)},-h_1^{(2)}\big)\big\},$$

we can prove the existence of three more collections of vectors satisfying the functional equation (4).

This implies that, for $\theta = \frac{11}{2}$, there exist six (2)-translation-invariant Gibbs measures.
Theorem 3 is proved.

Remark 2. On a Cayley tree of order k, $k \geq 6$, for $\theta = \frac{11}{2}$, by using the (2)-translation-invariant Gibbs measures described in Theorem 3, one can construct a Gibbs measure obtained by using the ART-structure.

Consider a Cayley tree of order $k = a + b + 2$, $a, B \in n$. Let

$$B(a,b) = \Big\{\theta \in \mathbb{R}_+ : \theta > 1 + 2\sqrt{2} \text{ and } ah_1^{(1)} + bh_1^{(2)} = 0\Big\}.$$

Note that the set $B(a,b)$ is nonempty because the case $a = 2$, $b = 1$ was considered in Theorem 2, i.e.,

$$\theta = \frac{11}{2} \in B(2,1).$$

We now prove the following theorem:

Theorem 4. *For the Potts model on a Cayley tree of order $k = a + b + 2$, $a, b \in n$, there exist at least six (2)-translation-invariant Gibbs measures for $q = 3$ and $\theta \in B(a,b)$.*

Proof. By using $\big(h_1^{(1)}, 0\big)$ and $\big(h_1^{(2)}, 0\big)$, we construct a collection of vectors h_x on V^k, $k = a + b + 2$, $a, b \in n$, satisfying the functional equation (4). We specify this collection of vectors as follows:

(l_3) Let $k = a + b + 2$, $a, B \in n$. If, at the vertex $x \in V^k$, we have $h_x = \big(h_1^{(1)}, 0\big)$, then we associate the vertices of $S_{a+2}(x)$ with the vector $h_x = \big(h_1^{(1)}, 0\big)$ and the remaining vertices of $S_b(x)$ with the vector $h_x = \big(h_1^{(2)}, 0\big)$. If, at the vertex $x \in V^k$, we have $h_x = \big(h_1^{(2)}, 0\big)$, then we associate the vertices of $S_{b+2}(x)$ with the vector $h_x = \big(h_1^{(2)}, 0\big)$ and the remaining vertices of $S_a(x)$ with the vector $h_x = \big(h_1^{(1)}, 0\big)$. As a result, we derive the following system of equations from (4):

$$\begin{aligned} h_1^{(1)} &= (a+2)\ln\frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + b\ln\frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}}, \\ h_1^{(2)} &= a\ln\frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + (b+2)\ln\frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}}. \end{aligned} \tag{13}$$

In view of (9), it follows from (13) that

$$a\ln\frac{\theta e^{h_1^{(1)}} + 2}{\theta + 1 + e^{h_1^{(1)}}} + b\ln\frac{\theta e^{h_1^{(2)}} + 2}{\theta + 1 + e^{h_1^{(2)}}} = 0. \tag{14}$$

Note that $h_1^{(1)}$ and $h_1^{(2)}$ depend on θ and are real for $\theta > \theta_{cr} = 1 + 2\sqrt{2}$ [see (19)]. We rewrite Eq. (14) in the form

$$ah_1^{(1)} + bh_1^{(2)} = 0. \tag{15}$$

Thus, for

$$\theta \in B(a,b) = \Big\{\theta \in \mathbb{R}_+ :\ \theta > 1 + 2\sqrt{2}\ \text{ and }\ ah_1^{(1)} + bh_1^{(2)} = 0\Big\},$$

the collection of vectors constructed by the rules (l_2) satisfies the functional equation (4).

As in the previous case, for the Potts model, a measure corresponding to the collection of vectors constructed according to the rules (l_3) is called a (2)-translation-invariant Gibbs measure. In a similar way, for the vectors $h_x = \big(0, h_1^{(i)}\big),\ i = 1, 2,$ we can prove the existence of one more (2)-translation-invariant Gibbs measure for $\theta \in B(a,b)$.

By using $\big(h_1^{(1)}, 0\big),\ \big(h_1^{(2)}, 0\big)$, and $\big(-h_1^{(1)}, -h_1^{(1)}\big)$, we can now construct a collection of vectors h_x on V^k satisfying the functional equation (4). We can specify this collection of vectors h_x as follows:

(l_4) If, at the vertex $x \in V^k$, we have $h_x = \big(-h_1^{(1)}, -h_1^{(1)}\big)$, then we associate the vertices of $S_2(x)$ with the vector $h_x = \big(-h_1^{(1)}, -h_1^{(1)}\big)$, the vertices of $S_a(x)$ with the vector $h_x = \big(h_1^{(1)}, 0\big)$, and the remaining vertices of $S_b(x)$ with the vector $h_x = \big(h_1^{(2)}, 0\big)$. If, at the vertex $x \in V^k$, we have $\big(h_1^{(1)}, 0\big)$ or $h_x = \big(h_1^{(2)}, 0\big)$, then we associate the vertices of $S(x)$ with the vectors $\big(h_1^{(1)}, 0\big)$ and $h_x = \big(h_1^{(2)}, 0\big)$ by the rules (l_3). As a result, we obtain the following system of equations from (4):

$$\begin{aligned}
-h_1^{(1)} &= 2\ln\frac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}} + a\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}},\\
-h_1^{(2)} &= 2\ln\frac{(\theta+1)e^{-h_1^{(1)}}+1}{\theta+2e^{-h_1^{(1)}}},\\
h_1^{(1)} &= (a+2)\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + b\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}},\\
h_1^{(2)} &= a\ln\frac{\theta e^{h_1^{(1)}}+2}{\theta+1+e^{h_1^{(1)}}} + (b+2)\ln\frac{\theta e^{h_1^{(2)}}+2}{\theta+1+e^{h_1^{(2)}}}.
\end{aligned}\tag{16}$$

By using (9), we derive Eq. (15) from (16). This equation is equivalent to (14). Thus, for $\theta \in B(a,b)$, the collection of vectors h_x constructed according to the rules (l_4) satisfies the functional equation (4). Similarly, for the set of vectors

$$\big\{\big(0, h_1^{(1)}\big),\ \big(0, h_1^{(2)}\big),\ \big(-h_1^{(1)}, -h_1^{(1)}\big)\big\},\quad \big\{\big(0, h_1^{(1)}\big), \big(0, h_1^{(2)}\big),\ \big(-h_1^{(2)}, -h_1^{(2)}\big)\big\},$$

$$\big\{\big(h_1^{(1)}, 0\big),\ \big(h_1^{(2)}, 0\big),\ \big(-h_1^{(2)}, -h_1^{(2)}\big)\big\},$$

we can prove the existence of three more collections of vectors satisfying the functional equation (4).

As a result, we conclude that six (2)-translation-invariant Gibbs measures exist for $\theta \in B(a,b)$ on the Cayley tree of order $k = a + b + 2,\ a, b \in n$.

Theorem 4 is proved.

5. Free Energies for the Gibbs Measures Obtained by Using the ART-Structure and for the (k_0)-Translation-Invariant Gibbs Measures

In this section, we find the free energies for the Gibbs measures obtained by using the ART-structure and for the (k_0)-translation-invariant Gibbs measures.

The following theorem specifying the generalized form of the free measure was proved in [15]:

Theorem 5. *For the collection of vectors satisfying conditions (4), the free energy is given by the formula*

$$E(\beta, h) = -\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{x\in V_n} a(x), \tag{17}$$

where

$$a(x) = \frac{1}{q\beta} \sum_{i=1}^{q} \ln\left(\sum_{u=1}^{q} \exp\left\{ (J\beta\sigma_i + h_x)\sigma_u \right\} \right). \tag{18}$$

In the present work, for the translation-invariant collection of vectors of the form

$$h = \Big(\underbrace{h_*, h_*, \ldots, h_*}_{m}, 0, 0, \ldots, 0 \Big), \quad m \ge 0, \tag{19}$$

we compute the free energy and consider the following cases:

Case $m = 0$. In this case, we find

$$h = \overline{0} = (0, 0, \ldots, 0) \in \mathbb{R}^{q-1}.$$

By using (17), we get

$$\begin{aligned} E_{TI}(\beta, \overline{0}) &= -a(x) \\ &= -\frac{1}{q\beta} \sum_{i=1}^{q} \ln\left(\sum_{u=1}^{q} \exp(J\beta\sigma_i\sigma_u) \right) \\ &= -\frac{1}{q\beta} \sum_{i=1}^{q} \ln\left(\exp(J\beta) + (q-1)\exp\left(\frac{J\beta}{1-q} \right) \right) \\ &= -\frac{1}{q\beta} q \ln\left(\exp(J\beta) + (q-1)\exp\left(\frac{J\beta}{1-q} \right) \right) \\ &= -J - \frac{1}{\beta} \ln\left(1 + (q-1)\exp\left(\frac{Jq\beta}{1-q} \right) \right). \end{aligned}$$

Case $m \neq 0$. In this case, for vectors of the form (19), we compute the free energy as follows:

$$E_{TI}(\beta, m, h_x) = -\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x) = -a(x)$$

$$= -\frac{1}{q\beta} \sum_{i=1}^{q} \ln\left(\sum_{u=1}^{q} \exp\left\{(J\beta\sigma_i + h_x)\sigma_u\right\} \right)$$

$$= -\frac{q-m}{q\beta} \ln\Bigg(m e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta - \frac{m}{q-1} h_*\right)}$$

$$+ (q-m-1) e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \Bigg)$$

$$- \frac{m}{q\beta} \ln\Bigg((m-1) e^{\left(-\frac{J\beta}{q-1} + \frac{q-m}{q-1} h_*\right)} + e^{\left(J\beta + \frac{q-m}{q-1} h_*\right)}$$

$$+ (q-m) e^{\left(-\frac{J\beta}{q-1} - \frac{m}{q-1} h_*\right)} \Bigg). \tag{20}$$

5.1. In this section, we compute the free energy of the Gibbs measure obtained by using the ART-structure and corresponding to the collection of vectors of the form (5). By $E_{\mathrm{ART}}(\beta, \tilde{h})$ we denote the free energy of the Gibbs measure obtained by using the ART-structure. By virtue of (17) and (18), we get

$$E_{\mathrm{ART}}(\beta, \tilde{h}) = -\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x)$$

$$= -\frac{1}{q\beta} \lim_{n\to\infty} \frac{|V_n^k| - |V_n^{k_0}|}{|V_n|} \sum_{i=1}^{q} \ln\left(\sum_{u=1}^{q} \exp\left\{(J\beta\sigma_i + h_x(\mu))\sigma_u\right\} \right)$$

$$- \frac{1}{q\beta} \lim_{n\to\infty} \frac{|V_n^{k_0}|}{|V_n|} \sum_{i=1}^{q} \ln\left(\sum_{u=1}^{q} \exp\left\{(J\beta\sigma_i + \bar{0})\sigma_u\right\} \right). \tag{21}$$

Since

$$\lim_{n\to\infty} \frac{|V_n^{k_0}|}{|V_n^k|} = \frac{k-1}{k_0-1} \lim_{n\to\infty} \frac{(k_0+1)k_0^n - 2}{(k+1)k^n - 2} = 0,$$

in view of the inequality

$$0 \le a(x) \le C_b,$$

we get

$$0 \le \sum_{x \in V_n^{k_0}} a(x) \le |V_n^{k_0}| C_b.$$

Thus, we can write

$$\lim_{n\to\infty}\frac{1}{|V_n^k|}\sum_{x\in V_n^{k_0}} a(x)=0.$$

This yields

$$E_{\mathrm{ART}}(\beta,\tilde{h})=E(\beta,h_x(\mu)). \tag{22}$$

If, in (5), we consider a translation-invariant collection of vectors of the form (19) as $h_x(\mu)$, then we get

$$E_{\mathrm{ART}}(\beta,\tilde{h})=E_{TI}(\beta,m,h_x), \tag{23}$$

i.e., the free energy of Gibbs measures obtained by using the ART-structure is equal to the free energy of translation-invariant Gibbs measures.

5.2. We now find the free energy for (2)-translation-invariant Gibbs measures constructed according to the rules (l_1).

We introduce the notation

$$\overline{h}_i=(h_1^{(i)},0),\quad i=1,2.$$

By $V_{n,i}^5,\ i=1,2$ (resp., $W_{n,i}^5,\ i=1,2$) we denote the sets of vertices V_n (resp., W_n) associated, according to the rules (l_1), with the vectors $\overline{h}_i,\ i=1,2$.

We can easily show that

$$|W_{1,1}^5|=4,\quad |W_{2,1}^5|=20=4\cdot 5,\quad |W_{3,1}^5|=100=4\cdot 5^2,\quad \ldots,\quad |W_{n,1}^5|=4\cdot 5^{n-1}.$$

It is clear that

$$|V_{n,1}^5|=1+|W_{1,1}^5|+|W_{2,1}^5|+|W_{3,1}^5|+\ldots+|W_{n,1}^5|$$

$$=1+4\cdot 5+4\cdot 5^2+\ldots+4\cdot 5^{n-1}=5^n.$$

It is known (see [4]) that, for a Cayley tree of the fifth order, we have

$$|V_n|=\frac{3\cdot 5^n-1}{2}.$$

This yields

$$|V_{n,2}^5|=|V_n|-|V_{n,1}^5|=\frac{5^n-1}{2}.$$

We now find the free energy

$$E_{(2)}(\beta,h)=-\lim_{n\to\infty}\frac{1}{|V_n|}\sum_{x\in V_n} a(x)$$

$$= -\frac{1}{q\beta}\lim_{n\to\infty}\frac{|V_{n,1}^5|}{|V_n|}\sum_{i=1}^{q}\ln\left(\sum_{u=1}^{q}\exp\left\{(J\beta\sigma_i+\overline{h}_1)\sigma_u\right\}\right)$$

$$-\frac{1}{q\beta}\lim_{n\to\infty}\frac{|V_{n,2}^5|}{|V_n|}\sum_{i=1}^{q}\ln\left(\sum_{u=1}^{q}\exp\left\{(J\beta\sigma_i+\overline{h}_2)\sigma_u\right\}\right).$$

Since

$$\lim_{n\to\infty}\frac{|V_{n,1}^5|}{|V_n|}=\frac{2}{3},\qquad \lim_{n\to\infty}\frac{|V_{n,2}^5|}{|V_n|}=\frac{1}{3},$$

the free energy of (2)-translation-invariant Gibbs measures constructed by the rules (l_1) takes the form

$$E_{(2)}(\beta,h)=\frac{2}{3}E_{TI}(\beta,1,\overline{h}_1)+\frac{1}{3}E_{TI}(\beta,1,\overline{h}_2). \tag{24}$$

Remark 3. If $\overline{h}_1=\overline{h}_2$, then the corresponding (2)-translation-invariant collection of vectors is translation-invariant. In this case, the free energy of the (2)-translation-invariant collection of vectors is equal to the free energy of the translation-invariant collection of vectors, i.e., the equality

$$E_{(2)}(\beta,h)=E_{TI}(\beta,1,\overline{h}_1)$$

holds.

Similarly, we can find the free energies of (2)-translation-invariant Gibbs measures obtained in Theorems 3 and 4.

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