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MONOGENIC FUNCTIONS WITH VALUES IN COMMUTATIVE COMPLEX ALGEBRAS OF THE SECOND RANK WITH UNIT AND A GENERALIZED BIHARMONIC EQUATION WITH SIMPLE NONZERO CHARACTERISTICS

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Among all two-dimensional algebras of the second rank with unit e over the field of complex numbers $\mathbb{C}$, we find a semisimple algebra $\mathbb{B}_0 := \{c_1 e + c_2\omega : c_k \in \mathbb{C},\ k = 1,2\},\ \omega^2 = e,$ containing bases $\{e_1, e_2\}$ such that the $\mathbb{B}_0$-valued "analytic" functions $\Phi(xe_1 + ye_2)$, where x and y are real variables, satisfy a homogeneous partial differential equation of the fourth order, which has only simple nonzero characteristics. The set of pairs $(\{e_1, e_2\}, \Phi)$ is described in the explicit form.

1. Statement of the Problems

Consider an equation

$$Lu(x,y) := \left(b_1 \frac{\partial^4}{\partial y^4} + b_2 \frac{\partial^4}{\partial x \partial y^3} + b_3 \frac{\partial^4}{\partial x^2 \partial y^2} + b_4 \frac{\partial^4}{\partial x^3 \partial y} + b_5 \frac{\partial^4}{\partial x^4}\right) u(x,y) = 0, \tag{1}$$

where the complex coefficients $b_k \in \mathbb{C},\ k = \overline{1,5},\ b_5 \neq 0,$ are such that the characteristic equation

$$l\,(s) := b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0, \quad s \in \mathbb{C}, \tag{2}$$

has four pairwise different roots (each root is simple):

$$\{s_1, s_2, s_3, s_4\} := \ker l, \tag{3}$$

where $s_k \in \mathbb{C} \setminus \{0\},\ s_k \neq s_m$ for $k \neq m,\ k,m \in \{1,\ldots,4\}.$ The relations $s_k \neq 0,\ k = \overline{1,4},$ are equivalent to the given condition $b_5 \neq 0.$ It is clear that the relation $b_1 \neq 0$ follows from the indicated condition. Thus,

$$b_1 b_5 \neq 0. \tag{4}$$

A solution of Eq. (1) in the domain D of the Cartesian plane xOy is defined as a single-valued function u with continuous partial derivatives up to the fourth order, inclusively, satisfying Eq. (1) in $D.$

Since special cases of Eq. (1) are elliptic equations ("close" to the biharmonic equation in a sense of Sec. 4) for the stress function in plane anisotropic media (see, e.g., [2–4]), we say that Eq. (1) a *generalized biharmonic equation* (this term is used, e.g., in [1, p. 67] for the equation for stress function in the anisotropic medium).

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By $\mathbb{B}_*$ we denote an associative algebra of the second rank with unit e commutative over the field of complex numbers $\mathbb{C}$. Let $\{e_1, e_2\}$ be a basis of $\mathbb{B}_*$ satisfying the relation

$$\mathcal{L}(e_1, e_2) := b_1(e_2)^4 + b_2 e_1(e_2)^3 + b_3(e_1)^2(e_2)^2 + b_4(e_1)^3 e_2 + b_5(e_1)^4 = 0. \tag{5}$$

We now state the problem of determination of all pairs $\mathbb{B}_*,\ \{e_1, e_2\}$ (see Sec. 2).

For the biharmonic equation, this problem and its solution are presented in [5]. In a special case of Eq. (1) ($b_1 = b_5 = 1,\ b_2 = b_4 = 0$, and $b_3 > 2$), this problem was posed and solved in [6].

We introduce the notation: $\mu_{e_1,e_2} := \{xe_1 + ye_2 \colon x, y \in \mathbb{R}\}$ (the linear span of the vectors e_1 and e_2 over the field of real numbers $\mathbb{R}$), $D_\zeta := \{\zeta = xe_1 + ye_2 \colon (x, y) \in D\} \subset \mu_{e_1,e_2}$, and $\zeta = xe_1 + ye_2 \in D_\zeta$ for $(x, y) \in D$.

In addition to conditions (5), we assume that the basis $\{e_1, e_2\}$ also satisfies the condition:

($\mathcal{MB}$) each nonzero element $h \in \mu_{e_1,e_2}$ is invertible (i.e., there exists an inverse element $h^{-1} \in \mathbb{B}_*$ such that $hh^{-1} = e$).

For each required basis $\{e_1, e_2\}$ simultaneously satisfying the conditions (5) and $\mathcal{MB}$, we consider functions *monogenic* in D_ζ, i.e., functions $\Phi \colon D_\zeta \to \mathbb{B}_*$ of the form

$$\Phi(\zeta) = U_1(x, y)\, e_1 + U_2(x, y)\, ie_1 + U_3(x, y)\, e_2 + U_4(x, y)\, ie_2 \quad \forall \zeta \in D_\zeta, \tag{6}$$

with the classical derivative $\Phi'(\zeta)$ at any point ζ in D_ζ:

$$\Phi'(\zeta) := \lim_{h \to 0, h \in \mu_{e_1,e_2}} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}.$$

We also denote each component $U_k \colon D \to \mathbb{R}$ in (6) by $\mathrm{U}_k[\Phi]$, i.e.,

$$\mathrm{U}_k[\Phi(\zeta)] := U_k(x, y), \qquad k \in \{1, \ldots, 4\}.$$

If a monogenic function Φ has continuous derivatives $\Phi^{(k)}(\zeta)$ up to the kth order, inclusively, $k \geq 4$, in the domain D_ζ, then, according to the relations

$$L\Phi(\zeta) = \mathcal{L}(e_1, e_2)\Phi^{(4)}(\zeta) \equiv 0$$

for any $\zeta \in D_\zeta$ {these relations are deduced by analogy with the corresponding relations in [6] (Sec. 6) for a special case of the operator L in Eq. (1)} and equality (6), we conclude that the components $U_k,\ k = \overline{1, 4}$, satisfy Eq. (1) in the domain D.

We state the problem of description of all monogenic functions and a subset of monogenic functions Φ whose components $\mathrm{U}_k[\Phi] = U_k,\ k = \overline{1, 4}$, are solutions of Eq. (1) (see Sec. 3).

Let D be a bounded and simply connected domain. Consider the problem of existence of monogenic functions Φ such that $\mathrm{U}_1[\Phi] = u$, where u is an arbitrary function from the space of solutions of Eq. (1). In the case where Eq. (1) is the equation for the stress function in a plane anisotropic medium, we also consider the problem of its reduction to equations $L(\widetilde{u}) = 0$ of the form (1) with the help of which the required monogenic functions Φ satisfying the relation $\mathrm{U}_1[\Phi] = \widetilde{u}$, can be found in the explicit form. This class of problems is investigated in Sec. 4.

Note that hypercomplex "analytic" functions $\Phi(xe_1 + ye_2)$ with values in finite-dimensional algebras over the field of real (of dimension four) or complex (of dimension two) numbers whose components satisfy equations of the form (1) were considered, e.g., in [7–14]. Despite the availability of numerous works, the complete description of the indicated triples $\mathbb{B}_*,\ \{e_1,e_2\},\ \Phi$ (or similar objects for the other definitions of "monogeneity") is unknown [the basis $\{e_1,e_2\}$ simultaneously satisfies the conditions (5) and $\mathcal{MB}$]. This is explained, in particular, by the fact that the class of Eqs. (1) is fairly broad.

In the present paper, we solve all posed problems in the complete and explicit form.

2. Commutative and Associative Algebras of the Second Rank and Their Bases Associated with Eq. (1)

It is known (see [15]) that there exist (to within an isomorphism) two associative algebras of the second rank with unit e commutative over the field of complex numbers $\mathbb{C}$:

$$\mathbb{B} := \{c_1 e + c_2\rho\colon\ c_k \in \mathbb{C},\ \ k = 1,2\}, \quad \rho^2 = 0, \tag{7}$$

$$\mathbb{B}_0 := \{c_1 e + c_2\omega\colon\ c_k \in \mathbb{C},\ \ k = 1,2\}, \quad \omega^2 = e. \tag{8}$$

It is clear that the algebra $\mathbb{B}_0$ is semisimple (for the definition, see, e.g., [16, p. 33]) and contains a basis of orthogonal idempotents $\{\mathfrak{I}_1,\mathfrak{I}_2\}$, where

$$\mathfrak{I}_1 = \frac{1}{2}\,(e+\omega), \qquad \mathfrak{I}_2 = \frac{1}{2}\,(e-\omega), \qquad \mathfrak{I}_1\mathfrak{I}_2 = 0, \qquad (\mathfrak{I}_k)^2 = \mathfrak{I}_k, \quad k = 1,2. \tag{9}$$

It is obvious that

$$\mathfrak{I}_1 + \mathfrak{I}_2 = e, \qquad \mathfrak{I}_1 - \mathfrak{I}_2 = \omega. \tag{10}$$

The element $w = c_1\mathfrak{I}_1 + c_2\mathfrak{I}_2$ from $\mathbb{B}_0$ is invertible if and only if $c_k \neq 0,\ k = 1,2.$ If this condition is satisfied, then the following equality is true for the inverse element:

$$w^{-1} = \frac{1}{c_1}\,\mathfrak{I}_1 + \frac{1}{c_2}\,\mathfrak{I}_2 \tag{11}$$

(see [17, p. 38]).

The theorem presented below gives the description of all couples $\mathbb{B}_*,\ \{e_1,e_2\},$ where the bases $\{e_1,e_2\}$ satisfy condition (5). In particular, it is established that $\mathbb{B}_* = \mathbb{B}_0$.

Theorem 1. *The algebra $\mathbb{B}$ does not contain any basis $\{e_1,e_2\}$ satisfying condition* (5)*. All pairs of basis elements of the algebra $\mathbb{B}_0$ satisfying condition* (5) *have the form*

$$e_1 = \alpha\,\mathfrak{I}_1 + \beta\,\mathfrak{I}_2, \qquad e_2 = \widetilde{s}_1\alpha\,\mathfrak{I}_1 + \widetilde{s}_2\beta\,\mathfrak{I}_2, \tag{12}$$

where $\widetilde{s}_k \in \ker l,\ k = 1,2,$ are such that $\widetilde{s}_1 \neq \widetilde{s}_2,$ and the complex numbers $\alpha \neq 0$ and $\beta \neq 0$ are chosen arbitrarily.

Proof. We seek pairs of basis elements $\{e_1,e_2\}$ of the form

$$e_k = \alpha_k e + \beta_k\rho \in \mathbb{B}, \quad k = 1,2, \tag{13}$$

where the unknown complex coefficients $\alpha_k,\ \beta_k,\ k=1,2,$ satisfy the relation

$$\Delta_{e_1e_2} := \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0. \tag{14}$$

It is easy to see that

$$(e_m)^k = (\alpha_m)^{k-1}\left(\alpha_m e + k\beta_m\rho\right), \quad k=\overline{1,4}, \quad m=\overline{1,2}. \tag{15}$$

Substituting (13) in (5) and taking into account (15), we get

$$\begin{aligned}\mathcal{L}(e_1,e_2) &= b_1\alpha_2^3\left(\alpha_2 e + 4\beta_2\rho\right) + b_2\left(\alpha_1 e + \beta_1\rho\right)\alpha_2^2\left(\alpha_2 e + 3\beta_2\rho\right)\\ &\quad + b_3\alpha_1\alpha_2\left(\alpha_1 e + 2\beta_1\rho\right)\left(\alpha_2 e + 2\beta_2\rho\right) + b_4\alpha_1^2\left(\alpha_1 e + 3\beta_1\rho\right)\left(\alpha_2 e + \beta_2\rho\right)\\ &\quad + b_5\alpha_1^3\left(\alpha_1 e + 4\beta_1\rho\right) = A_e\, e + A_\rho\, \rho,\end{aligned} \tag{16}$$

where

$$A_e := b_1\alpha_2^4 + b_2\alpha_2^3\alpha_1 + b_3\alpha_2^2\alpha_1^2 + b_4\alpha_2\alpha_1^3 + b_5\alpha_1^4,$$

$$\begin{aligned}A_\rho := &\left(b_2\beta_1 + 4b_1\beta_2\right)\alpha_2^3 + \left(3b_2\beta_2 + 2b_3\beta_1\right)\alpha_1\alpha_2^2\\ &+ \left(2b_3\beta_2 + 3b_4\beta_1\right)\alpha_1^2\alpha_2 + \alpha_1^3\left(b_4\beta_2 + 4b_5\beta_1\right).\end{aligned}$$

Hence, the required $\alpha_k, \beta_k \in \mathbb{C},\ k=1,2,$ must satisfy the following system:

$$A_e = 0, \qquad A_\rho = 0, \qquad \Delta_{e_1e_2} \neq 0. \tag{17}$$

Consider the first equation in system (17). According to (4), we get $\alpha_1 \neq 0$ [otherwise, $\alpha_1 = \alpha_2 = 0,$ which contradicts the third relation in (17)], and the equality

$$\frac{\alpha_2}{\alpha_1} = s_* \quad \forall s_* \in \ker l, \tag{18}$$

holds.

Dividing both sides of the second equation in (17) by α_1^3 and using (18), we get

$$-l_0(s_*)\beta_1 + l'(s_*)\beta_2 = 0, \tag{19}$$

where

$$l_0(s_*) := -\left(b_2 s_*^3 + 2b_3 s_*^2 + 3b_4 s_* + 4b_5\right)$$

and $l'(s_*)$ is the value of the derivative of the polynomial $l(s)$ from (2) for $s = s_*$. Since s_* is a simple root of Eq. (2), $l'(s_*) \neq 0$ and Eq. (19) is equivalent to the following equation:

$$\beta_2 = \frac{l_0(s_*)}{l'(s_*)}\beta_1. \tag{20}$$

Among the obtained couples $\{e_1, e_2\}$, it is necessary to select the set of linearly independent couples. To this end, we check the validity of the third relation in system (17). Substituting (18) and (20) in (14), we get

$$\Delta_{e_1e_2} = \left(\frac{l_0(s_*)}{l'(s_*)} - s_*\right)\alpha_1\beta_1 \neq 0. \tag{21}$$

If $\beta_1 = 0$, then condition (21) is not true. Thus, $\beta_1 \neq 0$ and, hence, $\beta_2 \neq 0$ according to (20). However, as shown above, $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. Hence, $\Delta_{e_1e_2}$ can be equal to zero only under the condition that

$$\frac{l_0(s_*)}{l'(s_*)} - s_* = 0.$$

We check whether it is possible. As a result of direct substitution, we get

$$\frac{l_0(s_*)}{l'(s_*)} - s_* = -\frac{4}{l'(s_*)} l\,(s_*) \equiv 0.$$

This enables us to conclude that the required bases do not exist in the algebra $\mathbb{B}$.

Thus, we find necessary bases in the algebra $\mathbb{B}_0$.

It is easy to see that the elements $e_k = \alpha_k\,\mathcal{I}_1 + \beta_k\,\mathcal{I}_2,\ k = 1, 2,$ satisfy the equalities

$$e_k^n = \alpha_k^n\,\mathcal{I}_1 + \beta_k^n\,\mathcal{I}_2, \quad n = \overline{1, 4}, \quad k = 1, 2. \tag{22}$$

Denote $(e_k)^0 := 1,\ k = 1, 2,\ \lambda^0 := 1$ for real λ. Then

$$\begin{aligned}\mathcal{L}(e_1, e_2) &= \sum_{k=1}^{5} b_k \left(\alpha_2^{5-k}\mathcal{I}_1 + \beta_2^{5-k}\mathcal{I}_2\right)\left(\alpha_1^{k-1}\mathcal{I}_1 + \beta_1^{k-1}\mathcal{I}_2\right) \\ &= \sum_{k=1}^{5} b_k \left(\alpha_2^{5-k}\alpha_1^{k-1}\mathcal{I}_1 + \beta_2^{5-k}\beta_1^{k-1}\mathcal{I}_2\right).\end{aligned}$$

Thus, the required system for the coefficients of the basis elements $e_k = \alpha_k\,\mathcal{I}_1 + \beta_k\,\mathcal{I}_2,\ k = 1, 2,$ has the form

$$\begin{gathered} A_e \equiv \sum_{k=1}^{5} b_k\alpha_2^{5-k}\alpha_1^{k-1} = 0, \qquad \sum_{k=1}^{5} b_k\beta_2^{5-k}\beta_1^{k-1} = 0, \\ \Delta_{e_1e_2} \equiv \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0. \end{gathered} \tag{23}$$

As in (17), we can show that $\alpha_1 \neq 0$. In a similar way, we consider the second equation in (23) and the relation $\Delta_{e_1e_2} \neq 0$. As a result, we obtain $\beta_1 \neq 0$. Moreover, by using inequality (4), we conclude that system (23) is equivalent to the system

$$l\left(\frac{\alpha_2}{\alpha_1}\right) = 0, \qquad l\left(\frac{\beta_2}{\beta_1}\right) = 0, \qquad \Delta_{e_1e_2} \neq 0. \tag{24}$$

The solutions of system (24) have the form

$$\frac{\alpha_2}{\alpha_1} = \widetilde{s}_1, \qquad \frac{\beta_2}{\beta_1} = \widetilde{s}_2 \quad \forall \widetilde{s}_k \in \ker l, \quad k = 1, 2, \quad \widetilde{s}_1 \neq \widetilde{s}_2. \tag{25}$$

Hence, all bases of the algebra $\mathbb{B}_0$ satisfying condition (5) can be represented in the form (12).

Theorem 1 is proved.

Remark 1. A special case of Theorem 1 ($b_1 = b_5 = 1,\ b_2 = b_4 = 0,$ and $b_3 > 2$) was obtained in [6].

In view of (9), as a result of solving (12) for $\mathcal{I}_k,\ k = 1, 2,$ we get

$$\alpha\,(\widetilde{s}_2 - \widetilde{s}_1)\,\mathcal{I}_1 = \widetilde{s}_2 e_1 - e_2, \qquad \beta\,(\widetilde{s}_2 - \widetilde{s}_1)\,\mathcal{I}_2 = -\widetilde{s}_1 e_1 + e_2. \tag{26}$$

Taking into account (9) and (26), we obtain the following multiplication table for the pairs of elements $e_k,\ k = 1, 2,$ of the bases $\{e_1, e_2\}$ in (12):

$$e_1^2 = \frac{1}{\widetilde{s}_2 - \widetilde{s}_1}\left((\widetilde{s}_2\alpha - \widetilde{s}_1\beta)\,e_1 + (\beta - \alpha)e_2\right), \tag{27}$$

$$e_2^2 = \frac{1}{\widetilde{s}_2 - \widetilde{s}_1}\left(\widetilde{s}_1\widetilde{s}_2\,(\widetilde{s}_1\alpha - \widetilde{s}_2\beta)\,e_1 + \left((\widetilde{s}_2)^2\beta - (\widetilde{s}_1)^2\alpha\right)e_2\right), \tag{28}$$

$$e_1 e_2 = \frac{1}{\widetilde{s}_2 - \widetilde{s}_1}\big(\widetilde{s}_1\widetilde{s}_2(\alpha - \beta)e_1 + (\widetilde{s}_2\beta - \widetilde{s}_1\alpha)\,e_2\big). \tag{29}$$

3. Monogenic Functions Associated with Eq. (1)

By using (11) and the conditions $\widetilde{s}_k \neq 0,\ k = 1, 2,$ we can easily show that bases (12) satisfy not only condition (5) but also the condition $\mathcal{MB}$ if and only if the pairs $\widetilde{s}_k \in \ker l,\ k = 1, 2,$ specifying the corresponding basis satisfy not only the conditions of Theorem 1 but also the condition

$$\operatorname{Im} \widetilde{s}_k \neq 0, \quad k = 1, 2. \tag{30}$$

Hence, we assume that the set of roots of Eq. (2) contains at least two different roots $\widetilde{s}_k \in \ker l,\ k = 1, 2,$ satisfying condition (30). Moreover, in the corresponding bases described in Theorem 1, the pair $\widetilde{s}_k \in \ker l,$ $k = 1, 2,$ satisfies this condition.

As in the case where a biharmonic operator is considered instead of the operator L (see [8, 18]), we establish the following theorem:

Theorem 2. *A function* $\Phi\colon D_\zeta \to \mathbb{B}_0$ *is monogenic in the domain* D_ζ *if and only if its components* $U_k\colon D \to \mathbb{R},\ k = \overline{1,4},$ *in decomposition* (6) *are differentiable in the domain* D *and the following analog of the Cauchy–Riemann conditions is true:*

$$\frac{\partial \Phi(\zeta)}{\partial y}\, e_1 - \frac{\partial \Phi(\zeta)}{\partial x}\, e_2 = 0 \quad \forall \zeta = x e_1 + y e_2 \in D_\zeta. \tag{31}$$

For each quadruple $\alpha,\ \beta,\ \widetilde{s}_1,\ \widetilde{s}_2$ in (12), we introduce the notation

$$A_1 := \beta - \alpha, \qquad A_2 := \frac{\alpha}{\widetilde{s}_2} - \frac{\beta}{\widetilde{s}_1}, \qquad B_1 := \widetilde{s}_2\beta - \widetilde{s}_1\alpha, \qquad B_2 := \frac{\widetilde{s}_1}{\widetilde{s}_2}\,\alpha - \frac{\widetilde{s}_2}{\widetilde{s}_1}\beta,$$

$$C_1 := \frac{\alpha}{\widetilde{s}_1} - \frac{\beta}{\widetilde{s}_2}, \qquad C_2 := \frac{\beta-\alpha}{\widetilde{s}_1\widetilde{s}_2}, \qquad D_1 := -A_1, \qquad D_2 = D_2 := -A_2,$$

$$F\left\{\widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}\left[U_n, U_m\right](x,y)$$
$$:= \frac{\widetilde{s}_2 - \widetilde{s}_1}{\widetilde{s}_1\widetilde{s}_2}\left(\frac{\partial U_n(x,y)}{\partial y}\, e_1^2 + \left(\frac{\partial U_m(x,y)}{\partial y} - \frac{\partial U_n(x,y)}{\partial x}\right) e_1e_2 - \frac{\partial U_m(x,y)}{\partial x}\, e_2^2\right) \tag{32}$$
$$\forall (x,y) \in D, \quad \text{where} \quad n, m \in \{1,2,3,4\}.$$

Substituting (27)–(29) in (32), we obtain

$$F\left\{\widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}\left[U_n, U_m\right](x,y)$$
$$= \sum_{k=1}^{2}\left(A_k\,\frac{\partial U_n(x,y)}{\partial x} + B_k\,\frac{\partial U_m(x,y)}{\partial x} + C_k\,\frac{\partial U_n(x,y)}{\partial y} + D_k\,\frac{\partial U_m(x,y)}{\partial y}\right) e_k \tag{33}$$
$$\forall (x,y) \in D, \quad n, m \in \{1,2,3,4\}.$$

Let $f_k,\ k = 1,2$, denote one of the functions Re, $-\mathrm{Re}$, Im, and $-\mathrm{Im}$. For any $k \in \{1,2\}$, we consider real-valued functions defined at each point $(x,y) \in D$ by the formulas

$$Q_k\left\{\Phi, f_1, f_2\right\}(x,y) := \sum_{j=1}^{4}\left(a_{k,j}\{f_1,f_2\}\,\frac{\partial U_j(x,y)}{\partial x} + b_{k,j}\{f_1,f_2\}\,\frac{\partial U_j(x,y)}{\partial y}\right),$$

where

$$U_j := \mathrm{U}_j[\Phi], \quad j = \overline{1,4},$$

$$a_{k,1}\{f_1,f_2\} := f_1(A_k), \qquad a_{k,2}\{f_1,f_2\} = f_2(A_k), \qquad a_{k,3}\{f_1,f_2\} := f_1(B_k),$$

$$a_{k,4}\{f_1,f_2\} := f_2(B_k), \qquad b_{k,1}\{f_1,f_2\} := f_1(C_k), \qquad b_{k,2}\{f_1,f_2\} := f_2(C_k),$$

$$b_{k,3}\{f_1,f_2\} := f_1(D_k), \qquad \text{and} \qquad b_{k,4}\{f_1,f_2\} := f_2(D_k).$$

Remark 2. In the componentwise form, equality (31) turns into a system of four equations for the components $U_k,\ k = \overline{1,4}$, of function (6). For the bases $\{e_1, e_2\}$ given by relation (12), this system has the form

$$Q_k\{\Phi, \mathrm{Re}, -\mathrm{Im}\}(x,y) = 0, \qquad Q_k\{\Phi, \mathrm{Im}, \mathrm{Re}\}(x,y) = 0 \quad \forall (x,y) \in D, \quad k = 1,2. \tag{34}$$

Indeed, for every $\zeta \in D_\zeta$, the equality

$$G\left\{\Phi, \widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}(x,y) := \frac{\widetilde{s}_2 - \widetilde{s}_1}{\widetilde{s}_1 \widetilde{s}_2}\left(\frac{\partial \Phi(\zeta)}{\partial y} e_1 - \frac{\partial \Phi(\zeta)}{\partial x} e_2\right)$$

$$= F\left\{\widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}[U_1, U_3](x,y) + iF\left\{\widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}[U_2, U_4](x,y) \qquad (35)$$

is true. Thus, substituting (33) with $n = 1,\ m = 3$ and $n = 2,\ m = 4$ in (35), we get

$$G\left\{\Phi, \widetilde{s}_1, \widetilde{s}_2, \alpha, \beta\right\}(x,y)$$

$$= \sum_{k=1}^{2} \big(Q_k\left\{\Phi, \mathrm{Re}, -\mathrm{Im}\right\}(x,y)\, e_k + Q_k\left\{\Phi, \mathrm{Im}, \mathrm{Re}\right\}(x,y)\, ie_k\big) \quad \forall (x,y) \in D,$$

which proves the required assertion.

Remark 3. The numerical coefficients of $\dfrac{\partial U_j}{\partial x}$ and $\dfrac{\partial U_j}{\partial y}$, $j = \overline{1,4}$, in system (34) are connected by the following relations:

$$a_{1,1}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{1,3}\{\mathrm{Re}, -\mathrm{Im}\} = a_{1,2}\{\mathrm{Im}, \mathrm{Re}\} = -b_{1,4}\{\mathrm{Im}, \mathrm{Re}\},$$

$$a_{1,2}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{1,4}\{\mathrm{Re}, -\mathrm{Im}\} = -a_{1,1}\{\mathrm{Im}, \mathrm{Re}\} = b_{1,3}\{\mathrm{Im}, \mathrm{Re}\},$$

$$a_{1,3}\{\mathrm{Re}, -\mathrm{Im}\} = a_{1,4}\{\mathrm{Im}, \mathrm{Re}\}, \qquad a_{1,4}\{\mathrm{Re}, -\mathrm{Im}\} = -a_{1,3}\{\mathrm{Im}, \mathrm{Re}\},$$

$$b_{1,1}\{\mathrm{Re}, -\mathrm{Im}\} = b_{1,2}\left\{\mathrm{Im}, \mathrm{Re}\right\}, \qquad b_{1,2}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{1,1}\{\mathrm{Im}, \mathrm{Re}\},$$

$$a_{2,1}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{2,3}\{\mathrm{Re}, -\mathrm{Im}\} = a_{2,2}\left\{\mathrm{Im}, \mathrm{Re}\right\} = -b_{2,4}\{\mathrm{Im}, \mathrm{Re}\},$$

$$a_{2,2}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{2,4}\{\mathrm{Re}, -\mathrm{Im}\} = -a_{2,1}\left\{\mathrm{Im}, \mathrm{Re}\right\} = b_{2,3}\{\mathrm{Im}, \mathrm{Re}\},$$

$$a_{2,3}\{\mathrm{Re}, -\mathrm{Im}\} = a_{2,4}\left\{\mathrm{Im}, \mathrm{Re}\right\}, \qquad a_{2,4}\{\mathrm{Re}, -\mathrm{Im}\} = -a_{2,3}\{\mathrm{Im}, \mathrm{Re}\},$$

$$b_{2,1}\{\mathrm{Re}, -\mathrm{Im}\} = b_{2,2}\{\mathrm{Im}, \mathrm{Re}\}, \qquad b_{2,2}\{\mathrm{Re}, -\mathrm{Im}\} = -b_{2,1}\{\mathrm{Im}, \mathrm{Re}\}.$$

By $\mathcal{M}_4\{D_\zeta\}$ we denote a subclass of monogenic functions $\Phi\colon\ D_\zeta \to \mathbb{B}_0$ with continuous derivatives $\Phi^{(k)}$ up to the k order, inclusively, where $k \geq 4$, in D_ζ.

By using Theorem 2, we obtain a criterion of belonging of a function Φ to $\mathcal{M}_4\{D_\zeta\}$, which is an analog of the corresponding statement for holomorphic functions $F(z)$ of complex variable z via the conjugate harmonicity of the components $\mathrm{Re}\, F(z)$ and $\mathrm{Im}\, F(z)$.

Lemma 1. *A function Φ belongs to $\mathcal{M}_4\{D_\zeta\}$ if and only if each function $U_k = \mathrm{U}_k[\Phi]$, $k = \overline{1,4}$, is a solution of Eq. (1) in the domain D and the quadruple of functions (U_1, U_2, U_3, U_4) satisfies relation (31).*

Proof. *Sufficiency.* Since each function $U_k = \mathrm{U}_k[\Phi],\ k = \overline{1,4}$, is a solution of Eq. (1), we conclude that $U_k(x,y),\ k = \overline{1,4}$, has continuous derivatives up to the fourth order, inclusively, in the domain D. It follows from Theorem 2 that Φ is a monogenic function in D_ζ and that the following equality is true:

$$\frac{\partial \Phi(\zeta)}{\partial x} = \Phi'(\zeta) e_1 \quad \forall \zeta \in D_\zeta, \tag{36}$$

where

$$\mathrm{U}_k\left[\frac{\partial \Phi(\zeta)}{\partial x}\right] = \frac{\partial U_k(x,y)}{\partial x}, \quad U_k = \mathrm{U}_k[\Phi], \quad k = \overline{1,4}.$$

Acting by the operator $(e_1)^{-1}\dfrac{\partial}{\partial x}$ on both sides of equality (31) and using (36), we conclude that the function $\Phi := \Phi' = (e_1)^{-1}\dfrac{\partial \Phi}{\partial x}$ satisfies condition (31) and is monogenic in the domain D_ζ. Applying this operation consecutively to Φ' and Φ'', we conclude that the function Φ possesses derivatives $\Phi^{(k)},\ 1 \le k \le 4$, up to the fourth order, inclusively, and moreover, the following equalities are true:

$$\Phi^{(k)}(\zeta) = \left((e_1)^{-1}\right)^k \frac{\partial^k \Phi(\zeta)}{\partial x^k} \quad \forall \zeta \in D_\zeta, \tag{37}$$

where

$$\mathrm{U}_j\left[\frac{\partial^k \Phi(\zeta)}{\partial x^k}\right] = \frac{\partial^k U_j(x,y)}{\partial x^k}, \quad k = \overline{1,4}, \quad j = \overline{1,4}.$$

According to (37), the function Φ has continuous derivatives in the domain D_ζ up to the fourth order, inclusively.

The proof of *necessity* is trivial. To this end, we use Theorem 2 and the fact that each function $U_k,\ k \in \{1,2,3,4\}$, satisfies Eq. (1) in view of the equalities

$$L\Phi(\zeta) = \mathcal{L}(e_1,e_2)\Phi^{(4)}(\zeta) \equiv 0, \qquad \mathrm{U}_k\left[L\Phi(\zeta)\right] = L\left(\mathrm{U}_k(x,y)\right) \quad \forall \zeta \in D_\zeta, \quad k = \overline{1,4},$$

which are proved by analogy with equalities (37).

Lemma 1 is proved.

We now introduce complex variables and the domains of their definition:

$$z_k := x + \widetilde{s}_k y, \qquad D_{z_k} := \{z_k \in \mathbb{C} \colon\, xe_1 + ye_2 \in D_\zeta\}, \quad k = 1,2. \tag{38}$$

The monogenic function $\Phi\colon D_\zeta \to \mathbb{B}_0$ can be expressed in terms of two holomorphic functions of the complex variables z_1 and z_2, respectively.

Theorem 3. *A function $\Phi\colon D_\zeta \to \mathbb{B}_0$ is monogenic in the domain D_ζ if and only if the following equality is true:*

$$\Phi(\zeta) = F_1(z_1)\mathfrak{I}_1 + F_2(z_2)\mathfrak{I}_2 \quad \forall \zeta \in D_\zeta, \tag{39}$$

where F_k is a holomorphic function of the complex variable z_k in the domain D_{z_k} for $k = 1,2$.

Proof. *Necessity.* Let $\Phi\colon D_\zeta \to \mathbb{B}_0$ be monogenic. It is necessary to prove that there exist holomorphic functions $F_k\colon D_{z_k} \to \mathbb{C},\ k=1,2,$ such that equality (39) is true. Substituting equalities (12) in relation (6), we get

$$\Phi(\zeta) = \alpha f_1(z_1)\mathcal{I}_1 + \beta f_2(z_2)\mathcal{I}_2 \quad \forall \zeta \in D_\zeta, \tag{40}$$

where

$$\begin{gathered} f_k(z_k) := U_1(x,y) + iU_2(x,y) + \widetilde{s}_k(U_3(x,y) + iU_4(x,y)) \\ \forall z_k = x + \widetilde{s}_k y \in D_{z_k}, \quad k=1,2. \end{gathered} \tag{41}$$

We prove that functions (41) are holomorphic functions of their complex variables in the domains $D_{z_k},\ k = 1,2.$ Writing an analog of the Cauchy–Riemann conditions (31) for function (40), we arrive at the equality

$$\alpha^2\,\mathrm{C}_{\widetilde{s}_1} f_1(z_1)\,\mathcal{I}_1 + \beta^2\,\mathrm{C}_{\widetilde{s}_2} f_2(z_2)\,\mathcal{I}_2 = 0 \quad \forall (x,y) \in D, \tag{42}$$

where

$$\mathrm{C}_{\widetilde{s}_k} := \frac{\partial}{\partial y} - \widetilde{s}_k\,\frac{\partial}{\partial x}, \quad k=1,2. \tag{43}$$

We now rewrite equality (42) in the componentwise form

$$\mathrm{C}_{\widetilde{s}_k} f_k(z_k) = 0 \quad \forall z_k \in D_{z_k}, \quad k=1,2. \tag{44}$$

Selecting the real and imaginary parts of the variables $z_k,\ k=1,2,$ in (38), we can write the equalities

$$z_k = \xi_k + i\eta_k, \qquad \xi_k := x + \operatorname{Re}\widetilde{s}_k\,y, \qquad \eta_k := \operatorname{Im}\widetilde{s}_k\,y, \quad k=1,2. \tag{45}$$

We determine the partial derivatives of the first order for functions (41) in the domain D as follows:

$$\frac{\partial f_k}{\partial y} = \operatorname{Re}\widetilde{s}_k\,\frac{\partial f_k}{\partial \xi_k} + \operatorname{Im}\widetilde{s}_k\,\frac{\partial f_k}{\partial \eta_k}, \qquad \frac{\partial f_k}{\partial x} = \frac{\partial f_k}{\partial \xi_k}, \quad k=1,2. \tag{46}$$

Substituting equalities (46) in (44), we obtain

$$0 \equiv \mathrm{C}_{\widetilde{s}_k} f_k(z_k) = \operatorname{Im}\widetilde{s}_k \left(\frac{\partial}{\partial \eta_k} - i\,\frac{\partial}{\partial \xi_k}\right) f_k(z_k) \quad \forall z_k \in D_{z_k}, \quad k=1,2. \tag{47}$$

Since $\operatorname{Im}\widetilde{s}_k \neq 0,\ k=1,2,$ we conclude that (47) gives the Cauchy–Riemann conditions for the complex-valued functions $f_k(z_k),\ k=1,2$:

$$\left(\frac{\partial}{\partial \eta_k} - i\,\frac{\partial}{\partial \xi_k}\right) f_k(z_k) \quad \forall z_k \in D_{z_k}, \quad k=1,2. \tag{48}$$

By using equalities (41), we get

$$\operatorname{Re} f_k(z_k) = U_1(x,y) + \operatorname{Re}\widetilde{s}_k\, U_3(x,y) - \operatorname{Im}\widetilde{s}_k\, U_4(x,y), \tag{49}$$

$$\operatorname{Im} f_k(z_k) = U_2(x,y) + \operatorname{Im}\widetilde{s}_k\, U_3(x,y) + \operatorname{Re}\widetilde{s}_k\, U_4(x,y). \tag{50}$$

By Theorem 2, the components $U_k,\ k = \overline{1,4}$, are differentiable in the domain D. Therefore, by using relations (49) and (50), we conclude that the functions $\operatorname{Re} f_k(z_k)$ and $\operatorname{Im} f_k(z_k),\ k = 1,2$, are differentiable in the domain

$$D_{\xi_k,\eta_k} := \big\{(\xi_k,\eta_k)\colon\ z_k = \xi_k + i\eta_k \in D_{z_k}\big\}, \qquad k = 1,2.$$

Redenoting $\alpha f_1(z_1)$ by $F_1(z_1)$ and $\beta f_2(z_1)$ by $F_2(z_2)$, we rewrite relation (40) in the form (39).

Necessity is proved.

Sufficiency. It is necessary to show that the function given by equality (39) (F_k is holomorphic in D_{z_k}, $k = 1,2$) is monogenic in D_ζ.

By using notation (43), equalities (9) and (12), and the holomorphy of the complex-valued functions

$$F_k(z_k)\colon\ D_{z_k} \to \mathbb{C}, \quad k = 1,2,$$

we arrive at the chain of equalities

$$\frac{\partial\Phi(\zeta)}{\partial y}\, e_1 - \frac{\partial\Phi(\zeta)}{\partial x}\, e_2 = (\alpha\mathcal{I}_1\mathrm{C}_{\widetilde{s}_1} - \beta\mathcal{I}_2\mathrm{C}_{\widetilde{s}_2})\,(F_1(z_1)\mathcal{I}_1 + F_2(z_2)\mathcal{I}_2)$$

$$= \alpha\mathcal{I}_1\mathrm{C}_{\widetilde{s}_1}\,(F_1(z_1))\,\mathcal{I}_1 - \beta\mathcal{I}_2\mathrm{C}_{\widetilde{s}_2}\,(F_2(z_2))\,\mathcal{I}_2 \equiv 0 \quad \forall\zeta \in D_\zeta.$$

Thus, function (39) satisfies an analog of the Cauchy–Riemann conditions (31).

Substituting (26) in (39), we obtain the equality

$$\Phi(\zeta) = \frac{1}{\widetilde{s}_2 - \widetilde{s}_1}\left(\left(\frac{\widetilde{s}_2}{\alpha}F_1(z_1) - \frac{\widetilde{s}_1}{\beta}F_2(z_2)\right)e_1 + \left(\frac{1}{\beta}\,F_2(z_2) - \frac{1}{\alpha}\,F_1(z_1)\right)e_2\right) \quad \forall\zeta \in D_\zeta. \tag{51}$$

It is clear that the functions $F_k(z_k),\ k = 1,2$, have continuous partial derivatives of the first order with respect to the variables x and y, respectively, in the domain D. Hence, the components $U_k = \mathrm{U}_k[\Phi]$ of function (51) have the same property. Sufficiency is proved.

Theorem 3 is proved.

Remark 4. The cases of Theorem 3 for the monogenic functions $\Phi(xe_1 + ye_2),\ e_1 = e$, also follow from [23, 24], which can be proved as for the special case of the operator L in [6], Sec. 3.

Remark 5. The cases of Theorem 3 for the monogenic functions associated with the corresponding equations of the form (1) were obtained in [6, 19, 20].

We say that D_ζ is *bounded* and has a *Jordan rectifiable boundary* ∂D_ζ if the domain of the complex plane $D_z = \{x + iy\colon\ (x,y) \in D\}$ is bounded and its boundary ∂D_z is the union of finitely many closed Jordan rectifiable curves; the direction of traversing these curves is chosen so that the domain D_z remains to the left.

Corollary 1. *Suppose that a bounded domain* D_ζ *has a Jordan rectifiable boundary* ∂D_ζ *and that a function* $\Phi\colon D_\zeta \to \mathbb{B}_0$ *is monogenic in the domain* D_ζ, *continuous in its closure* $D_\zeta \cup \partial D_\zeta$, *and given by relation (39). Then the following equalities are true:*

$$\Phi(\zeta) = \sum_{k=1}^{2} \mathcal{I}_k \frac{1}{2\pi i} \int\limits_{\partial D_{z_k}} \frac{F_k(t_k)}{t_k - z_k}\, dt_k \quad \forall \zeta \in D_\zeta,$$

$$\int\limits_{\partial D_\zeta} \Phi(\zeta)\, d\zeta = 0, \qquad \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial D_\zeta} \Phi(\vartheta)\, (\vartheta - \zeta)^{-1}\, d\vartheta \quad \forall \zeta \in D_\zeta,$$

where $\zeta = xe_1 + ye_2 \in D_\zeta,\ z_k \in D_{z_k},\ k = 1, 2.$

Corollary 2. *Every monogenic function* $\Phi\colon D_\zeta \to \mathbb{B}_0$ *has continuous derivatives* $\Phi^{(n)}$ *of any order* n, $n = 1, 2, \ldots,$ *in the domain* D_ζ. *The components* $U_k = \mathrm{U}_k[\Phi],\ k = \overline{1,4}$, *are infinitely continuously differentiable functions in the domain* D.

It follows from Corollary 2 and Lemma 1 that Φ belongs to $\mathcal{M}_4\{D_\zeta\}$ and each component $U_k = \mathrm{U}_k[\Phi]$, $k = \overline{1,4}$, is a particular solution of Eq. (1).

In the proof of Theorem 3, it is established that the monogenic function can be represented in the form (51). Without loss of generality, we can replace in this function

$$\frac{\widetilde{s}_2}{(\widetilde{s}_2 - \widetilde{s}_1)\,\alpha} F_1(z_1) \quad \text{by} \quad F_1(z_1) \qquad \text{and} \qquad -\frac{\widetilde{s}_1}{(\widetilde{s}_2 - \widetilde{s}_1)\,\beta} F_2(z_2) \quad \text{by} \quad F_2(z_2).$$

As a result, we obtain a representation of the monogenic function Φ in the basis $\{e_1, e_2\}$ from (12):

$$\Phi(\zeta) = (F_1(z_1) + F_2(z_2))\, e_1 - \left(\frac{F_1(z_1)}{\widetilde{s}_2} + \frac{F_2(z_2)}{\widetilde{s}_1}\right) e_2 \quad \forall \zeta \in D_\zeta. \tag{52}$$

Corollary 3. *Let* Φ *be an arbitrary monogenic function* $\Phi\colon D_\zeta \to \mathbb{B}_0$. *Particular solutions of Eq. (1) are functions of the form*

$$u(x, y) = \sum_{k=1}^{4} a_k \mathrm{U}_k[\Phi(\zeta)] \quad \forall (x, y) \in D,$$

where a_k *are arbitrary real constants and* $\mathrm{U}_k[\Phi(\zeta)]$ *is determined from (52),* $k = \overline{1,4}$.

4. The Case of at Least Two NonSelf-Adjoint Complex Characteristics

Assume that the set of roots of Eq. (2) contains at least two different roots $\widetilde{s}_k \in \ker l,\ k = 1, 2,$ satisfying the special case (30), namely,

$$\overline{\widetilde{s}_2} \neq \widetilde{s}_1, \quad \operatorname{Im} \widetilde{s}_k \neq 0, \quad k = 1, 2, \tag{53}$$

where $\overline{x + iy} := x - iy,\ x, y \in \mathbb{R}.$

It is clear that the validity of conditions (53) for $b_k \in \mathbb{R},\ k = \overline{1,5},$ in (1) is equivalent to the assertion that the set $\ker l$ consists of four pairwise different complex numbers

$$\ker l \equiv \{s_1, s_2, \overline{s_1}, \overline{s_2}\}, \tag{54}$$

where $s_k,\ k = 1,2,$ satisfy conditions (53) with $\widetilde{s}_k := s_k,\ k = 1,2.$

In this section, $\widetilde{s}_k,\ k = 1,2,$ is regarded as an arbitrary pair of different elements with $\widetilde{s}_k \in \ker l,\ k = 1,2,$ satisfying the conditions of Theorem 1 and relation (53).

If $b_k \in \mathbb{R},\ k = \overline{1,5},$ then it follows from equality (54) that there exist four pairs of given elements $\widetilde{s}_1,\ \widetilde{s}_2:$ $(s_1, s_2),\ (\overline{s_1}, \overline{s_2}),$ and the pairs obtained by permutations.

Let $b_k \in \mathbb{R},\ k = \overline{1,5},$ in (1). Then the formula for the general solution of Eq. (1) in case (53) for a bounded simply connected domain D is obtained in exactly the same way (the reasoning is analogous to that used in [2, pp. 109, 110]) as the corresponding formula for the general solution of Eq. (1) (see, e.g., [2, p. 136]), i.e., the equation for the stress function in a plane anisotropic medium:

$$u(x,y) = \mathrm{Re}\ (F_1(z_1) + F_2(z_2)) \quad \forall (x,y) \in D. \tag{55}$$

Here, $F_k\colon\ D_{z_k} \to \mathbb{C},\ k = 1,2,$ are arbitrary holomorphic functions of the corresponding variables. Then equality (55) can be rewritten in the form

$$u(x,y) = U_1[\Phi_u(\zeta)] \quad \forall \zeta \in D_\zeta,$$

where $\Phi_u := \Phi$ is given by relation (52) with the same $F_k(z_k),\ k = 1,2,$ as in (55).

The monogenic function (52) satisfies the conditions of Lemma 1.

By Φ_0 we denote a monogenic function $\Phi_0 := \Phi$ in (6) with $U_1 \equiv 0.$ Thus, the quadruple of its components

$$(U_1, U_2, U_3, U_4), \quad U_k = \mathrm{U}_k[\Phi_0], \quad k = \overline{1,4},$$

satisfies system (34) with $U_1 \equiv 0.$

Theorem 4. *Suppose that u is a solution of Eq. (1), where $b_k \in \mathbb{R},\ k = \overline{1,5},$ in the bounded simply connected domain D and the basis $\{e_1, e_2\}$ of the algebra $\mathbb{B}_0$ is given by relation (12). All monogenic functions $\Phi\colon\ D_\zeta \to \mathbb{B}_0$ such that $\mathrm{U}_1[\Phi] \equiv u$ have the form*

$$\Phi(\zeta) = \Phi_u(\zeta) + \Phi_0(\zeta) \quad \forall \zeta \in D_\zeta.$$

Consider the case where Eq. (1) is an equation for the stress function (see, e.g., [2, p. 140], where $a_{26} = a_{16} = 0,\ \overline{U} \equiv 0,$ and $F := u,$ for an orthotropic medium in the absence of bulk forces) in a special case of plane anisotropic medium (plane orthotropic medium) (see, e.g., [2, pp. 33, 34]), namely,

$$L(x,y) \equiv a_{11}\,\frac{\partial^4 u(x,y)}{\partial y^4} + (2a_{12} + a_{66})\,\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + a_{22}\,\frac{\partial^4 u(x,y)}{\partial x^4} = 0. \tag{56}$$

Here, the coefficients satisfy the relations (see, e.g., [3, p. 56])

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{66} > 0, \qquad -\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{11}a_{22}}. \tag{57}$$

In addition, the numbers playing the role of coefficients in Eq. (56) are the coefficients of equations of generalized Hooke's law for the orthotropic medium (see, e.g., [2, p. 34]). These equations have the form

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y, \qquad \varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y, \qquad \frac{\gamma_{xy}}{2} = a_{66}\tau_{xy},$$

where $\varepsilon_x,\ \dfrac{\gamma_{xy}}{2}$, and ε_y and $\sigma_x,\ \tau_{xy}$, and σ_y are the components of the strain [2, p. 16] and stress [2, p. 15] tensors, respectively.

It is known (see, e.g., [2, p. 113], where $a_{16} = a_{26} = 0$) that the characteristic equation (2) for Eq. (56) cannot have real roots. In view of the fact that the coefficients of Eq. (56) are real, we conclude that the condition of simplicity of roots of the characteristic equation (2) for Eq. (56) is equivalent to the existence of at least one pair of different $s_k \in \ker l,\ k = 1, 2,$ such that $\overline{s_2} \neq s_1$. Hence, (54) holds.

We now determine the class of orthotropies corresponding to the case in which the equation for the stress function [which, in turn, corresponds to condition (54)] has the form (56). It follows from [22, p. 50] (or [3, p. 53]) that this class of orthotropic plane media coincides with the class of general orthotropic plane media for which Eq. (56) cannot be reduced to a biharmonic equation by a nondegenerate change of variables [invertible transformation of the form $X = X(x, y),\ Y = Y(x, y),\ (x, y) \in D \times D$].

We perform the change of variables

$$\sqrt[4]{a_{22}}\,x \quad \text{by} \quad X =: x \qquad \text{and} \qquad \sqrt[4]{a_{11}}\,y \quad \text{by} \quad Y =: y.$$

Then Eq. (56) takes the form

$$L_p(x, y) := L(x, y) \equiv \frac{\partial^4 u(x, y)}{\partial y^4} + 2p\,\frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x, y)}{\partial x^4} = 0, \tag{58}$$

where

$$p := \frac{2a_{12} + a_{66}}{2\sqrt{a_{11}a_{22}}}. \tag{59}$$

We now determine the range of values of the right-hand side of (59). By using (57), we obtain

$$-1 + \varepsilon < p < 1 + \varepsilon, \qquad \varepsilon := \frac{a_{66}}{2\sqrt{a_{11}a_{22}}} > 0.$$

It is clear that the change of variables used to reduce Eq. (56) to Eq. (58) is nondegenerate. Equation (58) with $p = 1$ is a biharmonic equation. It is known (see, e.g., [22, p. 50]) that the necessary and sufficient condition for the existence of a nondegenerate change of independent variables that reduces an equation of the form (56) [moreover, an equation of the form (2)] to a biharmonic equation is the fact that the characteristic equation for Eq. (56) has two pairwise equal complex roots, i.e., roots of the form $S_k,\ \operatorname{Im} S_k \neq 0,\ k = 1, 2,$ of multiplicity two, which is impossible because the roots of the characteristic equation are simple. Hence, $p \neq 1$ and the following two cases are possible for p:

(i) $p \in (-1 + \varepsilon; 1) \subset (-1; 1)$,

(ii) $p \in (1; 1 + \varepsilon) \subset (1; \infty)$.

Cases 1 and 2 were considered in [19, 20] and [6, 21], respectively. In particular, for the operator $L := L_p$, Theorem 4 takes a closed form in a sense that the function Φ_0 can be found in the explicit form.

In the case where Eq. (1) is the equation for stress function in a plane anisotropic medium without volume forces, this equation is elliptic and takes the following form (see, e.g., [2, p. 140]):

$$L(x,y) \equiv a_{11}\,\frac{\partial^4 u(x,y)}{\partial y^4} - 2a_{16}\,\frac{\partial^4 u(x,y)}{\partial x \partial y^3} + (2a_{12}+a_{66})\,\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} - 2a_{26}\,\frac{\partial^4 u(x,y)}{\partial x^3 \partial y} + a_{22}\,\frac{\partial^4 u(x,y)}{\partial x^4} = 0, \tag{60}$$

where the real coefficients satisfy conditions similar to (57) (see, e.g., [3, 4]).

As follows from the results established in [22, p. 50] and the condition of simplicity of the roots of the characteristic equation for Eq. (60), it is impossible to find a nondegenerate change of independent variables that reduces Eq. (60) to the biharmonic equation

$$(\Delta_2)^2\, u(x,y) = 0, \qquad \Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

At the same time, Mikhlin gave an example of a nondegenerate change of independent variables of this kind (a composition of linear changes of variables and rotations of the coordinates axes) that reduces Eq. (60) to the form

$$\left(\frac{\partial^2}{\partial x^2} + k^2\,\frac{\partial^2}{\partial y^2}\right)\Delta_2 u(x,y) = 0, \tag{61}$$

where $k > 0,\ k \neq 1$ (see, e.g., [25] or [26], Chap. 5, Sec. 7).

Note that Eq. (61) is an equation of the form (56), which can be reduced to Eq. (58).

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