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ON LEIBNIZ ALGEBRAS WHOSE SUBALGEBRAS ARE EITHER IDEALS OR SELF-IDEALIZING SUBALGEBRAS

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A subalgebra S of a Leibniz algebra L is called self-idealizing in L if it coincides with its idealizer $\mathrm{I}_L(S)$. We study the structure of Leibniz algebras whose subalgebras are either ideals or self-idealizing subalgebras.

1. Introduction

Let L be an algebra over the field F with two binary operations $+$ and $[\,,\,]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a, b, c \in L.$ We also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

For the first time, Leibniz algebras appeared in the work by Blokh [3] who called them *D-algebras*. However, at that time, they did not arouse any significant interest and were not extensively developed. Only twenty years later, these algebras attracted attention of the researchers. The term “Leibniz algebras” appeared in the monograph by Loday [13] and in his work [14]. In [15], Loday and Pirashvili began to study the properties of Leibniz algebras. The theory of Leibniz algebras has been very extensively developed in many directions. Some results of this theory can be found in the monograph [1]. Note that the Lie algebras can be regarded as a special case of Leibniz algebras. Conversely, if L is a Leibniz algebra such that $[a,a] = 0$ for each element $a \in L,$ then it is a Lie algebra. Thus, the Lie algebras can be characterized as anticommutative Leibniz algebras. In this connection, we can mention a certain parallel with associative structures such as, e.g., groups and associative rings. In these structures, we can indicate a significant difference between the Abelian and non-Abelian groups and between commutative and noncommutative rings. The difference between the Lie and Leibniz algebras becomes noticeable even in the analysis of the first natural types of Leibniz algebras. Thus, the cyclic Lie algebras have the dimension 1, whereas the structure of cyclic Leibniz algebras is much more complicated [4]. Another example of the same kind: Lie algebras each subalgebra of which is an ideal are Abelian algebras, whereas in the case of Leibniz algebras, the situation is much more complicated [8]. Similar situations are also encountered in the investigation of the other types of Leibniz algebras (see, e.g., [7, 10–12]).

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In the present paper, the term “Leibniz algebra” is used for left Leibniz algebras that are not Lie algebras.

Let L be a Leibniz algebra over the field F and let A and K be subalgebras of the algebra L. A *left idealizer* of the subalgebra A in K is defined by the following rule:

$$\mathrm{I}_K^{\mathrm{left}}(A) = \{x \in K \mid [x,a] \in A \text{ for all elements } a \in A\}.$$

A left idealizer of A in K is a subalgebra in K. Indeed, let $x, y \in \mathrm{I}_K^{\mathrm{left}}(A),\ a \in A,\ \alpha \in F$. Then

$$[x-y,a] = [x,a] - [y,a] \in A, \qquad [\alpha x, a] = \alpha[x,a] \in A,$$

$$[[x,y],a] = [x,[y,a]] - [y,[x,a]] \in A.$$

Similarly, a *right idealizer* of the subalgebra A in K is given by the rule:

$$\mathrm{I}_K^{\mathrm{right}}(A) = \{x \in K \mid [a,x] \in A \text{ for all elements } a \in A\}.$$

Unlike the left idealizer, the right idealizer of A in K is not necessarily a subalgebra. Example 1.7 presented in [2] illustrates this assertion.

An *idealizer* of the subalgebra A in K is given by the rule

$$\mathrm{I}_K(A) = \{x \in K \mid [x,a],[a,x] \in A \text{ for all elements } a \in A\} = \mathrm{I}_K^{\mathrm{left}}(A) \cap \mathrm{I}_K^{\mathrm{right}}(A).$$

An idealizer of A in K is a subalgebra in K. Indeed, let $x, y \in \mathrm{I}_K(A),\ a \in A,\ \alpha \in F$. As above, we can prove that $x - y, \alpha x \in \mathrm{I}_K(A)$. Moreover,

$$[[x,y],a] = [x,[y,a]] - [y,[x,a]] \in A,$$

$$[a,[x,y]] = [[a,x],y] + [x,[a,y]] \in A.$$

For any subalgebra A of the Leibniz algebra L, we have the following increasing series:

$$A = A_0 \le A_1 \le \ldots A_\alpha \le A_{\alpha+1} \le \ldots A_\gamma,$$

where $A_1 = \mathrm{I}_L(A),\ A_{\alpha+1} = \mathrm{I}_L(A_\alpha)$ for all ordinal numbers $\alpha,\ A_\lambda = \bigcup_{\mu<\lambda} A_\mu$ for all limit ordinal numbers λ, and $A_\gamma = \mathrm{I}_L(A_\gamma)$. This series is called an *upper idealizer series* of the subalgebra A in L. If $\gamma = 1$, then we get two cases:

$$A_1 = \mathrm{I}_L(A) = L \qquad \text{or} \qquad A_1 = \mathrm{I}_L(A) = A.$$

The following types of subalgebras correspond to these two cases: A is an ideal of the algebra L and A is a self-idealizing subalgebra in L (i.e., $A = \mathrm{I}_L(A)$). Thus, the following natural question arises: *What can be said about a Leibniz algebra each subalgebra of which is either an ideal or a self-idealizing subalgebra?*

The aim of the present paper is to give a detailed description of these Leibniz algebras.

The first type of these algebras includes Leibniz algebras all subalgebras of which are ideals. The description of these algebras can be found in [8].

The Leibniz algebra L is called *extraspecial* if it satisfies the following conditions:

(i) the center $\zeta(L)$ of the algebra L is nontrivial and one-dimensional,

(ii) the quotient algebra $L/\zeta(L)$ is Abelian.

We say that an extraspecial algebra E is *strongly extraspecial* if

$$[x, x] \neq 0$$

for each element $x \notin \zeta(E)$.

The Leibniz algebra L all subalgebras of which are ideals has the following structure: $L = E \oplus Z$, where Z is a subalgebra of the center of algebra L and E is a strongly extraspecial algebra.

The procedure of investigation of Leibniz algebras whose subalgebras are either ideals or self-idealizing subalgebras is split into the following steps:

Let L be a Leibniz algebra over the field F. Then L contains a maximal locally nilpotent ideal [9] (Corollary C1). This ideal is called a *locally nilpotent radical* of the algebra L and denoted by $\mathrm{Ln}(L)$.

In the first stage, we investigate Leibniz algebras in which a locally nilpotent radical is Abelian and noncyclic.

Theorem A. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that a locally nilpotent radical A of the algebra L is Abelian and noncyclic. Then the following conditions are satisfied:*

(i) $A = \zeta^{\mathrm{left}}(L)$*;*

(ii) $L = A \oplus W$*, where W is a subalgebra of dimension 1, $W = Fw$, and $W = \mathrm{I}_L(W)$;*

(iii) there exists a nonzero element $\sigma \in F$ such that $[w, a] = \sigma a$ for each element of the algebra $a \in A$.

Conversely, if the Leibniz algebra L satisfies these conditions, then each subalgebra of the algebra L is either an ideal in L or a self-idealizing subalgebra in L.

In the next stage, it is natural to investigate the case where a locally nilpotent radical is non-Abelian and noncyclic. In this case, we obtain the following results:

Theorem B1. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that $L \neq \mathrm{Ln}(L)$ and the locally nilpotent radical $\mathrm{Ln}(L)$ is non-Abelian and noncyclic. If*

$$[\mathrm{Ln}(L), \mathrm{Ln}(L)] \leq \zeta(L),$$

then $\mathrm{char}(F) = 2$ and the following conditions are satisfied:

(i) $K = \mathrm{Ln}(L)$ *is a strongly extraspecial subalgebra; moreover, $[x, x] \neq 0$ for each element $x \notin \mathrm{Leib}(L)$;*

(ii) $[K, K] = \zeta(L) = \mathrm{Leib}(L)$*;*

(iii) $L = K + \langle v \rangle$*, where $[v, v] = \eta z$ for some nonzero element $\eta \in F$ and*

$$[v + \zeta(K), x + \zeta(K)] = x + \zeta(K)$$

for each element $x \in K \setminus \zeta(K)$.

Theorem B2. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that a locally nilpotent radical $\mathrm{Ln}(L) = K \neq L$ is non-Abelian and noncyclic. If $\zeta(L)$ does not contain $[K, K]$, then the following conditions are satisfied:*

(i) $\mathrm{char}(F) \neq 2$;

(ii) every subalgebra from K is an ideal of the algebra L;

(iii) $\mathrm{Ln}(L)$ *is a strongly extraspecial subalgebra;*

(iv) $[K, K] = \zeta(K) = \mathrm{Leib}(L) = \langle z\rangle$ *has dimension 1;*

(v) $L = K + \langle v\rangle$, *where* $[v + \zeta(K), x + \zeta(K)] = x + \zeta(K)$ *for each element* $x \in K \setminus \zeta(K)$, $[v, z] = 2z$, *and* $[v, v] = \nu z$ *for some element* $\nu \in F$ *(ν can be equal to zero).*

The last step is connected with the investigation of the case where the locally nilpotent radical is cyclic. In this case, its dimension is equal either to 1 or to 2. The following theorem describes the second case:

Theorem C. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that the locally nilpotent radical $\mathrm{Ln}(L)$ is cyclic and has dimension 2. Then either $L \neq \mathrm{Ln}(L)$ or $\mathrm{char}(F) = 2$ and L has the basis $\{z, a, v\}$ such that $[z, z] = [z, a] = [a, z] = [z, v] = [v, z] = 0$, $[a, a] = z$, $[v, v] = \eta z$, $[v, a] = a + \lambda z$, $[a, v] = a + \mu z$, $\eta, \lambda, \mu \in F$, and the polynomial $X^2 + (\mu + \lambda)X + \eta$ does not have solutions in the field F.*

The case where the dimension of locally nilpotent radical is equal to 1 is reduced to the study of Lie algebras whose Abelian subalgebras have dimension 1. This case requires separate investigation. In this connection, we note that an infinite-dimensional Lie algebra proper subalgebras of which have dimension 1 is, in a certain sense, an analog of the so-called Tarski monster from the group theory. The problem of existence of these Lie algebras is one of the most interesting and complicated unsolved problems of the general theory of Lie algebras.

2. Leibniz Algebras Whose Subalgebras Are Either Ideals or Self-Idealizing Subalgebras. The Case Where the Locally Nilpotent Radical Is Abelian

Every Leibniz algebra L contains a specific ideal. By $\mathrm{Leib}(L)$ we denote a subspace generated by all elements $[a, a]$, $a \in L$. It can be shown that $\mathrm{Leib}(L)$ is an ideal of the algebra L. It is called a *Leibniz kernel* of the algebra L.

Note that the quotient algebra $L/\mathrm{Leib}(L)$ is a Lie algebra. Conversely, if H is an ideal of the algebra L such that the quotient algebra L/H is Lie, then $\mathrm{Leib}(L) \leq H$.

Lemma 2.1. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that A and B are subalgebras of the algebra L such that B is an ideal in A. If the quotient algebra A/B is nontrivial and cyclic, then either $\dim_F(A/B) = 1$ or $A/B = Fa_1 \oplus Fa_2$ and*

$$[a_1, a_1] = a_2, \quad [a_2, a_1] = [a_2, a_2] = 0, \quad [a_1, a_2] = \lambda a_2,$$

where $\lambda \in \{0, 1\}$.

Proof. Since B is an ideal in A, we have $A \leq \mathrm{I}_L(B)$, i.e., $B \neq \mathrm{I}_L(B)$ and, hence, B is an ideal of the algebra L. Let a be an element from A such that $A/B = \langle a + B\rangle$. If $[a, a] \in B$, then the cyclic subalgebra $\langle a+B\rangle$ has dimension 1. We now assume that $d=[a,a]\notin B$. This means that the kernel $\mathrm{Leib}(A/B)$ is nontrivial. This subalgebra is Abelian and, in particular, each its subspace is a subalgebra. If we assume that $\dim_F(\mathrm{Leib}(A/B))>1$, then we get $\langle d+B\rangle\neq\mathrm{Leib}(A/B)$. In this case, the cyclic subalgebra $\langle d+B\rangle$ cannot be self-idealizing. Hence, $\langle d,B\rangle$ also cannot be self-idealizing. This means that $\langle d,B\rangle$ is an ideal of the algebra L. Therefore, $\langle d+B\rangle$ is an ideal in the quotient algebra L/B. In this case, $[a+B,d+B],[d+B,a+B]\in\langle d+B\rangle$. This means that

$$\langle a+B\rangle = F(a+B)\oplus F(d+B)$$

and, in particular, $\dim_F(\langle a+B\rangle)=2$. We now can apply the description of Leibniz algebras of dimension 2 (see, e.g., [5]).

Lemma 2.1 is proved.

Lemma 2.2. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that A and B are subalgebras of the algebra L such that B is an ideal in A and the quotient algebra A/B is locally nilpotent. If A/B is noncyclic, then every subalgebra from A that contains B is an ideal of the algebra L. In particular, A and B are also ideals of the algebra L.*

Proof. Since the quotient algebra A/B is nontrivial, as in the proof of Lemma 2.1, we can show that B is an ideal of the algebra L. Let a be an arbitrary element from A such that $a\notin B$. Since A/B is noncyclic, we get $\langle a+B\rangle\neq A/B$. Thus, we can choose an element b of A such that $b\notin\langle a,B\rangle$. Since A/B is locally nilpotent, we conclude that

$$\langle a+B,b+B\rangle/B = (\langle a,b\rangle + B)/B$$

is nilpotent. Then [2]

$$\mathrm{I}_{(\langle a,b\rangle+B)/B}(\langle a+B\rangle/B)\neq\langle a+B\rangle/B.$$

This means that $\mathrm{I}_A(\langle a,B\rangle)\neq\langle a,B\rangle$, i.e., $\langle a,B\rangle$ is an ideal of the algebra L.

Let

$$\mathfrak{S}=\{V\mid V \text{ is a subalgebra from } A \text{ such that } B\leq V \text{ and } V/B \text{ is cyclic}\}.$$

If $V\in\mathfrak{S}$, then, as shown above, V is an ideal of the algebra L. It is clear that subalgebras from $\mathfrak{S}$ generate A. This means that A is an ideal of the algebra L.

Lemma 2.2 is proved.

Corollary 2.1. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that A and B are subalgebras of the algebra L such that B is an ideal in A and the quotient algebra A/B is Abelian. If $\dim_F(A/B)>1$, then each subalgebra from A that contains B is an ideal of the algebra L. In particular, A and B are also ideals of the algebra L.*

Corollary 2.2. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. If A is an Abelian subalgebra of the algebra L such that $\dim_F(A)>1$, then each subalgebra from A is an ideal of the algebra L. In particular, A is also an ideal of the algebra L.*

Corollary 2.3. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that A is a locally nilpotent subalgebra of the algebra L. If A is noncyclic, then each subalgebra from A is an ideal of the algebra L. In particular, A is also an ideal of the algebra L.*

Let L be a Leibniz algebra over the field F, let M be a nonempty subset of L, and let H be a subalgebra of the algebra L. We set

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle\},$$

$$\mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [M, a] = \langle 0 \rangle\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the *left annihilator* of the subset M in H and the subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the *right annihilator* of the subset M in H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) = \big\{a \in H \mid [a, M] = [M, a] = \langle 0 \rangle\big\}$$

is called the *annihilator* of the subset M in H.

It is easy to see that all these subsets are subalgebras of the algebra L. Moreover, if M is a left ideal of the algebra L, then we can easily prove that $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of the algebra L. We can also prove that if M is an ideal in L, then $\mathrm{Ann}_L^{\mathrm{right}}(M)$ is a left ideal of the algebra L. Finally, $\mathrm{Ann}_L(M)$ is an ideal of the algebra L.

A *left* (resp., *right*) *center* $\zeta^{\mathrm{left}}(L)$ (resp., $\zeta^{\mathrm{right}}(L)$) of the Leibniz algebra L is defined by the rule

$$\zeta^{\mathrm{left}}(L) = \big\{x \in L \mid [x, y] = 0 \text{ for any element } y \in L\big\}$$

(or, resp.,

$$\zeta^{\mathrm{right}}(L) = \big\{x \in L \mid [y, x] = 0 \text{ for any element } y \in L\big\}\big).$$

It is easy to see that the left center of the algebra L is an ideal in L. However, in the general case, this cannot be said about the right center. Moreover, $\mathrm{Leib}(L) \le \zeta^{\mathrm{left}}(L)$ and, hence, $L/\zeta^{\mathrm{left}}(L)$ is a Lie algebra. The right center is a subalgebra of the algebra L. In the general case, the left and right centers are different. Moreover, they may even have different dimensions (see [6]).

The *center* $\zeta(L)$ of the algebra L is defined by the rule

$$\zeta(L) = \big\{x \in L \mid [x, y] = [y, x] = 0 \text{ for any element } y \in L\big\}.$$

The center is an ideal of the algebra L.

Let L be a Leibniz algebra over the field F. A linear transformation f of the algebra L is called *differentiation* if

$$f([a, b]) = [f(a), b] + [a, f(b)]$$

for all $a, b \in L$. By $\mathrm{End}_F(L)$ we denote the set of all linear transformations of the algebra L. Then $\mathrm{End}_F(L)$ is an associative algebra with respect to the operations $+$ and $\circ$. As usual, $\mathrm{End}_F(L)$ is a Lie algebra with respect to the operations $+$ and $[\,,\,]$, where $[f, g] = f \circ g - g \circ f$ for all $f, g \in \mathrm{End}_F(L)$. It can be shown that the subset $\mathrm{Der}(L)$ of all differentiations of the algebra L is a subalgebra of the Lie algebra $\mathrm{End}_F(L)$ (see, e.g., [5]).

Lemma 2.3. *Suppose that L is a Leibniz algebra over the field F and A is an Abelian ideal of the algebra L. If each subalgebra from A is an ideal of the algebra L, then, for any element $x \in L$, there exist elements $\lambda_x, \rho_x \in F$ such that $[x, a] = \lambda_x a$ and $[a, x] = \rho_x a$ for any element $a \in A$.*

Proof. If $\dim_F(A) = 1$, then everything is obvious. Assume that $\dim_F(A) > 1$. Since the ideal A is Abelian, the subspace Fa is a subalgebra in A for every element $a \in A$. Thus, a cyclic subalgebra $\langle a \rangle = Fa$ is an ideal of the algebra L. If $x \in L$, then $[x, a] = \alpha a$ (resp., $[a, x] = \beta a$) for some elements $\alpha, \beta \in F$. Since $\dim_F(A) > 1$, we can choose an element $c \in A$ such that a and c are linearly independent. Reasoning similarly, for the element c, we conclude that $[x, c] = \gamma c$ (resp., $[c, x] = \sigma c$) for some elements $\gamma, \sigma \in F$. Thus, we get

$$[x, a - c] = [x, a] - [x, c] = \alpha a - \gamma c \quad (\text{resp., } [a - c, x] = [a, x] - [c, x] = \beta a - \sigma c).$$

On the other hand, $a - c \in A$. Hence, $F(a - c)$ is an ideal of the algebra L. Therefore,

$$[x, a - c] = \eta(a - c) = \eta a - \eta c \quad (\text{resp., } [a - c, x] = \mu(a - c) = \mu a - \mu c)$$

for some $\eta, \mu \in F$. This means that $\alpha a - \gamma c = \eta a - \eta c$ (resp., $\beta a - \sigma c = \mu a - \mu c$). Thus,

$$\alpha = \eta = \gamma \quad (\text{resp., } \beta = \mu = \sigma).$$

In other words, for any element $x \in L$, there exist elements $\lambda_x, \rho_x \in F$ such that $[x, a] = \lambda_x a$ and $[a, x] = \rho_x a$ for any $a \in A$.

Lemma 2.3 is proved.

Lemma 2.4. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras and that A is a maximal Abelian ideal of the algebra L that contains* $\mathrm{Leib}(L)$. *Assume that* $\dim_F(A) > 1$. *Then the following conditions are satisfied:*

(i) $A = \zeta^{\mathrm{left}}(L)$*;*

(ii) $L/\mathrm{Ann}_L(A)$ *has dimension 1;*

(iii) for every $x \in L$, there exists an element $\sigma_x \in F$ such that $[x, a] = \sigma_x a$ for any $a \in A$;

(iv) each subalgebra from $\mathrm{Ann}_L(A)$ *is an ideal of the algebra L.*

Proof. For an element $x \in L$, we consider a mapping $\mathrm{l}_x \colon A \to A$ defined by the rule: $\mathrm{l}_x(a) = [x, a]$ for any element $a \in A$. Then the mapping l_x is the differentiation of the ideal A and the set $\{\mathrm{l}_x \mid x \in L\}$ is a subalgebra of the algebra $\mathrm{Der}(A)$ of all differentiations of the ideal A (see, e.g., [5]).

By Corollary 2.2, every subalgebra from A is an ideal of the algebra L. Thus, by Lemma 2.3, for any element $x \in L$, there exist elements $\sigma_x, \rho_x \in F$ such that $[x, a] = \sigma_x a$ and $[a, x] = \rho_x a$ for any element $a \in A$.

Since $\mathrm{Leib}(L) \le \zeta^{\mathrm{left}}(L)$, A contains an element a_0 such that $[a_0, x] = 0$ for any element $x \in L$. This means that $[a, x] = 0$ for each element $a \in A$. Since this is true for any element $x \in L$, we get $A \le \zeta^{\mathrm{left}}(L)$. Since $\zeta^{\mathrm{left}}(L)$ is an Abelian ideal of the algebra L and A is the maximal Abelian ideal of the algebra L, we find $A = \zeta^{\mathrm{left}}(L)$.

Consider a mapping $\delta \colon L \to F$ defined by the rule: $\delta(x) = \sigma_x$ for any element $x \in L$. For elements $x, y \in L$, we obtain

$$\sigma_{x+y} a = [x + y, a] = [x, a] + [y, a] = \sigma_x a + \sigma_y a = (\sigma_x + \sigma_y)a,$$

$$\sigma_{\beta x} a = [\beta x, a] = \beta[x, a] = \beta(\sigma_x a) = (\beta \sigma_x)a.$$

This implies that $\sigma_{x+y} = \sigma_x + \sigma_y$ and $\sigma_{\beta x} = \beta\sigma_x$ for all $x, y \in L,\ \beta \in F$. This means that the mapping δ is linear. Moreover,

$$\operatorname{Ker}(\delta) = \big\{x \in L \mid \delta(x) = \sigma_x = 0\big\},$$

i.e., $[x, a] = 0$ for each element $x \in A$. In other words,

$$\operatorname{Ker}(\delta) \leq \operatorname{Ann}_L^{\text{left}}(A).$$

The inverse inclusion is obvious. Therefore,

$$\operatorname{Ker}(\delta) = \operatorname{Ann}_L^{\text{left}}(A).$$

As indicated above, $\operatorname{Ann}_L^{\text{left}}(A)$ is a two-sided ideal of the algebra L. Hence, the quotient algebra $L/\operatorname{Ann}_L^{\text{left}}(A)$ is isomorphic to F. In particular, it has dimension 1.

It follows from the equality $A = \zeta^{\text{left}}(L)$ that $L = \operatorname{Ann}_L^{\text{right}}(A)$. Thus,

$$\operatorname{Ann}_L^{\text{left}}(A) = \operatorname{Ann}_L(A).$$

Let $z \in \operatorname{Ann}_L(A)$. If $z \in A$, then, as indicated above, the subalgebra $\langle z\rangle$ is an ideal of the algebra L. Assume that $z \notin A$. By Lemma 2.1, the dimension of the cyclic subalgebra $\langle a\rangle$ does not exceed 2. If $\dim_F(\langle z\rangle) = 1$, then $\langle z\rangle = Fz$ and $\langle z\rangle \cap A = \langle 0\rangle$. Thus, for any nonzero element $a \in A$, we obtain $[a, z] = [z, a] = 0$. This means that $a \in \mathrm{I}_L(\langle z\rangle)$ and, in particular, $\mathrm{I}_L(\langle z\rangle) \neq \langle z\rangle$. In this case, $\langle z\rangle$ is an ideal of the algebra L. We now assume that $\dim_F(\langle z\rangle) = 2$. Setting $v = [z, z]$, we get $v \in A$. Since $\dim_F(A) > 1$, one can choose an element $d \in A$ such that $Fv \cap Fd = \langle 0\rangle$. Then $Fd \cap \langle z\rangle = \langle 0\rangle$. Thus, we again obtain

$$[d, z] = [z, d] = 0.$$

This means that $d \in \mathrm{I}_L(\langle z\rangle)$. Hence, $\mathrm{I}_L(\langle z\rangle) \neq \langle z\rangle$. In other words, $\langle z\rangle$ is an ideal of the algebra L. Thus, each cyclic subalgebra from $\operatorname{Ann}_L(A)$ is an ideal of the algebra L. Therefore, every subalgebra from $\operatorname{Ann}_L(A)$ is an ideal of the algebra L.

Lemma 2.4 is proved.

Corollary 2.4. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Suppose that $\dim_F(\operatorname{Leib}(L)) > 1$. Then the dimension of the quotient algebra $L/\operatorname{Ln}(L)$ does not exceed 1 and every subalgebra from $\operatorname{Ln}(L)$ is an ideal of the algebra L.*

Proof. Let A be a maximal Abelian ideal from L that contains $\operatorname{Leib}(L)$. By Lemma 2.4, every subalgebra from $\operatorname{Ann}_L(A)$ is an ideal of the algebra L. Thus, the annihilator $\operatorname{Ann}_L(A)$ is nilpotent [8]. Hence,

$$\operatorname{Ann}_L(A) \leq \operatorname{Ln}(L).$$

By Lemma 2.4, the dimension of the quotient algebra $L/\operatorname{Ln}(L)$ does not exceed 1 and every subalgebra from $\operatorname{Ln}(L)$ is an ideal of the algebra L.

Proof of Theorem A. Since the kernel $\operatorname{Leib}(L)$ is Abelian, we get $\operatorname{Leib}(L) \leq A$. The locally nilpotent radical A is noncyclic and, hence, $\dim_F(A) > 1$. Since A is a maximal locally nilpotent ideal, A is a maximal

Abelian ideal of the algebra L. By Lemma 2.4, every subalgebra from A is an ideal of the algebra L, and the dimension of the quotient algebra L/A does not exceed 1. Assume that $L \neq A$. We choose an element v such that $L = A \oplus Fv$. By Lemma 2.4, $A = \zeta^{\text{left}}(L)$ and there exists an element $\sigma \in F$ such that $[v, a] = \sigma a$ for any element $a \in A$. Since $A \neq L$, we have $\sigma \neq 0$. Let b be an arbitrary element from A. As indicated above, the subalgebra Fb is an ideal of the algebra L. If d is an arbitrary element from Fb, then $d = \lambda b$ for some $\lambda \in F$. Hence,

$$d = \lambda(\sigma^{-1}\sigma)b = \lambda\sigma^{-1}(\sigma b) = \lambda\sigma^{-1}[v, b] = [v, \lambda\sigma^{-1}b] \in [v, Fb].$$

This means that $[v, Fb] = Fb$. Since this relation is true for every one-dimensional subalgebra of A, we get

$$A = [v, A].$$

Let x be an arbitrary element of the algebra L. Since $\dim_F(L/A) = 1$, the quotient algebra L/A is Abelian. This means that $[v, x] = c \in A$. As shown above, A contains an element u such that $c = [v, u]$. Then $[v, x] = [v, u]$ and, hence,

$$[v, x - u] = 0.$$

This means that $x - u \in \mathrm{Ann}_L^{\text{right}}(v)$, i.e., $x \in A + \mathrm{Ann}_L^{\text{right}}(v)$. Since x is an arbitrary element from L, we get

$$L = A + \mathrm{Ann}_L^{\text{right}}(v).$$

Let $a \in A \cap \mathrm{Ann}_L^{\text{right}}(v)$ and let $a \neq 0$. Then $[v, a] = 0$. On the other hand, $[v, a] = \sigma a$, where σ is nonzero. We arrive at a contradiction. This proves that

$$A \cap \mathrm{Ann}_L^{\text{right}}(v) = \langle 0 \rangle.$$

Setting $W = \mathrm{Ann}_L^{\text{right}}(v)$, we get $L = A \oplus W$. It follows from the isomorphism $W \cong L/A$ that the subalgebra W is Abelian and has dimension 1, i.e., $W = Fw$. We also have $w = \lambda v + a_1$ for any element $\lambda \in F$ and $a_1 \in A$. Since $w \notin A$, we get $\lambda \neq 0$. If we now replace w with $\lambda^{-1}w$, then we can assume that $w = v + a_2$. Thus,

$$[w, a] = [v, a] = \sigma a$$

for every element $a \in A$.

If we assume that $\mathrm{I}_L(W) \neq W$, then the subalgebra W is an ideal. However, in this case, the algebra L is Abelian, which is impossible. This contradiction shows that the subalgebra W is self-idealizing.

Conversely, assume that L is a Leibniz algebra satisfying all conditions imposed above. It follows from conditions (i) and (iii) that each cyclic subalgebra from A is an ideal of the algebra L. This means that each subalgebra from A is an ideal of the algebra L.

Let S be an arbitrary subalgebra of the algebra L. If A contains S, then, as shown above, S is an ideal of the algebra L. Assume that A does not contain S. Then S contains an element $\mu w + e$, where $0 \neq \mu \in F$ and $e \in A$. We have $L = A + S$. If we now assume that $\mathrm{I}_L(S) \neq S$, then we can choose an element $a \in A$ such that $[S, a] \leq S$ and $a \notin S$. Hence,

$$[\mu w + e, a] = \mu[w, a] = \mu\sigma a \in S.$$

Since $\mu\sigma \neq 0$, we get $a \in S$, which is impossible. This contradiction shows that $\mathrm{I}_L(S) = S$.

Theorem A is proved.

3. Leibniz Algebras Whose Subalgebras Are Either Ideals or Self-Idealizing Subalgebras. The Case Where the Locally Nilpotent Radical Is Not Abelian

Lemma 3.1. *Suppose that L is a Leibniz algebra over the field F and f is the differentiation of the algebra L. Then*

$$f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L), \quad f(\zeta^{\text{right}}(L)) \leq \zeta^{\text{right}}(L), \quad \textit{and} \quad f(\zeta(L)) \leq \zeta(L).$$

Proof. Let x be an arbitrary element of the algebra L and let $z \in \zeta^{\text{left}}(L)$. Then $[z, x] = 0$. Since differentiation is a linear mapping, we get $f([z, x]) = 0$. On the other hand,

$$0 = f(0) = f([z, x]) = [f(z), x] + [z, f(x)] = [f(z), x],$$

i.e., $f(z) \in \zeta^{\text{left}}(L)$. Let $z \in \zeta^{\text{right}}(L)$. Then $[x, z] = 0$ and

$$0 = f(0) = f([x, z]) = [f(x), z] + [x, f(z)] = [x, f(z)],$$

i.e., $f(z) \in \zeta^{\text{right}}(L)$. Combining both results, we get $f(\zeta(L)) \leq \zeta(L)$.

Lemma 3.1 is proved.

Lemma 3.2. *Suppose that L is a Leibniz algebra over the field F, where $\operatorname{char}(F) \neq 2$, $L = Fa_1 \oplus Fa_2$, $[a_1, a_1] = a_2$, and $[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = 0$. The linear mapping f is the differentiation of the algebra L if and only if $f(a_1) = \alpha a_1 + \beta a_2$ for some elements $\alpha, \beta \in F$ and $f(a_2) = 2\alpha a_2$.*

Proof. It follows from the equality $\zeta(L) = Fa_2$ and Lemma 3.1 that $f(a_2) = \gamma a_2$ for some element $\gamma \in F$. Thus,

$$\gamma a_2 = f(a_2) = f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)]$$

$$= [\alpha a_1 + \beta a_2, a_1] + [a_1, \alpha a_1 + \beta a_2] = \alpha[a_1, a_1] + \alpha[a_1, a_1] = 2\alpha a_2.$$

This yields $f(a_2) = 2\alpha a_2$.

Let $f(a_1) = \alpha a_1 + \beta a_2$ and let x and y be arbitrary elements of the algebra L. Then $x = \lambda a_1 + \mu a_2$ and $y = \sigma a_1 + \rho a_2$, which implies that

$$[x, y] = [\lambda a_1 + \mu a_2, \sigma a_1 + \rho a_2]$$

$$= \lambda\sigma[a_1, a_1] + \lambda\rho[a_1, a_2] + \mu\sigma[a_2, a_1] + \mu\rho[a_2, a_2] = \lambda\sigma a_2,$$

$$f(x) = f(\lambda a_1 + \mu a_2) = \lambda f(a_1) + \mu f(a_2) = \lambda(\alpha a_1 + \beta a)_2) + \mu(2\alpha a_2)$$

$$= \lambda\alpha a_1 + \lambda\beta a_2 + 2\mu\alpha a_2 = \lambda\alpha a_1 + (\lambda\beta + 2\mu\alpha)a_2,$$

$$f(y) = f(\sigma a_1 + \rho a_2) = \sigma f(a_1) + \rho f(a_2) = \sigma(\alpha a_1 + \beta a_2) + \rho(2\alpha a_2)$$

$$= \sigma\alpha a_1 + \sigma\beta a_2) + 2\rho\alpha a_2 = \sigma\alpha a_1 + (\sigma\beta + 2\rho\alpha)a_2,$$

$$2\alpha\lambda\sigma a_2 = \lambda\sigma f(a_2) = f(\lambda\sigma a_2) = f([x,y]) = [f(x),y] + [x, f(y)]$$

$$= [\lambda\alpha a_1 + (\lambda\beta + 2\mu\alpha)a_2, \sigma a_1 + \rho a_2] + [\lambda a_1 + \mu a_2, \sigma\alpha a_1 + (\sigma\beta + 2\rho\alpha)a_2]$$

$$= \lambda\alpha\sigma[a_1, a_1] + \lambda\sigma\alpha[a_1, a_1] = 2\alpha\lambda\sigma a_2.$$

Hence, every linear transformation f of the algebra L satisfying the conditions $f(a_1) = \alpha a_1 + \beta a_2$ and $f(a_2) = 2\alpha a_2$ is a differentiation of the algebra L.

Lemma 3.2 is proved.

By using similar reasoning, we can prove the following assertion:

Lemma 3.3. *Suppose that L is a Leibniz algebra over the field F, where $\mathrm{char}(F) = 2$, $L = Fa_1 \oplus Fa_2$, $[a_1, a_1] = a_2$, and $[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = 0$. The linear mapping f is a differentiation of the algebra L if and only if $f(a_2) = 0$.*

Corollary 3.1. *Suppose that L is a Leibniz algebra over the field F, where $\mathrm{char}(F) = 2$, $L = Fa_1 \oplus Fa_2$, and $[a_1, a_1] = a_2$, $[a_1, a_2] = [a_2, a_1] = [a_2, a_2] = 0$. Then the algebra of differentiations of the algebra L is isomorphic to a subalgebra of matrices from $M_2(F)$, which have the form $\alpha E_{11} + \beta E_{21}$, $\alpha, \beta \in F$. In particular, it is Abelian and has dimension 2.*

Proof. If f is a differentiation of the algebra L, then, by Lemma 3.3, $f(a_1) = \alpha a_1 + \beta a_2$ and $f(a_2) = 0$ for some elements $\alpha, \beta \in F$. Thus, the matrix of the mapping f in the basis $\{a_1, a_2\}$ has the form: $\alpha E_{11} + \beta E_{21}$, $\alpha, \beta \in F$. Conversely, if the linear mapping f has this matrix in the basis $\{a_1, a_2\}$, then f is a differentiation of the algebra L. This means that the algebra of differentiations of the algebra L is isomorphic to a subalgebra of matrices of the form $\alpha E_{11} + \beta E_{21}$, $\alpha, \beta \in F$.

Lemma 3.4. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that A is an ideal of the algebra L and that the quotient algebra B/A is a nontrivial subalgebra from L/A. If S/A is a subalgebra from L/A such that*

$$S/A \le \mathrm{Ann}_{L/A}(B/A)$$

and S/A does not contain B/A, then S is an ideal of the algebra L.

Proof. In the quotient algebra B/A, we choose an element zA such that $zA \notin S/A$. Since

$$S/A \le \mathrm{Ann}_{L/A}(B/A),$$

we get

$$[zA, S/A] = [S/A, zA] = \langle 0 \rangle.$$

This means that

$$[z, S], [S, z] \le A \le S.$$

The choice of an element z indicates that $z \notin S$. In other words, $\mathrm{I}_L(S) \ne S$. This implies that S is an ideal of the algebra L.

Lemma 3.4 is proved.

Lemma 3.5. *Suppose that L is a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that the locally nilpotent radical $\mathrm{Ln}(L) = K \neq L$ is non-Abelian and noncyclic. Then every subalgebra from K is an ideal of the algebra L. If $[K, K] \leq \zeta(L)$, then K is a strongly extraspecial algebra and $[K, K] = \zeta(K) = \mathrm{Leib}(L) \leq \zeta(L)$.*

Proof. By Corollary 2.3, every subalgebra from K is an ideal of the algebra L. Since the radical K is non-Abelian, $K = E \oplus Z$, where Z is a subalgebra of the center K and E is a strongly extraspecial algebra. Since the radical K is non-Abelian, the subalgebra E is nontrivial. In E, we choose an element y such that $z = [y, y] \neq 0$. Let Y be a subalgebra generated by the element y. Then $Y = Fy \oplus Fz$ and $[z, y] = [y, z] = 0$. In view of the previous remarks, we conclude that Y is an ideal of the algebra L. Since $[K, K] = [E, E]$ has dimension 1 and $z \in [E, E]$, we get $Fz = [K, K]$, i.e., $z \in \zeta(L)$.

We assume that the subalgebra Z is nontrivial and consider a subalgebra $\langle z\rangle \oplus Z$. Then this subalgebra and each of its subalgebras are ideals of the algebra L. By Lemma 2.3, $Z \leq \zeta(L)$. We consider a quotient algebra $L/\langle z\rangle$. Its ideal $K/\langle z\rangle$ is Abelian and, moreover, it is clear that $\dim_F(K/\langle z\rangle) > 1$. By Lemma 2.2, every subalgebra from $K/\langle z\rangle$ is an ideal in $L/\langle z\rangle$. In view of the fact that the factor $(\langle z\rangle \oplus Z)/\langle z\rangle$ is central in L, by virtue of Lemma 2.3, we conclude that the factor $K/\langle z\rangle$ is central in $L/\langle z\rangle$. Since $L \neq K$, we can choose an element v such that $v \notin K$. By Lemma 2.1, the subalgebra $V = \langle v\rangle$ has either dimension 1 or dimension 2. Assume that $\dim_F(V) = 1$. Then $V \cap K = \langle 0\rangle$ and, in particular, $z \notin V$ but $z \in \mathrm{I}_L(V)$. This means that V is an ideal of the algebra L. Since $\dim_F(V) = 1$, the subalgebra V is Abelian. However, in this case, $\mathrm{Ln}(L) = K$ contains V, which is impossible.

Now let $\dim_F(V) = 2$. If $[V, V] \cap K = \langle 0\rangle$, then we again establish that $z \notin [V, V]$ and $z \in \mathrm{I}_L([V, V])$. Thus, $[V, V]$ is an ideal of the algebra L. Since $\dim_F([V, V]) = 1$, the subalgebra $[V, V]$ is Abelian. However, in this case, $\mathrm{Ln}(L) = K$ contains $[V, V]$, which contradicts the condition $[V, V] \cap K = \langle 0\rangle$. Hence, $[V, V] \cap K \neq \langle 0\rangle$. If $\langle 0\rangle \neq [V, V] \cap \langle z\rangle$, then, in view of $\dim_F([V, V]) = 1$, we get $[V, V] = \langle z\rangle$. Thus,

$$\langle 0\rangle = V/\langle z\rangle \cap K/\langle z\rangle.$$

Since the factor $K/\langle z\rangle$ is central in $L/\langle z\rangle$, we find $\mathrm{I}_L(V) \neq V$, i.e., V is an ideal of the algebra L. The fact that $[V, V] = \langle z\rangle \leq \zeta(L)$ yields the nilpotency of the subalgebra V. However, in this case, K must also contain V, which is impossible. This contradiction proves that $\langle 0\rangle = [V, V] \cap \langle z\rangle$. Thus, $V \cap \langle z\rangle = \langle 0\rangle$. In this case, $z \notin V$ but $z \in \mathrm{I}_L(V)$. This means that V is an ideal of the algebra L. On the other hand,

$$([V, V] + \langle z\rangle)/\langle z\rangle \leq K/\langle z\rangle \leq \zeta(L/\langle z\rangle).$$

Hence, the quotient algebra $(V + \langle z\rangle)/\langle z\rangle$ is nilpotent. The inclusion $\langle z\rangle \leq \zeta(L)$ shows that $V + \langle z\rangle$ is also nilpotent. In particular, the subalgebra V itself is nilpotent. However, in this case, we again conclude that K must contain V. This contradiction proves the equality $Z = \langle 0\rangle$, which implies that $K = E$ is a strongly extraspecial Leibniz algebra.

We now assume that $\mathrm{Leib}(L) \neq \langle z\rangle$. Since $\mathrm{Leib}(L)$ is an Abelian ideal, K contains $\mathrm{Leib}(L)$. Then

$$\dim_F(\mathrm{Leib}(L)) > 1,$$

i.e., $\mathrm{Leib}(L)$ must contain an element $u \notin \langle z\rangle$. Since the kernel $\mathrm{Leib}(L)$ is Abelian, $[u, u] = 0$. On the other hand, K is a strongly extraspecial Leibniz algebra and, hence, $[u, u] \neq 0$. This contradiction proves that $\mathrm{Leib}(L) = \langle z\rangle$.

Lemma 3.5 is proved.

Corollary 3.2. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that the locally nilpotent radical* $\mathrm{Ln}(L) = K \neq L$ *is non-Abelian and noncyclic. If* $[K, K] \leq \zeta(L)$, *then* $\mathrm{char}(F) = 2$.

Proof. By Lemma 3.5, K is a strongly extraspecial algebra and $[K, K] = \zeta(K) = \mathrm{Leib}(L)$. Since the radical K is non-Abelian, the subalgebra E is nontrivial. In E, we choose an element y such that $z = [y, y] \neq 0$. Let Y be a subalgebra generated by the element y. Then $Y = Fy \oplus Fz$ and $[z, y] = [y, z] = 0$. Under the imposed conditions, Y is an ideal of the algebra L. As in Lemma 3.5, we can show that $\langle z \rangle = [K, K]$. Hence, $z \in \zeta(L)$.

For an element $x \in L$, we consider a mapping $\mathrm{l}_x \colon Y \to Y$ defined by the following rule: $\mathrm{l}_x(a) = [x, a]$ for any element $a \in Y$. Then the mapping l_x is a differentiation of the subalgebra Y.

We now assume that $\mathrm{char}(F) \neq 2$. Thus, by Lemma 3.2 and the fact that $\mathrm{l}_x(z) = 0$, we conclude that $\mathrm{l}_x(y) = \beta z$ for some element $\beta \in F$. In other words, $[x, y] \in \langle z \rangle$ for all elements $x \in L$. This means that

$$Y/\langle z \rangle \leq \zeta^{\mathrm{right}}(L/\langle z \rangle).$$

By Lemma 3.5, $\mathrm{Leib}(L) = \langle z \rangle$. This implies that $L/\langle z \rangle$ is a Lie algebra. Then $\zeta^{\mathrm{right}}(L/\langle z \rangle) = \zeta(L/\langle z \rangle)$, i.e.,

$$Y/\langle z \rangle \leq \zeta(L/\langle z \rangle).$$

By Lemma 2.2 every subalgebra of $K/\langle z \rangle$ is an ideal in $L/\langle z \rangle$. By using Lemma 2.3, we can show that the factor $K/\langle z \rangle$ is central in $L/\langle z \rangle$. Since $L/\langle z \rangle$ is a Lie algebra, every cyclic subalgebra $X/\langle z \rangle$ of $L/\langle z \rangle$ is one-dimensional. This means that either $K/\langle z \rangle$ contains $X/\langle z \rangle$ or $K/\langle z \rangle \cap X/\langle z \rangle = \langle 0 \rangle$. In the first case, Lemma 3.5 shows that X is an ideal of the algebra L. In the second case, it follows from Lemma 3.4 that X is an ideal of the algebra L. Thus, every cyclic subalgebra from $L/\langle z \rangle$ is an ideal in $L/\langle z \rangle$. Since $L/\langle z \rangle$ is a Lie algebra, the quotient algebra $L/\langle z \rangle$ is Abelian. Hence, L is nilpotent, which contradicts the condition $L \neq K$. This contradiction proves that $\mathrm{char}(F) = 2$.

Proof of Theorem B1. By Corollary 3.2, $\mathrm{char}(F) = 2$. By K we denote a locally nilpotent radical of the algebra L. By Lemma 3.5, every subalgebra from K is an ideal of the algebra L and K is a strongly extraspecial algebra. In K, we choose an element y such that $z = [y, y] \neq 0$. If Y is an algebra generated by the element y, then $Y = Fy \oplus Fz$ and $[z, y] = [y, z] = 0$. As already indicated, Y is an ideal of the algebra L. Moreover, it follows from Lemma 3.5 that

$$Z = Fz = \langle z \rangle = [K, K] = \mathrm{Leib}(L) \leq \zeta(L).$$

This means that the quotient algebra L/Z is not Abelian (otherwise, the algebra L is nilpotent, which contradicts the condition $\mathrm{Ln}(L) \neq L$). It follows from the equality $Z = \mathrm{Leib}(L)$ that L/Z is a Lie algebra.

Assume that L contains an element $w \notin Z$ such that $[w, w] = 0$. Then $\langle w \rangle = Fw$ and $\langle w \rangle \cap Z = \langle 0 \rangle$. It follows from the inclusion $Z \leq \zeta(L)$ that $z \in \mathrm{I}_L(\langle w \rangle)$, in particular, $\mathrm{I}_L(\langle w \rangle) \neq \langle w \rangle$. Hence, $\langle w \rangle$ is an ideal of the algebra L. Since the subalgebra $\langle w \rangle$ is Abelian, we get $\langle w \rangle \leq \mathrm{Ln}(L)$. On the other hand, as already indicated, L is a strongly extraspecial algebra. We arrive at a contradiction, which shows that L satisfies condition (i).

Since the radical K is noncyclic, we get $K \neq Y$. This means that $\dim_F(K/Z) > 1$. Let $A = \mathrm{Ann}_L^{\mathrm{left}}(Y)$. Since Y is an ideal of the algebra L, A is also an ideal of L. It follows from the inclusion $Z \leq \zeta(L)$ that $Z \leq A$. It is clear that $y \notin A$ and, hence, $A \cap Y = Z$. This means that the intersection $Y/Z \cap A/Z$ is trivial. Let a be an arbitrary element from A such that $a \notin Z$. Then $a \notin Y$ and $[a, y] = [y, a] = 0$. This means that

$\mathrm{I}_L(\langle a\rangle) \neq \langle a\rangle$ and, hence, $\langle a\rangle$ is an ideal of the algebra L. Since $[a,a] \in Z$, the subalgebra $\langle a\rangle$ is nilpotent. Therefore, $\langle a\rangle \leq \mathrm{Ln}(L)$. Hence, $A \leq \mathrm{Ln}(L)$. By virtue of Proposition 3.2 in [6] and Corollary 3.1, we get

$$\dim_F(L/A) \leq 2.$$

Thus,

$$\dim_F(L/(A+Y)) \leq 1.$$

Since $(A+Y) \leq \mathrm{Ln}(L)$ and $L \neq \mathrm{Ln}(L)$, we can show that

$$\dim_F(L/\mathrm{Ln}(L)) = 1.$$

We choose an element u such that $u \notin K$. Then $L = K \oplus Fu$. By Corollary 2.1, every subalgebra from K that contains Z is an ideal of the algebra L. According to Lemma 2.3, there exists an element $\alpha \in F$ such that

$$[a+Z, u+Z] = \alpha(a+Z) \quad \text{for any} \quad a \in K.$$

Note that $\alpha \neq 0$. Then $[a,u] = \alpha a + z_a$ for some element $z_a \in Z$. Let a and b be elements of K such that $a, b \notin Z$. Since $[a,b] \in Z$, we get $[[a,b],v] = 0$. This yields

$$0 = [[a,b],u] = [a,[b,u]] - [b,[a,u]]$$

$$= [a, \alpha b + z_b] - [b, \alpha a + z_a] = \alpha[a,b] - \alpha[b,a] = \alpha([a,b]-[b,a]).$$

Since $\alpha \neq 0$, we find $[a,b] = [b,a]$.

By Lemma 3.2, we get $[u,y] = \lambda y + \mu z$. In view of the fact that each subalgebra of K/Z is an ideal in L/Z, by virtue of Lemma 2.3, we conclude that $[u+Z, a+Z] = \lambda a + Z$ for each element $a \in K \setminus Z$. Since $u \notin K$, we have $\lambda \neq 0$. Thus, we can set $v = \lambda^{-1}u$. This implies that $[v+Z, a+Z] = a+Z$ for every element $a \in K \setminus Z$.

We now consider a cyclic subalgebra $\langle v\rangle$. By Lemma 2.1, either $\langle v\rangle = Fv$ or $\langle v\rangle = Fv \oplus F[v,v]$. In the first case, $\langle v\rangle \cap \mathrm{Leib}(L) = \langle 0\rangle$. The inclusion $\mathrm{Leib}(L) \leq \zeta(L)$ shows that $z \in \mathrm{I}_L(\langle v\rangle)$. Thus. $\langle v\rangle$ must be an ideal of the algebra L. In this case, $\langle v\rangle \cap K = \langle 0\rangle$. However, this means that $v \in \mathrm{Ann}_L(K)$, which is impossible. This contradiction proves that $[v,v] = \eta z$ for some nonzero element $\eta \in F$.

Theorem B1 is proved.

We now consider some details of the procedure of construction of the locally nilpotent radical $\mathrm{Ln}(L)$.

Let $\{y+Z, w_\lambda + Z \mid \lambda \in \Lambda\}$ be a basis of the quotient algebra K/Z. We set $a_1 = y$. Since K/Z is Abelian, we have $[w_\lambda, a_1] = \xi_\lambda z$ for a certain element $\xi_\lambda \in F$, $\lambda \in \Lambda$. If $\xi_\lambda = 0$, then we set $u_\lambda = w_\lambda$. Moreover, if $\xi_\lambda \neq 0$, then we set $u_\lambda = \xi_\lambda a_1 - w_\lambda$. This yields

$$[u_\lambda, a_1] = [\xi_\lambda a_1 - w_\lambda, a_1] = \xi_\lambda[a_1,a_1] - [w_\lambda, a_1] = \xi_\lambda z - \xi_\lambda z = 0.$$

It follows from the equality $[u_\lambda, a_1] = [a_1, u_\lambda]$ that $[a_1, u_\lambda] = 0$. It is clear that the elements

$$\{a_1 + Z, u_\lambda + Z \mid \lambda \in \Lambda\}$$

form a basis of the quotient algebra K/Z. Let U_1/Z be a subspace of K/Z generated by the elements

$$\{u_\lambda + Z \mid \lambda \in \Lambda\}.$$

In view of the Abelian property of $K/Z,\ U_1$ is a subalgebra of K and $[U_1, a_1] = [a_1, U_1] = \langle 0\rangle$.

By using similar reasoning and transfinite induction, we can construct a basis

$$\{a_\mu + Z \mid \mu \in M\}$$

such that $[a_\mu, a_\nu] = [a_\nu, a_\mu] = 0$ for all $\mu, \nu \in M,\ \mu \neq \nu$. Moreover, $[v, a_\mu] = a_\mu + \gamma_\mu z$ for all $\mu \in M$.

Proof of Theorem B2. By virtue of Corollary 2.3, every subalgebra from K is an ideal of the algebra L. Since the locally nilpotent radical K is not Abelian, we conclude that $K = E \oplus Z,\ Z$ is a subalgebra of the center of K, and E is a strongly extraspecial algebra. Since the radical K is not Abelian, the subalgebra E is nontrivial. In E, we choose an element y such that $z = [y, y] \neq 0$. Let Y be a subalgebra generated by the element y. Then $Y = Fy \oplus Fz$ and $[z, y] = [y, z] = 0$. Under the imposed conditions, the subalgebra Y is an ideal of the algebra L. Since $[K, K] = [E, E]$ has dimension 1 and $z \in [E, E]$, we find $Fz = [K, K]$. This means that $z \in \zeta(K)$.

For an element $x \in L$, we consider a mapping $\mathrm{l}_x \colon Y \to Y$ defined by the following rule: $\mathrm{l}_x(a) = [x, a]$ for any element $a \in Y$. Then the mapping l_x is a differentiation of the ideal Y. Since $z \notin \zeta(L)$, by virtue of Lemmas 3.2 and 3.3, we conclude that $\mathrm{char}(F) \neq 2$. Moreover, Lemma 3.2 shows that there exists an element $u \in A$ such that $[u, y] = \alpha y + \beta z,\ [u, z] = 2\alpha z$, and $\alpha \neq 0$. We set $v = \alpha^{-1}u$. Then $[v, y] = y + \gamma z$ and $[v, z] = 2z$, where $\gamma = \alpha^{-1}\beta$.

We assume that the subalgebra Z is nontrivial and consider $\langle z\rangle \oplus Z$. According to the results established above, this subalgebra and all its subalgebras are ideals of the algebra L. Thus, it follows from Lemma 2.3 that $[v, w] = 2w$ for any element $w \in Z$. We now consider the quotient algebra $L/\langle z\rangle$. Its ideal $K/\langle z\rangle$ is Abelian and, hence, $\dim_F(K/\langle z\rangle) > 1$. Since $Z \neq \langle z\rangle$, there exists an element $w \in Z$ such that $\langle w\rangle \cap Y = \langle 0\rangle$. By Lemma 2.2, every subalgebra from $K/\langle z\rangle$ is an ideal in $L/\langle z\rangle$. Then

$$[v + \langle z\rangle, y + \langle z\rangle] = [v, y] + \langle z\rangle = y + \langle z\rangle.$$

According to Lemma 2.3, we obtain $[v + \langle z\rangle, w + \langle z\rangle] = w + \langle z\rangle$. On the other hand,

$$[v + \langle z\rangle, w + \langle z\rangle] = [v, w] + \langle z\rangle = 2w + \langle z\rangle.$$

We arrive at a contradiction, which proves the equality $Z = \langle 0\rangle$. This means that $K = E$ is a strongly extraspecial algebra.

We now assume that $\mathrm{Leib}(L) \neq \langle z\rangle$. Since the kernel $\mathrm{Leib}(L)$ is an Abelian ideal, K contains $\mathrm{Leib}(L)$. Thus, $\dim_F(\mathrm{Leib}(L)) > 1$. Hence, $\mathrm{Leib}(L)$ must contain an element $u \notin \langle z\rangle$. Since the kernel $\mathrm{Leib}(L)$ is Abelian, we get $[u, u] = 0$. On the other hand, K is a strongly extraspecial Leibniz algebra and, hence, $[u, u] \neq 0$. This contradiction proves that $\mathrm{Leib}(L) = \langle z\rangle$.

Since $\langle z\rangle$ is an ideal of dimension 1, we get

$$L/\mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle) \cong F \qquad \text{and} \qquad \dim_F(L/\mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle)) = 1.$$

In view of the fact that $v \notin \mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle)$, we obtain

$$L = \mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle) + \langle v\rangle.$$

Since the left center of the algebra L contains the Leibniz kernel, we find $L = \mathrm{Ann}_L^{\mathrm{right}}(\langle z\rangle)$, i.e.,

$$\mathrm{Ann}_L(\langle z\rangle) = \mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle) \cap \mathrm{Ann}_L^{\mathrm{right}}(\langle z\rangle) = \mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle) \cap L = \mathrm{Ann}_L^{\mathrm{left}}(\langle z\rangle).$$

If $x \in \mathrm{Ann}_L(\langle z\rangle)$, then, by Lemma 3.2, $[x, y] \in \langle z\rangle$. Thus, by virtue of Lemma 3.4, $\langle x\rangle$ is an ideal of the algebra L. It follows from the equality $\mathrm{Leib}(L) = \langle z\rangle$ that the quotient algebra $L/\langle z\rangle$ is a Lie algebra. This means that $[x + \langle z\rangle, x + \langle z\rangle] = \langle z\rangle$, i.e., either $\langle x\rangle = Fx$ or $\langle x\rangle = Fx + Fz$. In the second case, $[x, x] \in \langle z\rangle$. It follows from the equality $[x, z] = 0$ that the subalgebra $\langle x\rangle$ is nilpotent. Since this subalgebra is an ideal, $\mathrm{Ln}(L) = K$ contains $\langle x\rangle$. Therefore, $\mathrm{Ann}_L(\langle z\rangle) \leq K$. Since $z \in \zeta(K)$, we get $K = \mathrm{Ann}_L(\langle z\rangle)$. This means that $L = K + \langle v\rangle$.

As shown above, $[v, x] = x + \xi_x z$ for any $x \in K \setminus \langle z\rangle$ and some $\xi_x \in F$. Since $L/\langle z\rangle$ is a Lie algebra, we find $[v, v] \in \langle z\rangle$, i.e., $[v, v] = \nu z$ for some $\nu \in F$ and $[v, z] = 2z$.

Theorem B2 is proved.

Example 3.1. Let L be a Leibniz algebra over the field F, let $\mathrm{char}(F) = 2$, and let L be generated by elements a, b, and v such that

$$[a, a] = z, \qquad [b, b] = \sigma z, \qquad [v, v] = \eta z,$$

$$[z, z] = [z, a] = [a, z] = [z, b] = [b, z] = [z, d] = [d, z] = 0,$$

$$[a, b] = [b, a] = 0, \qquad [a, v] = a, [v, a] = a + z,$$

$$[b, v] = b, \qquad [v, b] = b + z.$$

Moreover, the polynomials $X^2 + \sigma$ and $X^2 + \eta$ do not have roots in F. It is easy to see that L is a Leibniz algebra whose subalgebras are either ideals or self-idealizing subalgebras.

We say that the field F is 2-*closed* if the polynomial $X^2 + \alpha \in F[X]$ has a root in the field F for every nonzero element $\alpha \in F$. This means that the multiplicative group $\mathrm{U}(F)$ of the field F is 2-divisible. In particular, every finite field F with characteristic 2 is 2-closed. Indeed, $|F| = 2^n$ for some natural n. Hence, the number $|\mathrm{U}(F)| = 2^n - 1$ is odd. This means that the multiplicative group $\mathrm{U}(F)$ is 2-divisible. As a consequence, we show that each locally finite field F with characteristic 2 is 2-closed.

Corollary 3.3. *Let L be a Leibniz algebra over the field F whose subalgebras are either ideals or self-idealizing subalgebras. Assume that $\mathrm{char}(F) = 2$, the locally nilpotent radical $\mathrm{Ln}(L)$ is not Abelian, and $L \neq \mathrm{Ln}(L)$. If the field F is 2-closed, then the radical $\mathrm{Ln}(L)$ is cyclic.*

Proof. By K we denote a locally nilpotent radical of the algebra L and assume that K is noncyclic. By Lemma 3.5, every subalgebra from K is an ideal of the algebra L and K is a strongly extraspecial algebra. In K, we choose an element y such that $z = [y, y] \neq 0$. Let Y be a subalgebra generated by the element y. Then $Y = Fy \oplus Fz$ and $[z, y] = [y, z] = 0$. As shown above, Y is an ideal of the algebra L. By Lemma 3.5, we also get

$$Z = Fz = \langle z\rangle = \mathrm{Leib}(L) \leq \zeta(L).$$

Since the radical K is noncyclic, we have $K \neq Y$. By using the same reasoning as in the proof of Theorem B1, we can find an element $a \in K$ such that $a \notin Z$ and $[a, y] = [y, a] = 0$. Then $[a, a] = \gamma z$ for some element $\gamma \in F$.

Further, let $b = \lambda a + \mu y,\ \lambda, \mu \in F$. Then

$$[b, b] = [\lambda a + \mu y, \lambda a + \mu y]$$
$$= \lambda^2[a, a] + \lambda\mu[a, y] + \mu\lambda[y, a] + \mu^2[y, y] = \lambda^2\gamma z + \mu^2 z = (\lambda^2\gamma + \mu^2)z.$$

Since the field F is 2-closed, there exists an element $\sigma \in F$ such that $\sigma^2 = \gamma$. We set $\lambda = 1$ and $\mu = \sigma$. Thus,

$$a + \sigma y \notin Z \qquad \text{and} \qquad [a + \sigma y, a + \sigma y] = (\gamma + \sigma^2)z = (\gamma + \gamma)z = 0.$$

We arrive at a contradiction with the fact that K is a strongly extraspecial algebra. This contradiction shows that the radical K must be cyclic.

We now consider the next natural case in which a locally nilpotent radical is cyclic. In this case, by Lemma 2.1, it is either 1- or 2-dimensional.

Proof of Theorem C. By K we denote a locally nilpotent radical of the algebra L. According to Lemma 2.1, K has a basis $\{a, z\}$ such that $[a, a] = z$ and $[z, a] = [a, z] = 0$.

For an element $x \in L$, we consider a mapping $\mathrm{l}_x\colon K \to K$ defined by the rule: $\mathrm{l}_x(y) = [x, y]$ for every element $y \in K$. Then the mapping l_x is a differentiation of the radical K. By Lemmas 3.2 and 3.3, $\mathrm{l}_x(z) = 0$. In other words, $[x, z] = 0$ for all elements $x \in L$. This means that $z \in \zeta^{\mathrm{right}}(L)$. On the other hand, $z \in \mathrm{Leib}(L)$. Since $\mathrm{Leib}(L) \leq \zeta^{\mathrm{left}}(L)$, we get $z \in \zeta^{\mathrm{left}}(L)$. Thus, $z \in \zeta(L)$. Note that the kernel $\mathrm{Leib}(L)$ is Abelian. Hence, $\mathrm{Leib}(L) \leq K$. Since $Z = \langle z \rangle$ is a maximal Abelian subalgebra from K, we get

$$Z = \mathrm{Leib}(L).$$

Let $A = \mathrm{Ann}_L^{\mathrm{left}}(K)$. Since K is an ideal of the algebra L, A is also an ideal of L. The equality $Z = \zeta(K)$ implies that $Z \leq A$. It is clear that $a \notin A$. Hence, $A \cap K = Z$. Assume that K does not contain A. Then the quotient algebra A/Z is nontrivial and the intersection $K/Z \cap A/Z$ is trivial. Let d be an arbitrary element from A such that $d \notin Z$. Then $d \notin K$ and $[a, d] = [d, a] = 0$. This means that $\mathrm{I}_L(\langle d \rangle) \neq \langle d \rangle$ and, hence, $\langle d \rangle$ must be an ideal of the algebra L. Since $[d, d] \in \mathrm{Leib}(L) = Z$, the subalgebra $\langle d \rangle$ is nilpotent, i.e.,

$$\langle d \rangle \leq K.$$

We arrive at a contradiction, which proves that $A \leq K$.

Assume that $\mathrm{char}(F) \neq 2$. In this case, Lemma 3.2 shows that the algebra of differentiations of the radical K is 1-dimensional. By virtue of Proposition 3.2 in [6], we obtain $\dim_F(L/A) \leq 1$. It follows from the equality $A = Z$ that $L = K$.

We now assume that $\mathrm{char}(F) = 2$. According to Proposition 3.2 in [6], it follows from Corollary 3.1 and the equality $A = Z$ that $\dim_F(L/Z) \leq 2$. If $\dim_F(L/Z) = 1$, then $L = K$ and if $\dim_F(L/Z) = 2$, then L/Z is a Lie algebra of dimension 2. Since $L \neq K$, we can choose an element $v \notin K$ such that

$$[a + Z, v + Z] = [v + Z, a + Z] = a + Z.$$

This means that $[v, a] = a + \lambda z$ and $[a, v] = a + \mu z$ for some elements $\lambda, \mu \in F$.

Finally, we consider a cyclic subalgebra $\langle v\rangle$. By Lemma 2.1, either $\langle v\rangle = Fv$ or $\langle v\rangle = Fv \oplus F[v,v]$. In the first case, $\langle v\rangle \cap \mathrm{Leib}(L) = \langle 0\rangle$. The equality $\mathrm{Leib}(L) = \zeta(L)$ shows that $z \in \mathrm{I}_L(\langle v\rangle)$. Thus, $\langle v\rangle$ must be an ideal of the algebra L. In this case, $\langle v\rangle \cap K = \langle 0\rangle$. However, at the same time, $v \in \mathrm{Ann}_L(K)$, and we arrive at a contradiction, which shows that $[v,v] = \eta z$ for some nonzero element $\eta \in F$.

Let $b = \sigma a + \tau v,\ \sigma, \tau \in F$. Then

$$\begin{aligned}
[b,b] &= [\sigma a + \tau v, \sigma a + \tau v] \\
&= \sigma^2[a,a] + \sigma\tau[a,v] + \tau\sigma[v,a] + \tau^2[v,v] \\
&= \sigma^2 z + \sigma\tau(a + \mu z) + \tau\sigma(a + \lambda z) + \tau^2\eta z \\
&= \sigma^2 z + \sigma\tau a + \sigma\tau\mu z + \tau\sigma a + \tau\sigma\lambda z + \tau^2\eta z \\
&= (\sigma^2 + \sigma\tau\mu + \tau\sigma\lambda + \tau^2\eta)z = \tau^2(\sigma^2\tau^{-2} + \sigma\tau^{-1}(\mu+\lambda) + \eta)z.
\end{aligned}$$

Assume that the polynomial $X^2 + (\mu+\lambda)X + \eta$ has a root γ in F. Thus, by setting $\tau = 1$ and $\sigma = \gamma$, we conclude that $[b,b] = 0$ and, hence, $\langle b\rangle = Fb$. It is clear that $b \notin K$. As above, $z \in \mathrm{I}_L(\langle b\rangle)$. Hence, $\langle b\rangle$ must be an ideal. Since the subalgebra $\langle b\rangle$ is Abelian, it is contained in a locally nilpotent radical of the algebra L, which is impossible. This contradiction shows that the polynomial $X^2 + (\mu+\lambda)X + \eta$ does not have roots in the field F.

Theorem C is proved.

We now present an example of Leibniz algebra satisfying the conditions of Theorem C.

Example 3.2. Let L be a Leibniz algebra over the field F, where $\mathrm{char}(F) = 2$ and the field F is not 2-closed. In F, we choose an element η such that the polynomial $X^2 + \eta$ does not have roots in the field F. Let L be a vector space over the field F and let $\{z, a, v\}$ be a basis in L. We define the operation $[\,,\,]$ as follows:

$$[z,z] = [z,a] = [a,z] = [z,v] = [v,z] = 0, \quad [a,a] = z, \quad [v,v] = \eta z, \quad [v,a] = [a,v] = a.$$

It is possible to show that L is a Leibniz algebra all subalgebras of which are either ideals or self-idealizing subalgebras.

Proposition 3.1. *Let L be a Leibniz algebra over the field F all subalgebras of which are either ideals or self-idealizing subalgebras. If the locally nilpotent radical $\mathrm{Ln}(L)$ has dimension 1, then the quotient algebra $L/\mathrm{Ln}(L)$ does not contain Abelian subalgebras of dimension 2.*

Proof. By K we denote a locally nilpotent radical of the algebra L. Since the kernel $\mathrm{Leib}(L)$ is an Abelian ideal, we have $\mathrm{Leib}(L) \le K$. This means that $\mathrm{Leib}(L) = K$. Hence, L/K is a Lie algebra. Let $A = \mathrm{Ann}_L^{\mathrm{left}}(K)$. Since K is an ideal of the algebra L, A is also an ideal in L. In view of the fact that $\dim_F(K) = 1$, we obtain $\dim_F(L/A) = 1$.

Assume the contrary, i.e., let U/K be an Abelian subalgebra from L/K and let $\dim_F(U/K) > 1$. Then it follows from Corollary 2.1 that every subalgebra from U that contains K is an ideal of the algebra L. The equality $\dim_F(L/A) = 1$ proves that $U \cap A > K$. Since the quotient algebra $(U \cap A)/K$ is Abelian, the ideal $U \cap A$ is nilpotent. Hence, $U \cap A \le K$, which is impossible. This contradiction shows that each Abelian subalgebra from L/K is 1-dimensional.

Proposition 3.1 is proved.

Thus, it is possible to conclude that the investigation of Leibniz algebras whose subalgebras are either ideals or self-idealizing subalgebras is reduced to the study of Lie algebras Abelian subalgebras of which have dimension 1. This case requires separate investigations.

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