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ON THE DYNAMICS OF IMPULSIVE PREDATOR-PREY SYSTEMS WITH BEDDINGTON–DEANGELIS-TYPE FUNCTIONAL RESPONSE

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We study a two-dimensional predator-prey system with Beddington–DeAngelis-type functional response with pulses in a periodic environment. In the analyzed special case, we establish necessary and sufficient conditions for the considered system to have at least one w-periodic solution. This result is mainly based on the continuation theorem from the coincidence-degree theory. Moreover, in order to find the globally attractive w-periodic solution of the system, by using the analytic structure of the given system, we deduce an inequality playing the role of both necessary and sufficient condition.

1. Introduction

Population dynamics is an important branch of the mathematical ecology and biomathematics. Predator-prey systems is one of research fields dealing with this subject and numerous studies have been devoted to the analysis of dynamical systems of this type. Studying on these systems is important because it helps us to understand the future of the investigated species.

In the present paper, we investigate impulsive predator-prey dynamical systems because giving pulses to a system has numerous important examples in the real life. Thus, if you use pesticides against insect species, then we get an immediate decrease in the population. Moreover, if there is an immigration from one territory to another for the same species, then we also observe an immediate increase in population. All these phenomena can be mathematically expressed by using pulses and, hence, the corresponding type of equations is called impulsive differential equations. There are many studies dealing with this type of differential and difference equations. Note that the theory of these equations was investigated in [1, 17–19, 21, 25].

The other significant notion important for our investigation is the presence (or absence) of periodic environment because many things in the actual life have periodic structures. Therefore, it becomes important to consider dynamical systems in periodic environments. Global existence and the existence of positive periodic solutions are significant aspects of the theory of periodic predator-prey systems. Thus, the works [7–11, 14, 15, 20, 24] investigated these problems for nonautonomous predator-prey systems by using the coincidence-degree theory and the continuation theorem.

Another notion important for the present work is functional response. Thus, in what follows, we use the so-called Beddington–DeAngelis-type functional response due to some advantages of this type of response over the other functional responses, such as Holling-type, ratio-dependent, semiratio-dependent, monotype, etc. Beddington and DeAngelis used the Beddington–DeAngelis-type functional response according to their observations over the populations on fishes in the Adriatic Sea. Moreover, the advantages of this type of functional response are described in their works [2, 6].

In particular, the problem of singularity in the Holling-type and similar functional responses, when either predator or prey goes to extinction, was solved by using the Beddington–DeAngelis functional response. In view

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of this singularity problem, the authors of [22] could not directly use the results of coincidence-degree theory in their system. Thus, they have divided their system, applied the continuation theorem to a certain part of the system, and achieved their goal by using Brouwer's fixed-point theorem. In addition, they needed to use constant pulses in the variable that symbolizes predator in order to get globally stable w-periodic solutions. In our system, due to the advantages of the Beddington–DeAngelis-type functional response, we use different impulses for both prey and predator variables, which is more meaningful for the real life. We apply the continuation theorem directly to our system in order to get the w-periodic solution and establish its global stability. Moreover, we find a relationship between the extinction of preys and predators and determine its consequences. Nevertheless, there exist some difficulties in the application of the continuation theorem to the entire system, which are solved by using certain analytic techniques.

2. Preliminaries

The following data were taken from [3]: Let $L\colon \operatorname{Dom} L \subset X \to Y$ be a linear mapping and let $C\colon X \to Y$ be a continuous mapping, where X and Y are normed vector spaces. If

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$$

and $\operatorname{Im} L$ is closed in $Y,$ then the mapping L is called a Fredholm mapping of index zero. There exist continuous projections $U\colon X \to X$ and $V\colon Y \to Y.$ If L is a Fredholm mapping of index zero such that

$$\operatorname{Im} U = \operatorname{Ker} L \qquad \text{and} \qquad \operatorname{Im} L = \operatorname{Ker} V = \operatorname{Im}(I - V),$$

then we conclude that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} U}\colon (I-U)X \to \operatorname{Im} L$ is invertible. The inverse of this map is denoted by $K_U.$ A mapping C is called L-compact on Ω if $VC(\Omega)$ is bounded and $K_U(I-V)C\colon \Omega \to X$ is compact, where Ω is an open bounded subset of $X.$ Since $\operatorname{Im} V$ is isomorphic to $\operatorname{Ker} L,$ the isomorphism $J\colon \operatorname{Im} V \to \operatorname{Ker} L$ exists and the information presented above is important for the continuation theorem presented below.

Definition 1 [5]. *The codimension (quotient or factor dimension) of a subspace L of the vector space V is the dimension of the quotient space $V/L;$ it is denoted by $\operatorname{codim}_V L$ (or simply by $\operatorname{codim} L$) and is equal to the dimension of the orthogonal complement of L in $V;$ thus, we have*

$$\dim L + \operatorname{codim} L = \dim V.$$

Theorem 1 [12] (continuation theorem). *Suppose that L is a Fredholm mapping of index zero and C is L-compact on Ω. Assume that:*

(a) any y that satisfies $Ly = \lambda Cy$ does not lie on $\partial\Omega$ for all $\lambda \in (0,1);$

(b) $VCy \neq 0$ and the Brouwer degree $\deg\{JVC, \partial\Omega \cap \operatorname{Ker} L, 0\} \neq 0$ for each $y \in \partial\Omega \cap \operatorname{Ker} L.$

Then $Ly = Cy$ has at least one solution lying in $\operatorname{Dom} L \cap \partial\Omega.$

Definition 2 [26]. *A w-periodic semiflow $F(t)\colon X \to X$ (X is the initial-value space), in a sense that $F(t)x$ is continuous in $(t,x) \in [0,+\infty) \times X,$ $F(0) = I,$ and $F(t+w) = F(t)F(w)$ for all $t > 0,$ is generated by the solutions of a w-periodic system.*

Definition 3 [26]. *If there exists $\eta > 0$ such that, for any $x \in X_0$,*

$$\liminf_{t\to\infty} d(F(t)x, \partial X_0) \geq \eta,$$

then the periodic semiflow $F(t)$ is called uniformly persistent with respect to $(X_0, \partial X_0)$.

Definition 4 [13]. *Suppose that $F\colon \mathbb{R}^n \to \mathbb{R}^n$. If there exists a bounded set B such that, for any $x \in \mathbb{R}^n$, one can find an integer $n_0 = n_0(x, B)$ such that $F^n(x) \in B$ for all $n \geq n_0$, then the map F is called point dissipative.*

Lemma 1 [26]. *Assume that $S\colon X \to X$ is continuous and such that $S(X_0) \subset X_0$. Suppose that S is uniformly persistent with respect to $(X_0, \partial X_0)$, compact, and point dissipative. Then, for S in X_0 relative to strongly bounded sets in X_0, there exists a global attractor A_0 and S has a coexistence state $x_0 \in A_0$.*

Definition 5 [11]. *System (3) is called permanent if there exist positive constants $r_1,\ r_2,\ R_1$, and R_2 such that the solution $\big(\tilde{x}(t), \tilde{y}(t)\big)$ of system (3) satisfies*

$$r_1 \leq \lim_{t\to\infty} \inf\ \tilde{x}(t) \leq \lim_{t\to\infty} \sup\ \tilde{x}(t) \leq R_1,$$

$$r_2 \leq \lim_{t\to\infty} \inf\ \tilde{y}(t) \leq \lim_{t\to\infty} \sup\ \tilde{y}(t) \leq R_2.$$

Lemma 2 [22]. *Consider a system*

$$\begin{aligned} &\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t), \quad t \neq t_k, \\ &\tilde{x}(t_k^+) = (1 + g_k)\tilde{x}(t_k). \end{aligned} \tag{1}$$

Then system (1) admits a unique positive w-periodic solution if and only if

$$\int_0^w a(t)dt + \ln \prod_{i=1}^{q} (1 + g_i) > 0. \tag{2}$$

Moreover, this solution is globally asymptotically stable.

3. Main Result

We investigate the following equation:

$$\begin{aligned} &\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - c(t)\tilde{x}(t)E(t, \tilde{y}(t), \tilde{x}(t), \tilde{y}(t)), \quad t \neq t_k, \\ &\tilde{y}'(t) = -d(t)\tilde{y}(t) + f(t)\tilde{y}(t)E(t, \tilde{x}(t), \tilde{x}(t), \tilde{y}(t)), \quad t \neq t_k, \\ &\tilde{x}(t_k^+) = (1 + g_k)\tilde{x}(t_k), \\ &\tilde{y}(t_k^+) = (r_k)\tilde{y}(t_k), \end{aligned} \tag{3}$$

where

$$E\big(t, u(t), x(t), y(t)\big) = \frac{u(t)}{\alpha(t) + \beta(t)x(t) + m(t)y(t)}.$$

If we take $\tilde{x}(t) = \exp(x(t))$ and $\tilde{y}(t) = \exp(y(t))$ in Eq. (3), then we get the following system:

$$\begin{aligned} x'(t) &= a(t) - b(t)\exp(x(t)) - c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big), \quad t \neq t_k, \\ y'(t) &= -d(t) + f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big), \quad t \neq t_k, \\ \Delta x(t_k) &= \ln(1 + g_k), \\ \Delta y(t_k) &= \ln(r_k), \end{aligned} \tag{4}$$

where

$$t_{k+q} = t_k + w, \quad a(t+w) = a(t), \quad b(t+w) = b(t), \quad c(t+w) = c(t), \quad d(t+w) = d(t),$$

$$f(t+w) = f(t), \quad \alpha(t+w) = \alpha(t), \quad \beta(t+w) = \beta(t), \quad m(t+w) = m(t),$$

and $k,\ g_k,$ and r_k are constants such that $1 > g_k > -1$ and $r_k > 0.$ Here,

$$x(t_{k+q}) = x(t_k) + w, \quad \tilde{x}(t_{k+q}) = \tilde{x}(t_k) + w, \quad y(t_{k+q}) = y(t_k) + w, \quad \text{and} \quad \tilde{y}(t_{k+q}) = \tilde{y}(t_k) + w.$$

In equations (3) and (4), each coefficient function is from the class of continuous functions and all coefficient functions are positive.

Definition 6. *In system (4), we say that $x(t)$ $(y(t))$ [prey (predator)] goes to extinction if and only if* $\exp(x(t))$ $(\exp(y(t)))$ *tends to* 0 *as* t *tends to infinity for all solutions* $x(t)$ $(y(t))$*. Equivalently, we also say that prey (predator) goes to extinction if and only if* $\tilde{x}(t)$ $(\tilde{y}(t))$ *tends to zero as* t *tends to infinity for all solutions of system (3).*

Lemma 3. *Assume that*

$$\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i > 0 \tag{5}$$

is satisfied. If $y(t)$ does not go to extinction, then $x(t)$ also does not go to extinction.

Proof. The statement of the above lemma is equivalent to the following statement: Assume that (5) is satisfied. In this case, if $x(t)$ goes to extinction, then $y(t)$ also goes to extinction. By using the second equation in system (4) and taking the integral of this equation from 0 to $t,$ we obtain

$$\exp(y(t)) = \exp(y(0))\prod_{ti<t}(r_i)\exp\left(\int_0^t -d(s) + f(s)E(t, \exp(x(s)), \exp(x(s)), \exp(y(s)))ds\right). \tag{6}$$

If $x(t)$ goes to extinction, then $\exp(x(t))$ tends to 0 as t tends to infinity. Since all coefficient functions are positive, $f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)$ also tends to 0 as t tends to infinity. For sufficiently large t, the integral

$$\int_0^t -d(s) + \ln \prod_{i=1}^{q} r_i + f(s)E\big(t, \exp(x(s)), \exp(x(s)), \exp(y(s))\big)ds$$

becomes negative and the right-hand side of equation (6) tends to 0 as t tends to infinity, which means that $\exp(y(t))$ tends to 0 as t tends to infinity. Thus, $y(t)$ goes to extinction. Hence, we are done.

3.1. Permanence and Extinction of the Solutions.

Lemma 4. *If inequalities (2) and (5) are satisfied, then for given system (3),*

$$\liminf_{t\to\infty} \tilde{x}(t) \geq r_1$$

for some $\tilde{r}_1 > 0$.

Proof. Assume that prey goes to extinction. Then, by Lemma 3, predator also goes to extinction. Thus, for sufficiently large $T > 0$, there exists $\epsilon_0 > 0$ such that, for any $t > T$,

$$\tilde{y}(t) < \epsilon_0.$$

If

$$\int_0^w a(t)dt + \ln \prod_{k=1}^{q} (1 + g_k) > 0$$

for sufficiently small ϵ_0, then we get

$$\int_0^w a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\tilde{x}(t)dt + \ln \prod_{k=1}^{q} (1 + g_k) > 0. \tag{7}$$

In addition, for sufficiently small ϵ_0, the following inequality is true:

$$\tilde{x}'(t) > \tilde{x}(t)\left(a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\tilde{x}(t)\right).$$

Further, we consider a system

$$\begin{aligned} \bar{x}'(t) &= \bar{x}(t)\left(a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\bar{x}(t)\right), \\ \bar{x}(t_k^+) &= (1 + g_k)\bar{x}(t_k). \end{aligned} \tag{8}$$

Since inequality (7) is true, we can apply Lemma 2 to system (8). Thus, system (8) has a globally attractive w-periodic solution $\check{x}^*(t)$. By the comparison theorem for impulsive differential equations, we get $\tilde{x}(t) > \check{x}^*(t)$.

Therefore, prey does not go to extinction, which is a contradiction. Hence,

$$\liminf_{t\to\infty} \tilde{x}(t) \geq \tilde{r}_1$$

for some $\tilde{r}_1 > 0$.

Lemma 5. *Assume that inequality (2) is satisfied. Predator in system (3) goes to extinction if and only if*

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t)+\beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^{q}(r_k)\right) \leq 0, \tag{9}$$

where $x^*(t)$ is a unique, positive, globally attractive, and w-periodic solution of the system (1).

Proof. By taking the contrapositive of the necessary part of the lemma, we conclude that if (9) does not hold, then predator does not go to extinction. From now on, we proceed by contradiction. Thus, we assume that (9) does not hold and predator goes to extinction. If we arrive at a contradiction, then we are able to get the desired result for the first part of the lemma. Here, we suppose that system (3) satisfies the equation

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t)+\beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^{q}(r_k)\right) > 0. \tag{10}$$

Then there exists $\tilde{\epsilon} > 0$ such that

$$\int_0^w -d(t) + \frac{f(t)(x^*(t)-\tilde{\epsilon})}{\alpha(t)+\beta(t)(x^*(t)-\tilde{\epsilon})+m(t)\tilde{\epsilon}} dt + \ln\left(\prod_{k=1}^{q}(r_k)\right) > 0. \tag{11}$$

Consider a system

$$\begin{aligned} \tilde{x}'(t) &= \tilde{x}(t)(a(t) - \frac{2\gamma c(t)}{\alpha(t)+2\gamma m(t)} - b(t)\tilde{x}(t)), \\ \tilde{x}(t_k^+) &= (1+g_k)\big(\tilde{x}(t_k)\big). \end{aligned} \tag{12}$$

where γ is a positive constant. It is clear that

$$a(t) - \frac{2\gamma c(t)}{\alpha(t)+2\gamma m(t)} > 0$$

for sufficiently small γ. Thus, system (12) has a globally attractive, unique, and w-periodic solution from Lemma 2 for sufficiently small γ. Assume that x_γ is a globally attractive solution of system (12). Thus, $x_\gamma(t) \to x^*(t)$ as $\gamma \to 0$. Then there exists $\hat{\gamma}$ such that $x_{\hat{\gamma}}(t) \geq x^*(t) - \tilde{\epsilon}/2$ and $2\hat{\gamma} < \tilde{\epsilon}$. Since predator goes to extinction, we conclude that

$$\limsup_{t\to\infty} \tilde{y}(t) < \hat{\gamma}.$$

Hence, there exists T such that, for any $t > T$,

$$\tilde{y(t)} < 2\hat{\gamma} < \tilde{\epsilon}.$$

Since $\tilde{y(t)} < 2\hat{\gamma}$, we get

$$\tilde{x}'(t) > \tilde{x}(t)(a(t) - \frac{2\gamma c(t)}{\alpha(t) + 2\gamma m(t)} - b(t)\tilde{x}(t)).$$

By the comparison theorem for impulsive differential equations, $\tilde{x}(t) > x^*(t) - \tilde{\epsilon}$. Therefore, we arrive at the system

$$\tilde{y}'(t) \geq \tilde{y}(t)\left(-d(t) + \frac{f(t)(x^*(t) - \tilde{\epsilon})}{\alpha(t) + \beta(t)(x^*(t) - \tilde{\epsilon}) + m(t)\tilde{\epsilon}}\right),$$

$$\tilde{y}(t_k^+) = (r_k)(\tilde{y}(t_k)).$$

Here,

$$\tilde{y}(t) \geq \tilde{y}(0) \exp\left(\int_0^t -d(s) + \frac{f(s)(x^*(s) - \tilde{\epsilon})}{\alpha(s) + \beta(s)(x^*(s) - \tilde{\epsilon}) + m(s)\tilde{\epsilon}} ds + \ln\left(\prod_{0<t_k<t} (r_k)\right)\right). \tag{13}$$

Since inequality (11) is satisfied, the right-hand side of inequality (13) is always positive for sufficiently large t and does not go to zero as t tends to infinity. Therefore, $\tilde{y}(t)$ always becomes positive and does not go to zero. In other words, predator does not go to extinction. Hence, we have proved that if predator goes to extinction, then inequality (9) is true.

To prove the converse result, we again proceed by contradiction. Assume that inequality (9) holds and predator does not go to extinction. Then

$$\lim \inf_{t\to\infty} \tilde{y}(t) \geq \tilde{r}_2.$$

Since (2) is true, then by Lemma 6,

$$\lim \sup_{t\to\infty} \tilde{x}(t) \leq R_1.$$

Thus, we get

$$\tilde{x}'(t) \leq \tilde{x}(t)\left(a(t) - \frac{c(t)\tilde{r}_2}{\alpha(t) + \beta(t)R_1 + m(t)\tilde{r}_2} - b(t)\tilde{x}(t)\right),$$

$$\tilde{x}(t_k^+) = (1 + g_k)\big(\tilde{x}(t_k)\big).$$

By the comparison theorem for impulsive differential equations, we have $\tilde{x}(t) \leq x^*(t)$. Therefore, the following inequality is true:

$$\begin{aligned} &\tilde{y}'(t) \leq \tilde{y}(t)\left(-d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t) + m(t)\tilde{r}_2}\right), \\ &\tilde{y}(t_k^+) = (r_k)(\tilde{y}(t_k)). \end{aligned} \tag{14}$$

Since all coefficient functions in this system are positive, we find

$$\frac{f(t)x^*(t)}{\alpha(t)+\beta x^*(t)+m(t)\tilde{r}_2} \le \frac{f(t)x^*(t)}{\alpha(t)+\beta(t)x^*(t)} - \mu$$

for some $\mu > 0$. Then

$$\tilde{y}(t) \le y(0)\exp\left(\int_0^t -d(t)+\frac{f(t)x^*(t)}{\alpha(t)+\beta(t)x^*(t)} - \mu dt + \ln\left(\prod_{0\le t_k\le t}(r_k)\right)\right). \tag{15}$$

Since inequality (9) holds for sufficiently large $t,$ the argument of the exponential function in inequality (15) is negative. Thus, as t tends to infinity, $\tilde{y}(t)$ tends to zero, which means that predator goes to extinction, which is a contradiction.

Lemma 5 is proved.

Lemma 6. *If inequalities (2) and (5) are satisfied, then there exist positive constants* R_1 *and* R_2 *such that*

$$\begin{aligned}&\lim_{t\to\infty}\sup x(t) \le R_1,\\ &\lim_{t\to\infty}\sup y(t) \le R_2.\end{aligned} \tag{16}$$

Proof. First, we consider system (3). Then the following inequality is true:

$$\begin{aligned}&\tilde{x}'(t) \le a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t), \quad t \ne t_k,\\ &\tilde{x}(t_k^+) = (1+g_k)\tilde{x}(t_k).\end{aligned} \tag{17}$$

Suppose that (2) holds and consider the following equations:

$$\begin{aligned}&\tilde{u}'(t) = a(t)\tilde{u}(t) - b(t)\tilde{u}^2(t), \quad t \ne t_k,\\ &\tilde{u}(t_k^+) = (1+g_k)\tilde{u}(t_k).\end{aligned} \tag{18}$$

By Lemma 2, system (18) possesses a unique positive globally attractive (or globally asypmtotically stable) w-periodic solution $\bar{u}(t)$. By using the comparison theorem for impulsive differential equations from [1], we conclude that

$$\tilde{x}(t) \le u(t).$$

The attractivity of $\bar{u}(t)$ implies that there exists $T > 0$ such that

$$u(t) \le \bar{u}(t) + 1 \quad \text{for} \quad t > T.$$

Therefore, it is clear that $\tilde{x}(t)$ is bounded above by a positive constant R_1.

Second, we consider system (3). The coefficient functions in system (3) are bounded, positive, and w-periodic Thus, the following inequality is true:

$$\begin{aligned} &\tilde{y}'(t) \le -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)}{m(t)} \le \frac{f^M R_1}{m^L} - d(t)\tilde{y}(t), \quad t \ne t_k, \\ &\tilde{y}(t_k^+) = (r_k)\tilde{y}(t_k). \end{aligned} \tag{19}$$

Hence, we get

$$\tilde{y}(t) \le \tilde{y}(0) \prod_{0<t_k<t} r_k \exp\left(\int_0^t -d(s)ds \right) + \int_0^t \prod_{s<t_k<t} r_k \exp\left(\int_s^t -d(\sigma)d\sigma \right) \frac{f^M R_1}{m^L} ds.$$

We can rewrite the last inequality as follows:

$$\begin{aligned} \tilde{y}(t) \le \tilde{y}(0) &\exp\left(\int_0^t -d(s)ds + \ln\left(\prod_{0<t_k<t} r_k \right) \right) \\ &+ \frac{f^M R_1}{m^L} \int_0^t \exp\left(\int_s^t -d(\sigma)d\sigma + \ln\left(\prod_{s<t_k<t} r_k \right) \right) ds. \end{aligned} \tag{20}$$

For sufficiently large $t,$ the argument of the exponential function in the first term and the second term in inequality (20) are negative. Hence, if we take

$$\frac{f^M R_1}{m^L} = M,$$

then

$$\begin{aligned} \tilde{y}(t) &\le \tilde{y}(0)e^{1+Dw-ct} + Me^{1+Dw} \int_0^t e^{c(s-t)} ds \\ &\le \tilde{y}(0)\left(e^{1+Dw-ct} + \frac{Me^{1+Dw}}{c}(1-e^{-ct}) \right) \\ &\le \tilde{y}(0)\left(e^{1+Dw} + \frac{Me^{1+Dw}}{c} \right). \end{aligned}$$

Here, $D = \max\big\{|d(t)| : \, t \in [0, w]\big\}$ and

$$c = \min\left\{ \left(\int_0^w d(s)ds - \ln\left(\prod_{k=1}^q r_k \right) \right) \Big/ \; w, \; 1/w \right\}.$$

Thus, we have a positive constant R_2 such that $\tilde{y}(t)$ is bounded above by a positive constant $R_2.$

Lemma 7. *If inequalities (2) and (10) are satisfied for the unique positive globally attractive w-periodic solution $x^*(t)$ of system (1), then*

$$\lim_{t\to\infty} \tilde{y}(t) > \tilde{r}_2$$

for some positive $\tilde{r}_2$.

Proof. This result is an immediate consequence of Lemma 5.

Lemma 8. *Assume that inequalities (2) and (5) are satisfied. Then system (3) is permanent if and only if inequality (10) is satisfied. Therefore, by Theorem 2, this system has at least one w-periodic solution.*

Proof. This is an immediate consequence of Lemmas 3, 6, and 5.

Lemma 9. *Assume that (2) is satisfied for system (3). If at least one solution $\tilde{y}(t)$ does not tend to 0 as t tends to infinity, then all solutions $\tilde{y}(t)$ does not tend to 0 as t tends to infinity.*

Proof. This is a proof by contradiction. Assume that there exist two solutions of system (3), $\big(\tilde{x}(t), \tilde{y}(t)\big)$ and $(\hat{x}(t), \hat{y}(t))$, such that $\tilde{y}(t)$ does not tend to 0 as t tends to infinity and $\hat{y}(t)$ tends to 0 as t tends to infinity. Since $\hat{y}(t)$ tends to 0 as t tends to infinity, we conclude that $\hat{x}(t)$ tends to x^* as t tends to infinity. According to Definition 6, predator does not go to extinction and, as a consequence of Lemma 5, the inequality

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^{q} (r_k)\right) > 0$$

is true. Thus, by using Lemma 7, we get $\hat{y}(t) > \hat{r}$, for some positive $\hat{r}$, which is a contradiction. Hence, the proof is completed.

3.2. w-Periodicity of the Solutions.

Theorem 2. *Assume that all coefficient functions in system (4) are bounded, positive, w-periodic, and belong to $C(\mathbb{R}, \mathbb{R}^2)$ and that inequalities (2) and (5) are satisfied. Then there exists at least one w-periodic solution if and only if $y(t)$ does not go to extinction.*

Proof. We have

$$X := \big\{(p, z)^\mathsf{T} \in PC(\mathbb{R}, \mathbb{R}^2) :\ p(t+w) = p(t), z(t+w) = z(t)\big\}$$

with the norm

$$\|(p, z)^\mathsf{T}\| = \sup_{t\in[0,w]} \big(|p(t)|, |z(t)|\big)$$

and

$$Y := \Big\{\big[(p, z)^\mathsf{T}, (d_1, f_1)^\mathsf{T}, \ldots, (d_q, f_q)^\mathsf{T}\big] \in PC(\mathbb{R}, \mathbb{R}^2) \times (\mathbb{R}^2)^q,\ p(t+w) = p(t), z(t+w) = z(t)\Big\}$$

with the norm

$$\Big\|\big[(p, z)^\mathsf{T}, (d_1, f_1)^\mathsf{T}, \ldots, (d_q, f_q)^\mathsf{T}\big]\Big\| = \sup_{t\in[0,w]} \Big(\|(p, z)^\mathsf{T}\|, \big\|(d_1, f_1)^\mathsf{T}\big\|, \ldots, \big\|(d_q, f_q)^\mathsf{T}\big\|\Big).$$

We define the mappings $L\colon \operatorname{Dom} L \subset X \to Y$ such that

$$L\big((p,z)^\intercal\big) = \big((p',z')^\intercal, (\Delta p(t_1), \Delta z(t_1))^\intercal, \dots, (\Delta p(t_q), \Delta z(t_q))^\intercal\big)$$

and $C\colon X \to Y$ such that

$$C\big((p,z)^\intercal\big) = \left(\begin{bmatrix} a(t) - b(t)\exp(p(t)) - c(t)E\big(t, z(t), p(t), z(t)\big) \\ -d(t) + f(t)E\big(t, p(t), p(t), z(t)\big) \end{bmatrix}, \begin{bmatrix} \ln(1+g_1) \\ \ln(p_1) \end{bmatrix}, \dots, \begin{bmatrix} \ln(1+g_q) \\ \ln(p_q) \end{bmatrix} \right).$$

Then $\operatorname{Ker} L = \big\{(p,z)^\intercal$ such that $(p,z)^\intercal = (c_1, c_2)^\intercal\big\}$, c_1 and c_2 are constants, and

$$\operatorname{Im} L = \left\{ \big[(p,z)^\intercal, (d_1, f_1)^\intercal, \dots, (d_q, f_q)^\intercal\big] : \begin{bmatrix} \int_0^w p(s)ds + \sum_{i=1}^q d_i \\ \int_0^w z(s)ds + \sum_{i=1}^q f_i \end{bmatrix} = (0,0)^\intercal \right\}.$$

Note that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$.

This can shown in the following way: It is clear that the sum of any element from $\operatorname{Im} L$ and $\operatorname{Ker} L$ lies in Y. Without loss of generality, we take $p \in Y$ and

$$\int_\kappa^{w+\kappa} p(t)dt + \sum_{i=1}^q d_i = I \neq 0.$$

Further, we define a new function

$$g = p - \frac{I}{w}.$$

Then $\frac{I}{w}$ is constant because, for all κ, $\int_\kappa^{w+\kappa} p(t)dt$ is always the same by the definition of periodic time scales, the pulses are constant, and there is the same number of pulses in the interval $[\kappa, w+\kappa]$ for all κ. If we take the integral of g from κ to $w+\kappa$, then we get

$$\int_\kappa^{w+\kappa} g(t)dt + \sum_{i=1}^q d_i = \int_\kappa^{w+\kappa} p(t)dt + \sum_{i=1}^q d_i - I = 0.$$

Thus, $p \in Y$ can be written as the sum of $g \in \operatorname{Im} L$ and $\frac{I}{w} \in \operatorname{Ker} L$ because $\frac{I}{w}$ is constant. Similar steps are used for z. Indeed, $(p,z)^\intercal \in Y$ can be represent as the sum of an element from $\operatorname{Im} L$ and an element from $\operatorname{Ker} L$. Moreover, it is easy to see that any element in Y can be uniquely expressed as the sum of an element $\operatorname{Ker} L$ and an element from $\operatorname{Im} L$. Hence, $\operatorname{codim} \operatorname{Im} L$ is also equal to 2, and we get the desired result. Therefore, L is a Fredholm mapping of index zero.

There exist continuous projectors $U:\ X \to X$ and $V:\ Y \to Y$ such that

$$U\big((p,z)^{\mathsf T}\big) = \frac{1}{w}\begin{bmatrix} \int\limits_0^w p(s)ds \\ \int\limits_0^w z(s)ds \end{bmatrix}$$

and

$$V\big((p,z)^{\mathsf T},(d_1,f_1)^{\mathsf T},\dots,(d_q,f_q)^{\mathsf T}\big) = \frac{1}{w}\left(\begin{bmatrix} \int\limits_0^w p(s)ds + \sum\limits_{i=1}^q d_i \\ \int\limits_0^w z(s)ds + \sum\limits_{i=1}^q f_i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

The generalized inverse $K_U = \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} U$ is given, i.e.,

$$K_U\big((p,z)^{\mathsf T},(d_1,f_1)^{\mathsf T},\dots,(d_q,f_q)^{\mathsf T}\big)$$
$$= \begin{bmatrix} \int\limits_0^t p(s)ds + \sum\limits_{t>t_i} d_i - \frac{1}{w}\int\limits_0^w\int\limits_0^t p(s)dsdt - \sum\limits_{i=1}^q d_i + \frac{1}{w}\sum\limits_{i=1}^q d_i t_i \\ \int\limits_0^t z(s)ds + \sum\limits_{t>t_i} f_i - \frac{1}{w}\int\limits_0^w\int\limits_0^t z(s)dsdt - \sum\limits_{i=1}^q f_i + \frac{1}{w}\sum\limits_{i=1}^q f_i t_i \end{bmatrix},$$

$$VC\big((p,z)^{\mathsf T}\big)$$
$$= \frac{1}{w}\left(\begin{bmatrix} \int\limits_0^w a(s) - b(s)\exp(p(s)) - c(s)E(s,z(s),p(s),z(s))ds + \ln\prod\limits_{i=1}^q(1+g_i) \\ \int\limits_0^w -d(s) + f(s)E(s,p(s),p(s),z(s))ds + \ln\prod\limits_{i=1}^q(r_i) \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

Let

$$a(t) - b(t)\exp(p(t)) - c(t)E\big(t,z(t),p(t),z(t)\big) = C_1(t),$$

$$-d(t) + f(t)E\big(t,p(t),p(t),z(t)\big) = C_2(t),$$

$$\frac{1}{w}\int\limits_0^w a(s) - b(s)\exp(p(s)) - c(s)E(s,z(s),p(s),z(s))ds = \bar{C}_1,$$

and

$$\frac{1}{w}\int_0^w -d(s)+f(s)E(s,p(s),p(s),z(s))ds=\bar{C}_2,$$

$$K_U(I-V)C\big((p,z)^\intercal\big)=K_U\left(\begin{bmatrix}C_1(t)-\bar{C}_1\\ C_2(t)-\bar{C}_2\end{bmatrix},\begin{bmatrix}\ln(1+g_1)\\ \ln(p_1)\end{bmatrix},\ldots,\begin{bmatrix}\ln(1+g_q)\\ \ln(p_q)\end{bmatrix}\right)$$

$$=\begin{bmatrix}\int_0^t C_1(s)-\bar{C}_1ds+\ln\prod_{t>t_i}(1+g_i)-\frac{1}{w}\int_0^w\int_0^t C_1(s)\\ \quad -\bar{C}_1dsdt-\ln\prod_{i=1}^q(1+g_i)+\frac{1}{w}\sum_{i=1}^q\ln(1+g_i)t_i\\ \int_0^t C_2(s)-\bar{C}_2ds+\ln\prod_{t>t_i}r_i-\frac{1}{w}\int_0^w\int_0^t C_2(s)\\ \quad -\bar{C}_2dsdt-\ln\prod_{i=1}^q r_i+\frac{1}{w}\sum_{i=1}^q\ln(r_i)t_i\end{bmatrix}.$$

Clearly, VC and $K_U(I-V)C$ are continuous. Since X and Y are Banach spaces, by using the Arzela–Ascoli theorem, we conclude that $K_U(I-V)C(\bar{\Omega})$ is compact for any open bounded set $\Omega\subset X$. In addition, $VC(\bar{\Omega})$ is bounded. Thus, C is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega\subset X$.

We are now going to use the continuation theorem. In order to be able to apply this theorem, it is necessary to investigate the following system:

$$\begin{aligned}
&x'(t)=\lambda\big[a(t)-b(t)\exp(x(t))-c(t)E\big(t,\exp(y(t)),\exp(x(t)),\exp(y(t))\big)\big],\quad t\neq t_k,\\
&y'(t)=\lambda\big[-d(t)+f(t)E\big(t,\exp(x(t)),\exp(x(t)),\exp(y(t))\big)\big],\quad t\neq t_k,\\
&\Delta x(t_k)=\lambda\ln(1+g_k),\\
&\Delta y(t_k)=\lambda\ln(r_k),
\end{aligned}\tag{21}$$

$$\begin{aligned}
&\int_0^w a(t)dt+\ln\prod_{i=1}^q(1+g_i)=\int_0^w b(t)\exp(x(t))+c(t)E\big(t,\exp(y(t)),\exp(x(t)),\exp(y(t))\big)dt,\\
&\int_0^w d(t)dt-\ln\prod_{i=1}^q(r_i)=\int_0^w f(t)E\big(t,\exp(x(t)),\exp(x(t)),\exp(y(t))\big)dt.
\end{aligned}\tag{22}$$

By using (21) and (22), we find

$$\int_0^w |x'(t)|dt\leq\lambda\left[\int_0^w|a(t)|dt\right.$$

$$+ \int_0^w b(t)\exp(x(t)) + c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)dt\Bigg]$$

$$\leq \lambda\left[\int_0^w |a(t)|dt + \int_0^w a(t)dt + \ln\prod_{i=1}^{q}(1+g_i)\right] \leq M_1, \tag{23}$$

where

$$M_1 := 2\int_0^w a(t)dt + \ln\prod_{i=1}^{q}(1+g_i),$$

and

$$\int_0^w |y'(t)|dt \leq \lambda\left[\int_0^w |d(t)|dt + \int_0^w f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)dt\right]$$

$$\leq \lambda\left[\int_0^w |d(t)|dt + \int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i\right] \leq M_2, \tag{24}$$

where

$$M_2 := 2\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i.$$

Since $(x,y)^\intercal \in X$ and there are q constant pulses, we can say that there exist $\eta_i,\ \xi_i,\ i = 1, 2,$ such that

$$\begin{aligned} x(\xi_1) &= \min\Big\{\inf_{t\in[0,t_1]} x(t), \inf_{t\in(t_1,t_2]} x(t), \ldots, \inf_{t\in(t_q,w]} x(t)\Big\}, \\ x(\eta_1) &= \max\Big\{\sup_{t\in[0,t_1]} x(t), \sup_{t\in(t_1,t_2]} x(t), \ldots, \sup_{t\in(t_q,w]} x(t)\Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} y(\xi_2) &= \min\Big\{\inf_{t\in[0,t_1]} y(t), \inf_{t\in(t_1,t_2]} y(t), \ldots, \inf_{t\in(t_q,w]} y(t)\Big\}, \\ y(\eta_2) &= \max\Big\{\sup_{t\in[0,t_1]} y(t), \sup_{t\in(t_1,t_2]} y(t), \ldots, \sup_{t\in(t_q,w]} y(t)\Big\}. \end{aligned} \tag{26}$$

By the first equation in (22) and (23), we get $x(\xi_1) < l_1,$ where

$$l_1 := \ln\left(\frac{\int_0^w a(t)dt + \ln\prod_{i=1}^{q}(1+g_i)}{\int_0^w b(t)dt}\right).$$

Since $x(\xi_1)$ is the infimum of $x(t)$ for $t \in [0,w]$, then there exists $t_1 \in [0,w]$ such that

$$x(\xi_1) \le x(t_1) < l_1.$$

By using the first inequality in Lemma 1, we obtain

$$\begin{aligned} x(t) &\le x(t_1) + \int_0^w |x'(t)|dt \\ &\le x(t_1) + \left(2\int_0^w a(t)dt + \ln \prod_{i=1}^{q}(1+g_i)\right) < H_1 := l_1 + M_1. \end{aligned} \tag{27}$$

It follows from the second equation in (22) that $x(\eta_1) \ge l_2$, where

$$l_2 : \ = \ln\left(\frac{\int_0^w d(t)dt - \ln \prod_{i=1}^{q} r_i}{\int_0^w \big(f(t)/\alpha(t)\big)dt}\right).$$

Since $x(\eta_1)$ is the supremum of $x(t)$ for $t \in [0,w]$, there exists $t_2 \in [0,w]$ such that

$$x(\eta_1) \ge x(t_2) > l_2.$$

By using second inequality in Lemma 1, we find

$$\begin{aligned} x(t) &\ge x(t_2) - \int_0^w |x'(t)|dt \\ &\ge x(t_2) - \left(2\int_0^w a(t)dt + \ln \prod_{i=1}^{q}(1+g_i)\right) > H_2 := l_2 - M_1. \end{aligned} \tag{28}$$

By (27) and (28), we get

$$\max_{t\in[0,w]} |x(t)| \le B_1 := \max\{|H_1|, |H_2|\}.$$

By using the equality

$$\begin{aligned} &f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big) \\ &\qquad = f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)\exp(x(t)-y(t)), \end{aligned}$$

we conclude that

$$\int_0^w d(t)dt - \ln \prod_{i=1}^{q} r_i < \int_0^w (f(t)/m(t))[\exp(x(t)-y(t))]dt$$

$$\leq \left[\exp(x(\eta_1) - y(\xi_2))\right] \int_0^w (f(t)/m(t))dt.$$

Since (27) is true, we have the following inequality for each $t \in [0, w]$:

$$y(\xi_2) < H_1 - \ln\left(\frac{\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i}{\int_0^w (f(t)/m(t))dt}\right) := l_3.$$

Since $y(\xi_2)$ is the infimum of $y(t)$ for $t \in [0, w]$, there exists $t_3 \in [0, w]$ such that

$$y(\xi_2) \leq y(t_3) < l_3.$$

By using first equation in Lemma 1, we obtain

$$y(t) \leq y(t_3) + \int_0^w |y'(t)|dt$$

$$\leq y(t_3) + \left(2\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i\right) < H_3 := l_3 + M_2. \tag{29}$$

Here, all coefficient functions in $f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)$ are positive and $y(t)$ does not go to extinction. Thus, by Lemma 9, in view of the fact that systems (3) and (4) are equivalent, for all solutions $y(t)$, we conclude that as t tends to infinity, $\exp(y(t))$ does not tend to 0. Hence, we find

$$\frac{f(t)}{m(t)} > f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big) > 0.$$

Therefore, there exists $k \in \mathbb{N}$ such that

$$f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big) > \frac{1}{k}\frac{f(t)}{m(t)} > 0,$$

$$\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i = \int_0^w f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)\left[\exp(x(t) - y(t))\right]dt$$

$$\geq \left[\exp(x(\xi_1) - y(\eta_2))\right] \int_0^w f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)dt$$

$$> \left[\exp(x(\xi_1) - y(\eta_2))\right]\frac{1}{k}\int_0^w \frac{f(t)}{m(t)}dt.$$

This yields

$$y(\eta_2)) > x(\xi_1) - \ln\left(\frac{\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i}{1/k\dfrac{f(t)}{m(t)}}\right).$$

By virtue of (28), we get

$$y(\eta_2)) > H_2 - \ln\left(\frac{\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i}{1/k\dfrac{f(t)}{m(t)}}\right) := l_4.$$

Since $y(\eta_2)$ is the supremum of $y(t)$ for $t \in [0,w]$, there exists $t_4 \in [0,w]$ such that

$$y(\eta_2) \geq y(t_4) > l_4.$$

If we use the second inequality in Lemma 1, then we get

$$\begin{aligned} y(t) &\geq y(t_4) - \int_0^w |x'(t)|dt \\ &\geq y(t_4) - \left(2\int_0^w d(t)dt - \ln\prod_{i=1}^q r_i\right) > H_4 := l_4 - M_2. \end{aligned} \tag{30}$$

By (29) and (30), we obtain

$$\max_{t\in[0,w]} |y(t)| \leq B_2 := \max\left\{|H_3|, |H_4|\right\}.$$

Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then

$$\max_{t\in[0,w]} \left\|(x,y)^\mathsf{T}\right\| < M.$$

Also let $\Omega = \left\{\left\|(x,y)^\mathsf{T}\right\| \in X : \left\|(x,y)^\mathsf{T}\right\| < M\right\}$ and let Ω satisfy the requirement (a) in Theorem 1. If $\left\|(x,y)^\mathsf{T}\right\| \in \operatorname{Ker} L \cap \partial\Omega$ and $\left\|(x,y)^\mathsf{T}\right\|$ is a constant with $\left\|(x,y)^\mathsf{T}\right\| = M$, then

$$\begin{aligned} &VC\left((x,y)^\mathsf{T}\right) \\ &= \left(\begin{bmatrix} \int_0^w a(s) - b(s)\exp(x) - c(s)E(s,y(s),x(s),y(s))ds + \ln\prod_{i=1}^q (1+g_i) \\ \int_0^w -d(s) + f(s)E(s,x(s),x(s),y(s))ds + \ln\prod_{i=1}^q (r_i) \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ &\neq \left((0,0)^\mathsf{T}, \ldots, (0,0)^\mathsf{T}\right), \end{aligned}$$

$$JVC\big((x,y)^{\intercal}\big) = \begin{bmatrix} \displaystyle\int_0^w a(s) - b(s)\exp(x(s)) - c(s)E(s,y(s),x(s),y(s))ds + \ln\prod_{i=1}^{q}(1+g_i) \\ \displaystyle\int_0^w -d(s) + f(s)E(s,x(s),x(s),y(s))ds + \ln\prod_{i=1}^{q}(r_i) \end{bmatrix},$$

where $J\colon \operatorname{Im} V \to \operatorname{Ker} L$ is such that $J\big((x,y)^{\intercal},(0,0)^{\intercal},\ldots,(0,0)^{\intercal}\big) = (x,y)^{\intercal}$.

We define a homotopy $H_\nu = \nu(JVC) + (1-\nu)G$, where

$$G\big((x,y)^{\intercal}\big) = \begin{bmatrix} \displaystyle\int_0^w a(s) - b(s)\exp(x)ds + \ln\prod_{i=1}^{q}(1+g_i) \\ \displaystyle\int_0^w d(s) - f(s)E(s,x,x,y)ds + \ln\prod_{i=1}^{q}(r_i) \end{bmatrix}.$$

Since H_ν is a homotopy, then, for each $\nu \in [0,1]$, the Brouwer degrees

$$\deg(JVC, \Omega\cap\operatorname{Ker} L, 0), \quad \deg(G, \Omega\cap\operatorname{Ker} L, 0), \quad \text{and} \quad \deg(\nu(JVC) + (1-\nu)G, \Omega\cap\operatorname{Ker} L, 0)$$

are equal. Thus, it is sufficient to find one of these Brouwer degrees.

Let DJ_G be the determinant of the Jacobian of G. Since $(x,y)^{\intercal} \in \operatorname{Ker} L$, then the Jacobian of G is

$$\begin{bmatrix} \displaystyle -e^x\int_0^w b(s)ds & 0 \\ \displaystyle\int_0^w -f(s)E(s,e^x,e^x,e^y)ds + \int_0^w \frac{(e^x)^2 f(s)\beta(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}ds & \displaystyle -\int_0^w \frac{e^x e^y f(s)m(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}ds \end{bmatrix}.$$

All functions in the Jacobian of G are positive. Thus, $\operatorname{sign} DJ_G$ is always positive. Hence,

$$\begin{aligned}\deg(JVC, \Omega\cap\operatorname{Ker} L, 0) &= \deg(G, \Omega\cap\operatorname{Ker} L, 0) \\ &= \sum_{(x,y)^{\intercal}\in G^{-1}((0,0)^{\intercal})} \operatorname{sign} DJ_G\big((x,y)^{\intercal}\big) \neq 0.\end{aligned}$$

Thus, all conditions of Theorem 1 are satisfied. Therefore, system (4) has at least one positive w-periodic solution.

If the analyzed system (4) has at least one periodic solution, then, for at least one solution $y(t)$, the function $\exp(y(t))$, does not go to zero as t goes to infinity, which means that $y(t)$ does not go to extinction. Thus, by using Lemma 9 and the fact that systems (3) and (4) are equivalent, for all solutions $y(t)$, the function $\exp(y(t))$ does not go to zero as t tends to infinity. Hence, we are done.

Since systems (4) and (3) are equivalent, we conclude that if one of them has at least one w-periodic solution, then the other system also has a solution of this kind.

3.3. Some Simple Facts Obtained from the Results Presented in Subsections 3.1 and 3.2.

Remark 1. Assume that the inequalities (5) and

$$\int_0^w a(t) - \frac{c(t)}{m(t)} dt + \ln \prod_{i=1}^{q} (1 + g_i) > 0 \tag{31}$$

are satisfied. Consider a system

$$\begin{aligned} \tilde{v}'(t) &= (a(t) - \frac{c(t)}{m(t)})\tilde{v}(t) - b(t)\tilde{v}^2(t), \quad t \neq t_k, \\ \tilde{v}(t_k^+) &= (1 + g_k)\tilde{v}(t_k). \end{aligned} \tag{32}$$

By using system (32) and Lemma 2, we get

$$\int_0^w a(t) - \frac{c(t)}{m(t)} dt + \ln \prod_{i=1}^{q} (1 + g_i) = \int_0^w b(t) \exp(v(t)) dt.$$

Here, $\tilde{v}(t) = \exp(v(t))$. Therefore,

$$l_1 := \frac{\int_0^w a(t)dt + \ln \prod_{i=1}^{q} (1 + g_i)}{\int_0^w b(t)dt} \leq \tilde{v}(\xi_1),$$

where $\tilde{v}(\xi_1)$ is the supremum of $\tilde{v}$.

If we consider system (32) and set $\tilde{v}(t) = \exp(v(t))$ in this system, then we get

$$\begin{aligned} \int_0^w |v'(t)| dt &\leq \left[\int_0^w a(t)dt + \int_0^w b(t) \exp(u(t)) \right] \\ &\leq \left[\int_0^w a(t)dt + \int_0^w a(t)dt + \ln \prod_{i=1}^{q} (1 + g_i) \right]. \end{aligned}$$

By Lemma 2, the supremum of $\tilde{v}(t)$ and, therefore, the supremum of $v(t)$ exists. Since $\tilde{v}(\xi_1)$ is the supremum of $\tilde{v}$, by the definition of $v(t)$, $v(\xi_1)$ is the supremum of $v(t)$ for $t \in [0, w]$. Thus, there exists $t_1 \in [0, w]$ such that

$$v(\xi_1) \geq v(t_1) > l_1.$$

By using Lemma 2.4 in [4], we find

$$x(t) \geq v(t) \geq v(t_1) - \int\limits_0^w |v'(t)|dt$$

$$\geq \ln\left(\frac{\int_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)dt}\right) - \left(2\int\limits_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)\right).$$

The following corollary is obtained from Lemma 5:

Corollary 1. *If, in addition to (31) and (5), the following inequality is satisfied:*

$$\left(\frac{\int_0^w a(t) - \frac{c(t)}{m(t)}dt + \ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)dt}\right) \exp\left[-\left(2\int\limits_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)\right)\right]$$

$$\times\left(\int\limits_0^w f(t)dt - \beta^u\left(\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q (r_i)\right)\right) - \alpha^u\left(\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q (r_i)\right) > 0,$$

then system (3) has at least one w-periodic solution.

This is the same result as in Theorem 2 from [16] for the continuous case.

3.4. Some Examples.

Example 1.

$$\begin{aligned}
x'(t) &= (2\sin(2\pi t)+3) - (0.2\sin(2\pi t)+0.4)\exp(x) \\
&\quad - \frac{(5+2\cos(2\pi t))\exp(y)}{(\sin(2\pi t)+1.2)+(1+0.5\sin(2\pi t))\exp(x)+\exp(y)}, \\
y'(t) &= -(0.5\sin(2\pi t)+1.5) \\
&\quad + \frac{(0.8\cos(2\pi t)+4.45)\exp(x)}{(\sin(2\pi t)+1.2)+(1+0.5\sin(2\pi t))\exp(x)+\exp(y)}, \\
\Delta x(t_k^+) &= \ln(1+g_k), \\
\Delta y(t_k^+) &= \ln(r_k).
\end{aligned} \tag{33}$$

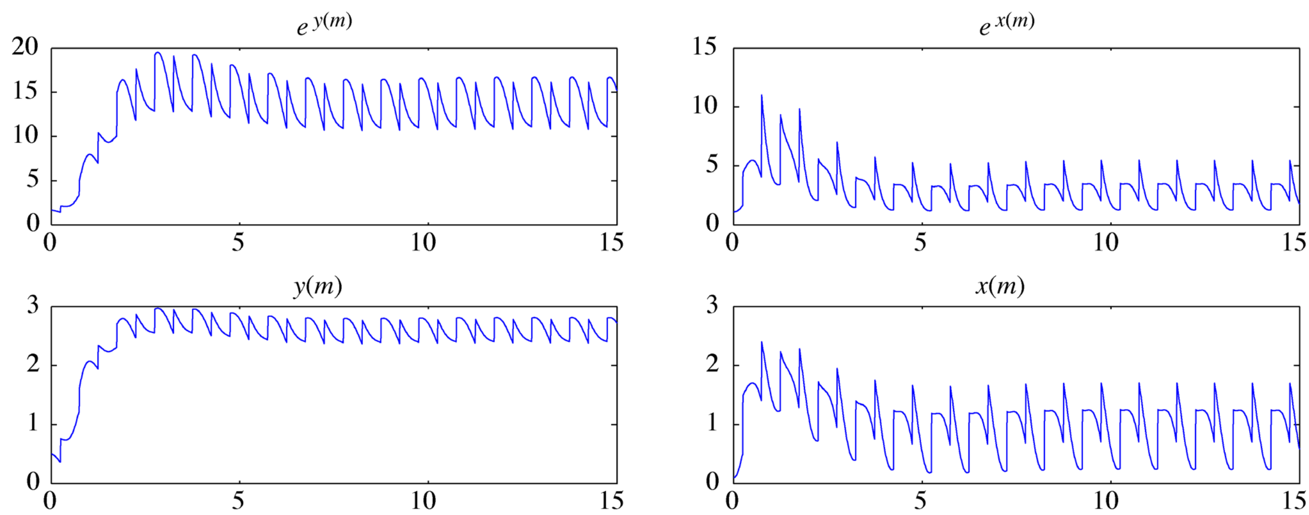

Fig. 1

Points of pulses:

$$t_1 = 2k + 1/4, \quad t_2 = 2k + 3/4 \quad \text{for} \quad k = 1, 2, 3, \ldots \quad \text{and} \quad q = 2.$$

Moreover,

$$g_1 = e^1 - 1, \quad g_2 = e^1 - 1, \quad r_1 = e^{0.4}, \quad \text{and} \quad r_2 = e^{0.4}.$$

First, we consider a system

$$x'(t) = (2\sin(2\pi t) + 3) - (0.2\sin(2\pi t) + 0.4)\exp(x),$$

$$\Delta x(t_k^+) = \ln(1 + g_k),$$

$$g_1 = e^1 - 1, \qquad g_2 = e^1 - 1.$$

Thus, by using the Mathlab program, we can find $x^* > 6.5$. Further, as a result of simple calculations, we can easily find that system (33) satisfies inequality (10). Moreover, by Lemma 8 system (33) has at least one 1-periodic solution; note that Fig. 1 $(x(0) = 0.1,\ y(0) = 0.5)$ also supports this result.

If, in system (33) from Example 1, we take

$$g_1 = e^{0.3} - 1, \quad g_2 = e^{0.3} - 1, \quad r_1 = e^{-1.7}, \quad \text{and} \quad r_2 = e^{-1.7},$$

then inequality (9) is satisfied and, by Lemma 5, we obtain the plots presented in Fig. 2 $(x(0) = 0.3,\ y(0) = 0.7)$.

This result shows us the importance of pulses. If we take pulses as follows:

$$g_1 = e^{0.3} - 1, \quad g_2 = e^{0.3} - 1, \quad r_1 = e^{-1.7}, \quad \text{and} \quad r_2 = e^{-1.7},$$

then, despite the fact that the system without pulses is identical, we conclude that predator goes to extinction because system (33) does not satisfy inequality (10).

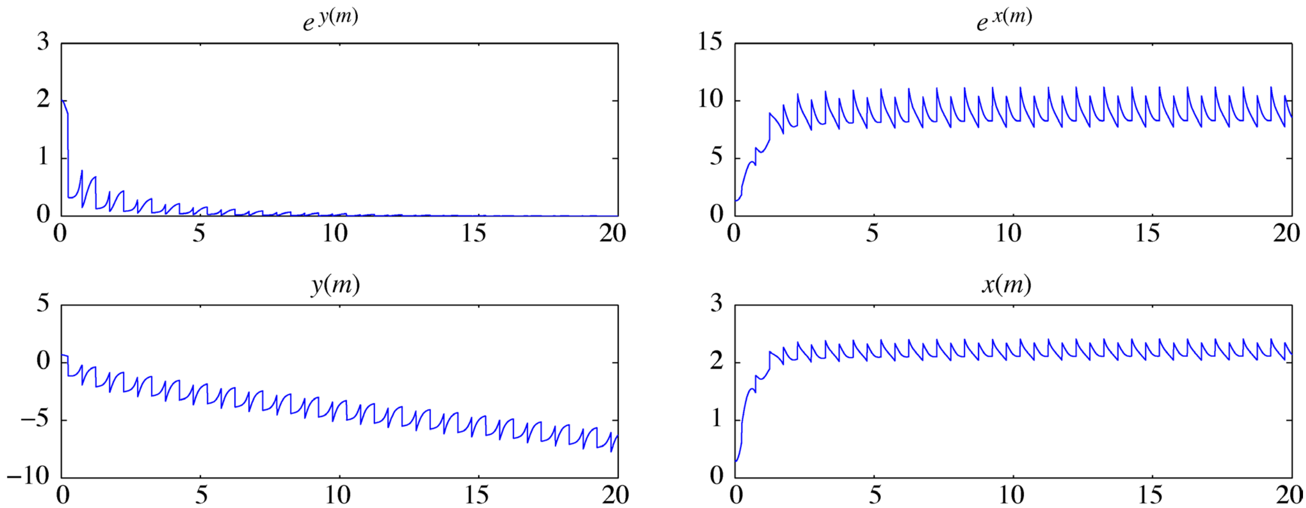

Fig. 2

The following example is for Corollary 1.

Example 2.

$$x' = (0.2\sin(2\pi t) + 0.3) - (0.2\sin(2\pi t) + 0.2)\exp(x)$$
$$- \frac{(0.1 + 0.1\cos(2\pi t))\exp(y)}{(0.5\sin(2\pi t) + 0.7) + (1 + 0.5\cos(2\pi t))\exp(x) + \exp(y)}, \quad t \neq t_k,$$
$$y'(t) = -(0.3\sin(2\pi t) + 1)$$
$$+ \frac{(4\cos(2\pi t) + 6.5)\exp(x)}{(0.5\sin(2\pi t) + 0.7) + (1 + 0.5\cos(2\pi t))\exp(x) + \exp(y)}, \quad t \neq t_k,$$
$$\Delta x(t_k) = \ln(1 + g_k),$$
$$\Delta y(t_k) = \ln(r_k).$$

Points of the pulses:

$$t_1 = 2k + 1/4, \quad t_2 = 2k + 3/4, \quad \text{and} \quad q = 2;$$

moreover,

$$g_1 = e^{-0.01} - 1, \quad g_2 = e^{-0.01} - 1, \quad p_1 = e^{0.1}, \quad \text{and} \quad p_2 = e^{0.1}.$$

Example 2 satisfies the condition of Corollary 1. Therefore, it has at least one w-periodic solution; moreover, Fig. 3 $(x(0) = 0.1,\ y(0) = 0.3)$ supports this result.

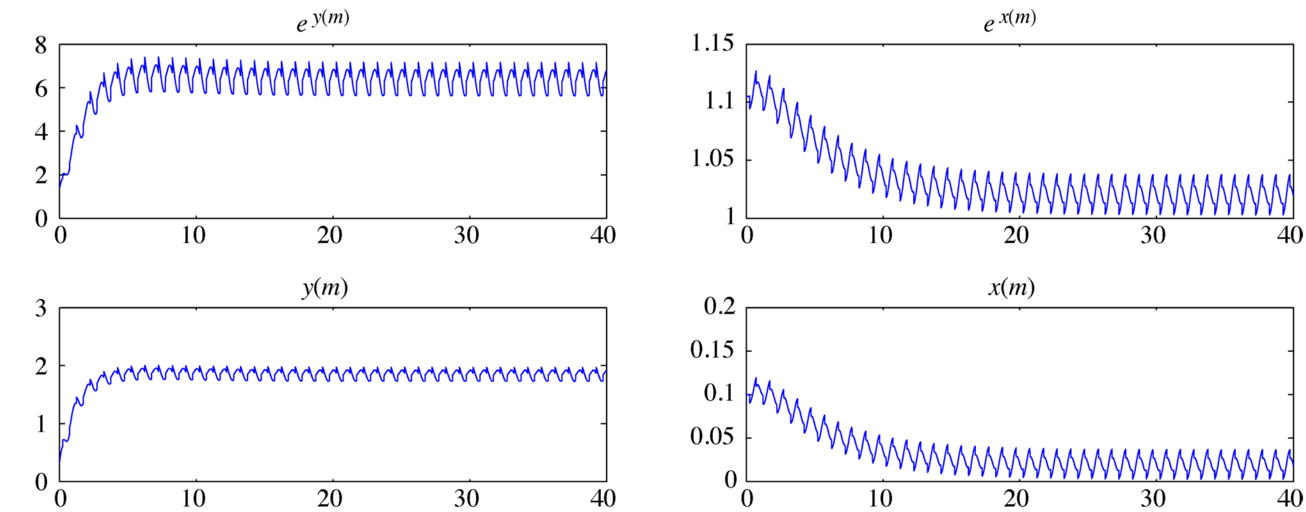

Fig. 3

3.5. Global Attractivity of the Solutions.

Theorem 3. *If inequalities (2), (5) and (10) are satisfied, then the w-periodic solution of the system (3) is globally attractive (globally asymptotically stable).*

Proof. The proof is very similar to the proof of Theorem 4.4 in [22]. To get the required result, we apply Lemma 1. Consider the following ordinary differential equation:

$$
\begin{gathered}
z'(t) = F(t, z(t)), \\
z(t_k^+) - z(t_k) = I_k(z(t_k)), \\
z(0) = \phi.
\end{gathered} \tag{34}
$$

Here,

$$
F \in C([0,\infty) \times \mathbb{R}^2, \mathbb{R}^2), \quad \phi \in \mathbb{R}^2, \qquad F(t+w, u) = F(t, u), \quad I_k \in C(\mathbb{R}^2, \mathbb{R}^2)
$$

and there exists an integer q such that $I_{k+q} = I_k$ and $t_{k+q} = t_k + w$. Thus, the operator that solves system (34) can be represented as follows:

$$
\hat{T}(t)z = ze^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \big[F(s, \hat{T}(s)z) + \lambda \hat{T}(s)z\big] ds + \sum_{0<t_k<t} e^{-\lambda(t-t_k)} I_k(\hat{T}(t_k)z),
$$

where λ is a positive constant. It is clear that $T(0) = I$. We can also show that

$$
u(s) = \begin{cases} T(s)z, & 0 \le s \le w, \\ T(s-w)T(w)z, & w \le s \le t + w, \end{cases}
$$

is the solution of system (34) with the initial value $u(0) = z$; here, $s \in [0, t + w]$. By the uniqueness theorem, system (34) possesses a unique solution. Therefore,

$$T(t + w)z = u(t + w) = T(t)T(w)z.$$

This is true for $t \neq t_k$. For $t = t_k$, we obtain

$$T(t_k^+ + w)z = T(t_k + w)z + I_k(T(t_k + w)z)$$

$$= T(t_k)T(w)z + I_k(T(t_k)T(w)z) = T(t_k^+)T(w)z.$$

To apply Lemma 1, we assume that $S = T(w)$ and

$$S^2 = SoS = T(w)oT(w) = T(2w).$$

Here, the analyzed system (34) is a periodic system. Hence, we can apply the Arzela–Ascoli theorem for impulsive differential equations and the result from [1]. This enables us to conclude that $T(t)$ is a compact operator.

If we take $X_i^+ = \{z_i : z_i \in \mathbb{R},\ z_i \geq 0\}$ for $i = 1, 2$ and $X_{i0}^+ = \{z_i : z_i \in \mathbb{R},\ z_i > 0\}$ for $i = 1, 2$, then we get $X = X_1^+ \times X_2^+$, $X = X_{1_0}^+ \times X_{2_0}^+$, and $\delta X_0 = X/X_0$. If system (3) satisfies inequalities (2), (5), and (10), then it becomes permanent. Therefore, S satisfies the conditions of Lemma 1. Hence, S admits a global attractor, which means that the system has a globally asymptotically stable (or globally attractive) w-periodic solution.

Corollary 2. *Assume that all coefficient functions in system (3) are bounded positive w-periodic functions from $PC(\mathbb{R}, \mathbb{R}^2)$. Then a globally attractive w-periodic solution of system (3) exists if and only if inequalities (2), (5), and (10) are satisfied.*

***Proof*.** The proof immediately follows from Theorems 2 and 3.

Example 1 satisfies all inequalities (2), (5), and (10). Therefore, it has a w-periodic globally asymptotically stable solution.

REFERENCES

1. D. Bainov and P. Simeonov, "Impulsive differential equations: periodic solutions and applications," *Pitman Monographs and Surveys in Pure and Applied Mathematics*, **66**, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York (1993).
2. J. R. Beddington, "Mutual interference between parasites or predators and its effect on searching efficiency," *J. Anim. Ecol.*, **44**, 331–340 (1975).
3. M. Bohner and A. Peterson, *Dynamic Equations on Times Scales: an Introduction with Applications*, Birkhäuser Boston, Boston, MA (2001).
4. M. Bohner, M. Fan, and J. Zhang, "Existence of periodic solutions in predator-prey and competition dynamic systems," *Nonlin. Anal.: Real World Appl.*, **7**, 1193–1204 (2006).
5. N. Bourbaki, *Elements of Mathematics. Algebra, Part I: Chapters 1–3*, Hermann, Paris; Addison-Wesley Publ., Reading Mass. (1974).
6. D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, "A model for trophic interaction," *Ecology*, **56**, 881–892 (1975).
7. M. Fan and S. Agarwal, "Periodic solutions for a class of discrete time competition systems," *Nonlin. Stud.*, **9**, No. 3, 249–261 (2002).
8. M. Fan and K. Wang, "Global periodic solutions of a generalized n-species Gilpin–Ayala competition model," *Comput. Math. Appl.*, **40**, No. 10-11, 1141–1151 (2000).
9. M. Fan and K. Wang, "Periodicity in a delayed ratio-dependent predator-prey system," *J. Math. Anal. Appl.*, **262**, No. 1, 179–190 (2001).
10. M. Fan and Q. Wang, "Periodic solutions of a class of nonautonomous discrete time semi-ratio-dependent predator-prey systems," *Discrete Contin. Dyn. Syst. Ser. B*, **4**, No. 3, 563–574 (2004).

11. Q. Fang, X. Li, and M. Cao, "Dynamics of a discrete predator-prey system with Beddington–DeAngelis function response," *Appl. Math.*, **3**, 389–394 (2012).
12. R. E. Gaines and J. L. Mawhin, "Coincidence degree, and nonlinear differential equations," *Lecture Notes Math.*, **568**, Springer-Verlag, Berlin–New York (1977).
13. J. K. Hale, "Asymptotic behavior of dissipative systems," *Math. Surveys Monogr.*, **25**, Amer. Math. Soc., Providence, R.I. (1988).
14. H. F. Huo, "Periodic solutions for a semi-ratio-dependent predator-prey system with functional responses," *Appl. Math. Lett.*, **18**, 313–320 (2005).
15. Y. K. Li, "Periodic solutions of a periodic delay predator-prey system," *Proc. Amer. Math. Soc.*, **127**, No. 5, 1331–1335 (1999).
16. A. F. Guvenilir, B. Kaymakcalan, and N. N. Pelen, "Impulsive predator-prey dynamic systems with Beddington–DeAngelis type functional response on the unification of discrete and continuous systems," *Appl. Math.*, **6** (2015); DOI: 10.4236/am.2015.69147.
17. A. M. Samoilenko and N. A. Perestyuk, "Impulsive differential equations," *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, Vol. 14, World Scientific Publ., River Edge, NJ (1995).
18. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, "Theory of impulsive differential equations," *Series in Modern Applied Mathematics, 6*, World Scientific Publ., Teaneck, NJ (1989).
19. S. Tang, Y. Xiao, L. Chen, and R. A. Cheke, "Integrated pest management models and their dynamical behaviour," *Bull. Math. Biol.*, **67**, No. 1, 115–135 (2005).
20. Q. Wang, M. Fan, and K. Wang, "Dynamics of a class of nonautonomous semi-ratio-dependent predator-prey systems with functional responses," *J. Math. Anal. Appl.*, **278**, No. 2, 443–471 (2003).
21. P. Wang, "Boundary value problems for first order impulsive difference equations," *Int. J. Difference Equat.*, **1**, No. 2, 249–259 (2006).
22. W. Wang, J. Shen, and J. Nieto, "Permanence and periodic solution of predator-prey system with Holling type functional response and impulses," *Discrete Dyn. Nat. Soc.*, **2007**, Article ID 81756 (2007), 15 p.
23. C. Wei and L. Chen, "Periodic solution of prey-predator model with Beddington–DeAngelis functional response and impulsive state feedback control," *J. Appl. Math.*, **2012** (2012), 17 p.
24. R. Xu, M. A. J. Chaplain, and F. A. Davidson, "Periodic solutions for a predator-prey model with Holling-type functional response and time delays," *Appl. Math. Comput.*, **161**, No. 2, 637–654 (2005).
25. Z. Xiang, Y. Li, and X. Song, "Dynamic analysis of a pest management SEI model with saturation incidence concerning impulsive control strategy," *Nonlin. Anal.*, **10**, No. 4, 2335–2345 (2009).
26. X.-Q. Zhao, "Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications," *Can. Appl. Math. Quart.*, **3**, No. 4, 473–495 (1995).