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SOME NEW RESULTS ON THE STRONG CONVERGENCE OF FEJÉR MEANS WITH RESPECT TO VILENKIN SYSTEMS

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We prove some new strong convergence theorems for partial sums and Fejér means with respect to the Vilenkin system.

1. Introduction

For the definitions and notation used in this introduction, we refer the reader to Section 2.

It is well known (for details, see, e.g., [1, 8, 10]) that the Vilenkin system does not form a basis in the space $L_1(G_m)$. Moreover, there is a function in the martingale Hardy space $H_1(G_m)$ such that the partial sums of f are not bounded in the $L_1(G_m)$-norm. However, for all $p > 0$ and $f \in H_p$, there exists an absolute constant c_p such that

$$\|S_{M_k} f\|_p \leq c_p \|f\|_{H_p}. \tag{1}$$

In [5] (see also [11]), the following strong convergence result was obtained for all $f \in H_1(G_m)$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0.$$

This yields

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \leq \|f\|_{H_1}, \quad n = 2, 3, \ldots.$$

In [19], it was proved that, for any $f \in H_1$, there exists an absolute constant c such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n \log n} \sum_{k=1}^{n} \|S_k f\|_1 \leq \|f\|_{H_1}, \quad n = 1, 2, 3, \ldots.$$

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Moreover, for any nondecreasing function $\varphi\colon \mathbb{N}_+ \to [1,\infty)$, satisfying the condition

$$\limsup_{n\to\infty} \frac{\log n}{\varphi_n} = +\infty,$$

there exists a function $f \in \mathbb{N}_1$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|S_k f\|_1 = \infty.$$

For the Vilenkin system, Simon [12] proved that there is an absolute constant c_p depending only on p and such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p$$

for all $f \in H_p(G_m)$, where $0<p<1$. In [16], it was proved that, for any nondecreasing function $\Phi\colon \mathbb{N}_+ \to [1,\infty)$ satisfying the condition $\lim_{n\to\infty} \Phi(n) = +\infty$, there exists a martingale $f \in H_p(G_m)$ such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_{\text{weak-}L_p}^p \Phi(k)}{k^{2-p}} = \infty \quad \text{for} \quad 0<p<1.$$

Strong convergence theorems for two-dimensional partial sums were investigated by Weisz [23], Goginava [6], Gogoladze [7], and Tephnadze [18] (see also [9]).

Weisz [24] studied the norm convergence of Fejér means of the Walsh–Fourier series and proved the following theorem:

Theorem W1 (Weisz). *Let $p > 1/2$ and $f \in H_p$. Then*

$$\|\sigma_k f\|_p \leq c_p \|f\|_{H_p}.$$

Moreover, Weisz [24] also proved that, for all $p > 0$ and $f \in H_p$, there exists an absolute constant c_p such that

$$\|\sigma_{M_k} f\|_p \leq c_p \|f\|_{H_p}. \tag{2}$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad 1/2 < p < \infty.$$

If Theorem W1 holds for $0 < p \leq \dfrac{1}{2}$, then we get

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad 0 < p < 1/2, \tag{3}$$

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_{1/2}^{1/2}}{k}\leq c\|f\|_{H_{1/2}}^{1/2}, \tag{4}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\|\sigma_k f\|_{1/2}^{1/2}\leq c\|f\|_{H_{1/2}}^{1/2}. \tag{5}$$

However, in [14] (see also [2, 3]) it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In particular, there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n\in\mathbb{N}}\|\sigma_n f\|_{1/2}=+\infty.$$

In [4] (see also [17]) it was proved that (3) and (4) hold despite the fact that Theorem W1 is not true for $0 < p \leq 1/2$.

Moreover, in [4] it was proved that if $0 < p < 1/2$ and $\Phi\colon \mathbb{N}_+ \to [1,\infty)$ is an arbitrary nondecreasing function satisfying the condition

$$\limsup_{k\to\infty}\frac{k^{2-2p}}{\Phi_k}=\infty,$$

then there exists a martingale $f \in H_p$ such that

$$\sum_{m=1}^{\infty}\frac{\|\sigma_m f\|_{\text{weak-}L_p}^{p}}{\Phi_m}=\infty.$$

On the other hand, inequality (5) is not true for the Walsh system (see [17]). In particular, it was proved that there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{m=1}^{n}\|\sigma_m f\|_{1/2}^{1/2}=\infty. \tag{6}$$

In the present paper, we prove a more general result for the bounded Vilenkin system. In a special case, we also obtain (6).

The present paper is organized as follows: In order not to interrupt our subsequent discussions, some definitions and notation are presented in Section 2. For the proofs of our main results we need several auxiliary lemmas, some of them are new and of independent interest. These results are presented in Section 3. The main result and its proof can be found in Section 4.

2. Definitions and Notation

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Also let $m := (m_0, m_1, \ldots)$ denote the sequence of positive integers not smaller than 2.

By

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

we denote the additive group of integers modulo m_k.

A group G_m is defined as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad j \in Z_{m_k},$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we say that G_m is a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In the present paper, we discuss only bounded Vilenkin groups.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots), \quad x_k \in Z_{m_k}.$$

It is easy to introduce a base for the neighborhood of G_m, namely,

$$I_0(x) := G_m,$$

and

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad x \in G_m, \quad n \in \mathbb{N}.$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m, \quad n \in \mathbb{N}.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad k \in \mathbb{N},$$

then every $n \in \mathbb{N}$ can be uniquely expressed as follows:

$$n = \sum_{k=0}^{\infty} n_j M_j,$$

where $n_j \in Z_{m_j}$, $j \in \mathbb{N}$, and only finitely many n_j differ from zero. Let

$$\|n\| := \max\{j \in \mathbb{N}, n_j \neq 0\}.$$

For a natural number

$$n = \sum_{j=1}^{\infty} n_j M_j,$$

we define

$$\delta_j = \operatorname{sign} n_j = \operatorname{sign}(\ominus n_j), \quad \delta_j^* = \| \ominus n_j - 1\| \delta_j,$$

where $\ominus$ is the inverse operation for

$$a_k \oplus b_k = (a_k + b_k) \bmod m_k.$$

We define functions v and v^* as follows:

$$v(n) = \sum_{j=0}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=0}^{\infty} \delta_j^*.$$

The nth Lebesgue constant is defined in the following way:

$$L_n = \|D_n\|_1.$$

The norm (or quasinorm) in the space $L_p(G_m)$ is defined as

$$\|f\|_p := \left(\int_{G_m} \|f(x)\|^p d\mu(x) \right)^{1/p}, \quad 0 < p < \infty.$$

The weak-$L_p(G_m)$ space consists of all measurable functions f such that

$$\|f\|_{\text{weak-}L_p(G_m)} := \sup_{\lambda>0} \lambda^p \mu\{f > \lambda\} < +\infty.$$

On G_m, we now introduce an orthonormal system, which is called the Vilenkin system.

First, we define a complex-valued function $r_k(x) \colon G_m \to \mathbb{C}$, as a generalized Rademacher function, as follows:

$$r_k(x) := \exp\left(2\pi\imath x_k/m_k\right), \quad \imath^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.$$

Further, we define the Vilenkin system $\psi := (\psi_n \colon n \in \mathbb{N})$ on G_m by

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

In a special case, for $m \equiv 2$, this system is called the Walsh–Paley system.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (for details, see, e.g., [1, 10, 20]).

If $f \in L_1(G_m)$, then we can define the Fourier coefficients, partial sums of the Fourier series, Fejér means, and Dirichlet and Fejér kernels with respect to the Vilenkin system in the ordinary way:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k d\mu, \quad k \in \mathbb{N},$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad n \in \mathbb{N}_+, \quad S_0 f := 0,$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \quad n \in \mathbb{N}_+,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+,$$

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k, \quad n \in \mathbb{N}_+.$$

Recall that (for details, see, e.g., [1, 8])

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n, \end{cases} \tag{7}$$

and

$$D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n - 1} \psi_{k M_n} = D_{M_n} \sum_{k=0}^{s_n - 1} r_n^k, \quad 1 \le s_n \le m_n - 1. \tag{8}$$

The σ-algebra generated by the intervals $\{I_n(x) \colon x \in G_m\}$ is denoted by F_n, $n \in \mathbb{N}$. By $f = (f_n, n \in \mathbb{N})$ we denote a martingale with respect to F_n, $n \in \mathbb{N}$ (for details, see, e.g., [21]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f_n|.$$

In the case where $f \in L_1(G_m)$, the maximal functions are also given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales such that

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to see that the sequence $(S_{M_n} f \colon n \in \mathbb{N})$ is a martingale. If $f = (f_n, n \in \mathbb{N})$ is martingale, then the Vilenkin–Fourier coefficients should be defined in a somewhat different way:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f_k(x) \overline{\psi}_i(x) \, d\mu(x).$$

The Vilenkin–Fourier coefficients of $f \in L_1(G_m)$ are the same as for the martingale $(S_{M_n} f \colon n \in \mathbb{N})$ obtained from f.

A bounded measurable function a is a p-atom if there exists an interval I such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \text{supp}\,(a) \subset I.$$

3. Auxiliary Lemmas

Lemma 1 [21, 22]. *A martingale $f = (f_n, n \in \mathbb{N})$ is in $H_p,\ 0 < p \le 1,$ if and only if there exist a sequence $(a_k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k,\ k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n \quad a.e., \tag{9}$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (9).

By using the atomic decomposition of $f \in H_p$ martingales, we can construct a counterexample, which plays a central role in proving the sharpness of our main results. In the present paper, it is used several times.

Lemma 2 [13]. *Let $n \in \mathbb{N}$ and $1 \le s_n \le m_n - 1$. Then*

$$s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left(\sum_{t=0}^{l-1} r_n^t \right) M_n D_{M_n} + \left(\sum_{l=0}^{s_n-1} r_n^l \right) M_n K_{M_n}$$

and

$$|s_n M_n K_{s_n M_n}(x)| \ge \frac{M_n^2}{2\pi} \quad for \quad x \in I_{n+1}(e_{n-1} + e_n).$$

Moreover, if $x \in I_t / I_{t+1},\ x - x_t e_t \notin I_n$, and $n > t$, then

$$K_{s_n M_n}(x) = 0. \tag{10}$$

Lemma 3 [4]. *Let*

$$n = \sum_{i=1}^{r} s_{n_i} M_{n_i},$$

where $n_{n_1} > n_{n_2} > \ldots > n_{n_r} \ge 0$ and $1 \le s_{n_i} < m_{n_i}$ for all $1 \le i \le r$, and let

$$n^{(k)} = n - \sum_{i=1}^{k} s_{n_i} M_{n_i},$$

where $0 < k \le r$. Then

$$nK_n = \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) s_{n_k} M_{n_k} K_{s_{n_k} M_{n_k}} + \sum_{k=1}^{r-1} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) n^{(k)} D_{s_{n_k} M_{n_k}}.$$

Lemma 4. *Let*

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{m_i} n_k M_k,$$

where

$$0\leq l_1\leq m_1\leq l_2-2<l_2\leq m_2\leq\ldots\leq l_s-2<l_s\leq m_s.$$

Then

$$n\left|K_n(x)\right|\geq cM_{l_i}^2 \quad \textit{for} \quad x\in I_{l_i+1}\left(e_{l_i-1}+e_{l_i}\right),$$

where $\lambda=\sup_{n\in\mathbb{N}} m_n$ and c is an absolute constant.

Proof. Assume that $x\in I_{l_i+1}(e_{l_i-1}+e_{l_i})$. Combining (10), (7), and (8), we obtain

$$D_{l_i}=0$$

and

$$D_{s_{n_k}M_{s_{n_k}}}=K_{s_{n_k}M_{s_{n_k}}}=0, \quad s_{n_k}>l_i.$$

Since $s_{n_1}>s_{n_2}>\ldots>s_{n_r}\geq 0$, we find

$$n^{(k)}=n-\sum_{i=1}^{k} s_{n_i}M_{n_i}=\sum_{i=k+1}^{s} s_{n_i}M_{n_i}\leq\sum_{i=0}^{n_{k+1}}(m_i-1)M_i=m_{n_{k+1}}M_{n_{k+1}}-1\leq M_{n_k}.$$

According to Lemma 3, we find

$$n\left|K_n\right|\geq\left|s_{l_i}M_{l_i}K_{s_{l_i}M_{l_i}}\right|-\sum_{r=1}^{i-1}\sum_{k=l_r}^{m_r}\left|s_kM_kK_{s_kM_k}\right|-\sum_{r=1}^{i-1}\sum_{k=l_r}^{m_r}\left|M_kD_{s_kM_k}\right|=I_1-I_2-I_3.$$

Let $x\in I_{l_i+1}(e_{l_i-1}+e_{l_i})$ and $1\leq s_{l_i}\leq m_{l_i}-1$. By using Lemma 2, we get

$$I_1=\left|s_{l_i}M_{l_i}K_{s_{l_i}M_{l_i}}\right|\geq\frac{M_{l_i}^2}{2\pi}\geq\frac{2M_{l_i}^2}{9}.$$

It is easy to see that

$$\begin{aligned}\sum_{s=0}^{k} n_s^2M_s^2&\leq\sum_{s=0}^{k}(m_s-1)^2M_s^2\\&\leq\sum_{s=0}^{k} m_s^2M_s^2-2\sum_{s=0}^{k} m_sM_s^2+\sum_{s=0}^{k} M_s^2\\&=\sum_{s=0}^{k} M_{s+1}^2-2\sum_{s=0}^{k} M_{s+1}M_s+\sum_{s=0}^{k} M_s^2\end{aligned}$$

$$= M_{k+1}^2 + 2\sum_{s=0}^{k} M_s^2 - 2\sum_{s=0}^{k} M_{s+1}M_s - M_0^2 \le M_{k+1}^2 - 1$$

and

$$\sum_{s=0}^{k} n_s M_s \le \sum_{s=0}^{k} (m_s - 1)M_s = m_k M_k - m_0 M_0 \le M_{k+1} - 2.$$

Note that $m_{i-1} \le l_i - 2.$ Thus, if we use the estimates presented above, then we obtain

$$\begin{aligned} I_2 &\le \sum_{s=0}^{l_i-2} |n_s M_s K_{n_s M_s}(x)| \le \sum_{s=0}^{l_i-2} n_s M_s \frac{n_s M_s + 1}{2} \\ &\le \frac{(m_{l_i-2}-1)M_{l_i-2}}{2} \sum_{s=0}^{l_i-2} (n_s M_s + 1) \\ &\le \frac{(m_{l_i-2}-1)M_{l_i-2}}{2} M_{l_i-1} + \frac{(m_{l_i-2}-1)M_{l_i-2}}{2} l_i \\ &\le \frac{M_{l_i-1}^2}{2} - \frac{M_{l_i-2}M_{l_i-1}}{2} + M_{l_i-1} l_i. \end{aligned} \tag{11}$$

For $I_3,$ we obtain

$$I_3 \le \sum_{k=0}^{l_i-2} |M_k D_{n_k M_k}(x)| \le \sum_{k=0}^{l_i-2} n_k M_k^2 \le M_{l_i-2} \sum_{k=0}^{l_i-2} n_k M_k \le M_{l_i-1}M_{l_i-2} - 2M_{l_i-2}. \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned} n|K_n(x)| &\ge I_1 - I_2 - I_3 \\ &\ge \frac{M_{l_i}^2}{2\pi} + \frac{3}{2} + 2M_{l_i-2} - \frac{M_{l_i-1}M_{l_i-2}}{2} - \frac{M_{l_i-1}^2}{2} - M_{l_i-1} l_i \\ &\ge \frac{M_{l_i}^2}{2\pi} - \frac{M_{l_i}^2}{16} - \frac{M_{l_i}^2}{8} + \frac{7}{2} - M_{l_i-1} l_i \\ &\ge \frac{2M_{l_i}^2}{9} - \frac{3M_{l_i}^2}{16} + \frac{7}{2} - M_{l_i-1} l_i \ge \frac{M_{l_i}^2}{144} - M_{l_i-1} l_i. \end{aligned}$$

Suppose that $l_i \ge 4.$ Then

$$n|K_n(x)| \ge \frac{M_{l_i}^2}{36} - \frac{M_{l_i}}{4} \ge \frac{M_{l_i}^2}{36} - \frac{M_{l_i}^2}{64} \ge \frac{5M_{l_i}^2}{36 \cdot 16} \ge \frac{M_{l_i}^2}{144}.$$

Lemma 4 is proved.

4. Main Result

The main result of this paper is the following theorem:

Theorem 1.

1. *Let $f \in H_{1/2}$. Then there exists an absolute constant c such that*

$$\sup_{n\in\mathbb{N}} \frac{1}{n\log n}\sum_{k=1}^{n}\|\sigma_k f\|_{H_{1/2}}^{1/2} \le c\|f\|_{H_{1/2}}^{1/2}, \quad n=1,2,3,\dots.$$

2. *Let $\varphi\colon \mathbb{N}_+ \to [1,\infty)$ be a nondecreasing function satisfying the condition*

$$\limsup_{n\to\infty} \frac{\log n}{\varphi_n} = +\infty. \tag{13}$$

Then there exists a function $f \in H_{1/2}$ such that

$$\sup_{n\in\mathbb{N}_+} \frac{1}{n\varphi_n}\sum_{k=1}^{n}\|\sigma_k f\|_{H_{1/2}}^{1/2} = \infty.$$

Corollary 1. *There exists a martingale $f \in H_{1/2}$ such that*

$$\sup_{n\in\mathbb{N}_+} \frac{1}{n}\sum_{k=1}^{n}\|\sigma_k f\|_{1/2}^{1/2} = \infty.$$

Proof of Theorem 1. 1. In [15], it was proved that there exists an absolute constant c, such that

$$\|\sigma_k f\|_{H_{1/2}}^{1/2} \le c\log k\|f\|_{H_{1/2}}^{1/2}, \quad k=1,2,\dots.$$

Hence,

$$\frac{1}{n\log n}\sum_{k=1}^{n}\|\sigma_k f\|_{H_{1/2}}^{1/2} \le \frac{c\|f\|_{H_{1/2}}^{1/2}}{n\log n}\sum_{k=1}^{n}\log k \le c\|f\|_{H_{1/2}}^{1/2}.$$

2. Under condition (13), there exists an increasing sequence of positive integers $\{\alpha_k\colon k\in\mathbb{N}\}$ such that

$$\limsup_{k\to\infty} \frac{\log M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}} = +\infty$$

and

$$\sum_{k=0}^{\infty} \frac{\varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2} M_{\alpha_k}} < c < \infty. \tag{14}$$

Let $f=(f_n, n\in\mathbb{N})$ be martingale defined by

$$f_n := \sum_{\{k;2\alpha_k<n\}} \lambda_k a_k,$$

where

$$a_k = M_{\alpha_k} r_{\alpha_k} D_{M_{\alpha_k}} = M_{\alpha_k}(D_{2M_{\alpha_k}} - D_{M_{\alpha_k}})$$

and

$$\lambda_k = \frac{\varphi_{2M_{\alpha_k}}}{\log M_{\alpha_k}}.$$

Note that

$$S_{2^A} a_k = \begin{cases} a_k, & \alpha_k < A, \\ 0, & \alpha_k \geq A, \end{cases}$$

$$\operatorname{supp}(a_k) = I_{\alpha_k}, \quad \int\limits_{I_{\alpha_k}} a_k d\mu = 0, \quad \|a_k\|_\infty \leq M^2_{\alpha_k} = \mu(\operatorname{supp} a_k)^{-2}.$$

Thus, if we apply Lemma 1 and (14), then we conclude that $f\in H_{1/2}$.

Moreover,

$$\widehat{f}(j) = \begin{cases} M_{\alpha_k}\lambda_k, & j\in\{M_{\alpha_k},\ldots,2M_{\alpha_k}-1\}, \quad k\in\mathbb{N}, \\ 0, & j\notin \bigcup\limits_{k=1}^{\infty}\{M_{\alpha_k},\ldots,2M_{\alpha_k}-1\}. \end{cases} \tag{15}$$

We have

$$\sigma_n f = \frac{1}{n}\sum_{j=0}^{M_{\alpha_k}-1} S_j f + \frac{1}{n}\sum_{j=M_{\alpha_k}}^{n-1} S_j f = I + II. \tag{16}$$

Let $M_{\alpha_k}\leq j<2M_{\alpha_k}$. Note that

$$D_{j+M_{\alpha_k}} = D_{M_{\alpha_k}} + \psi_{M_{\alpha_k}} D_j \quad \text{for} \quad j\leq M_{\alpha_k}.$$

Hence, if we apply (15), then we obtain

$$S_j f = S_{M_{\alpha_k}} f + \sum_{v=M_{\alpha_k}}^{j-1} \widehat{f}(v)\psi_v = S_{M_{\alpha_k}} f + M_{\alpha_k}\lambda_k \sum_{v=M_{\alpha_k}}^{j-1}\psi_v$$

$$= S_{M_{\alpha_k}} f + M_{\alpha_k}\lambda_k (D_j - D_{M_{\alpha_k}}) = S_{M_{\alpha_k}} f + \lambda_k \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}}. \tag{17}$$

According to (17), for $II,$ we conclude that

$$II=\frac{n-M_{\alpha_k}}{n}S_{M_{\alpha_k}}f+\frac{\lambda_k M_{\alpha_k}}{n}\sum_{j=M_{2\alpha_k}}^{n-1}\psi_{M_{\alpha_k}}D_{j-M_{\alpha_k}}=II_1+II_2.$$

Further, we can estimate II_2 as follows:

$$\begin{aligned}|II_2|&=\frac{\lambda_k M_{\alpha_k}}{n}\left|\psi_{M_{\alpha_k}}\sum_{j=0}^{n-M_{\alpha_k}-1}D_j\right|\\&=\frac{\lambda_k M_{\alpha_k}}{n}(n-M_{\alpha_k})|K_{n-M_{\alpha_k}}|\\&\geq\lambda_k(n-M_{\alpha_k})|K_{n-M_{\alpha_k}}|.\end{aligned}$$

Let

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{m_i}M_k,$$

where

$$0\leq l_1\leq m_1\leq l_2-2<l_2\leq m_2\leq\ldots\leq l_s-2<l_s\leq m_s.$$

Applying Lemma 4, we get

$$|II_2|\geq c\lambda_k\left|(n-M_{\alpha_k})K_{n-M_{\alpha_k}}(x)\right|\geq c\lambda_k M_{l_i}^2\quad\text{for}\quad x\in I_{l_i+1}(e_{l_i-1}+e_{l_i}).$$

Hence,

$$\begin{aligned}\int_{G_m}|II_2|^{1/2}d\mu&\geq\sum_{i=1}^{s-1}\int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})}|II_2|^{1/2}d\mu\\&\geq c\sum_{i=1}^{s-1}\int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})}\lambda_k^{1/2}M_{l_i}d\mu\geq c\lambda_k^{1/2}(s-1)\geq c\lambda_k^{1/2}v(n-M_{\alpha_k}).\end{aligned}\tag{18}$$

In view of (1), (2), and (16), we find

$$\|I\|^{1/2}=\left\|\frac{M_{\alpha_k}}{n}\sigma_{M_{\alpha_k}}f\right\|_{1/2}^{1/2}\leq\left\|\sigma_{M_{\alpha_k}}f\right\|_{1/2}^{1/2}\leq c\,\|f\|_{H_{1/2}}^{1/2}\tag{19}$$

and

$$\|II_1\|^{1/2}=\left\|\frac{n-M_{\alpha_k}}{n}S_{M_{\alpha_k}}f\right\|_{1/2}^{1/2}\leq\left\|S_{M_{\alpha_k}}f\right\|_{1/2}^{1/2}\leq c\,\|f\|_{H_{1/2}}^{1/2}.\tag{20}$$

Combining (18)–(20), we obtain

$$\|\sigma_n f\|_{1/2}^{1/2} \geq \|II_2\|_{1/2}^{1/2} - \|II_1\|_{1/2}^{1/2} - \|I\|_{1/2}^{1/2} \geq c\lambda_k^{1/2} v(n - M_{\alpha_k}) - c\|f\|_{H_{1/2}}^{1/2}.$$

By using the estimates presented above, we conclude that

$$\begin{aligned}\sup_{n\in\mathbb{N}_+} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} &\geq \frac{1}{M_{\alpha_k+1}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k}\leq l\leq 2M_{\alpha_k}\}} \|\sigma_l f\|_{1/2}^{1/2} \\ &\geq \frac{c}{M_{\alpha_k+1}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k}\leq l\leq 2M_{\alpha_k}\}} \left(\lambda_k^{1/2} v\left(l - M_{\alpha_k}\right) - c\|f\|_{H_{1/2}}^{1/2}\right) \\ &\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}} v(l) - \frac{c\|f\|_{H_{1/2}}^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{\{M_{\alpha_k}\leq l\leq 2M_{\alpha_k}\}} 1 \\ &\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k}\varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}-1} v(l) - c \geq c\,\frac{\log^{1/2} M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}^{1/2}} \to \infty \quad \text{as} \quad k \to \infty.\end{aligned}$$

Theorem 1 is proved.

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