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HOLTSMARK FLUCTUATIONS OF NONSTATIONARY GRAVITATIONAL FIELDS

V. A. Litovchenko UDC 517.937, 519.21

We construct Holtsmark distributions of nonstationary fluctuations of the local interactions of moving objects for a system in which the action of gravitation influence is governed by a power law. We deduce a pseudodifferential equation with Riesz operator of fractional differentiation corresponding to this process. We also clarify the general nature of stable symmetric Lévy random processes.

1. Introduction

One of the main problems of celestial mechanics is connected with the analysis of the nature of force interactions between the objects of one or another star system. The force $\mathcal{F}$ acting upon a specific star of the system has two components: the first component K corresponds to the influence of the entire system (regarded as a whole) and the second component F reflects the local influence of the immediate surrounding: $\mathcal{F} = K + F$.

The influence of the entire system can be described by the gravitational potential $\mathcal{R}(r;t)$ [1] traditionally obtained as a result of integration of the weighted density $n(r;m;t)$ characterizing the mean space distribution of stars of different masses m at time t. The force K (per unit mass) with which the system (as a whole) acts upon a given star $Z(0)$ is given by the formula

$$K(r;t) = -\mathrm{grad}\,\mathcal{R}(r;t).$$

The force $K(r;t)$ is a slowly varying function of the space and time variables because the corresponding potential $\mathcal{R}(r;t)$ characterizes a "smoothed" distribution of substance in the star system. The other force $F(t)$ (per unit mass) is characterized by relatively fast and sharp variations caused by the instantaneous changes in the local distribution of stars surrounding Z at time t. Since the quantity $F(t)$ is fluctuating, we can speak only about its probable values.

The statistical properties of $F(t)$ were investigated by Holtsmark [1, 2]. His investigations were based on the classical Newton gravitation law of "inverse squares," namely,

$$F(t) = \sum_{j=1}^{N(t)} F_j = G \sum_{j=1}^{N(t)} \frac{m_j}{|r_j|^2} r_j^{\circ},$$

where

G is the gravitational constant,

m_j is the mass of a typical star "field,"

r_j is the radius-vector of its position relative to the analyzed star Z placed at the origin of coordinates,

$r_j^{\circ} := r_j/|r_j|$ is the unit vector of r_j,

Yu. Fedkovych Chernivtsi National University, Chernivtsi, Ukraine; e-mail: v.litovchenko@chnu.edu.ua.

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and

$N(t)$ is the number of stars that form local surrounding of Z at time t.

For a constant mean density $n(r; m; t) \equiv n$ of the space distribution of stars, by using the equality

$$N = \frac{4}{3}\pi R^3 n \quad (\forall R > 0)$$

and the classical methods of the probability theory and integral calculus, Holtsmark determined the stationary distribution $W(F)$ of the quantity F in the form

$$W(F) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} e^{-i(\xi,F) - a|\xi|^{3/2}} d\xi \equiv \mathbb{F}^{-1}\big[e^{-a|\xi|^{3/2}}\big](F).$$

Here,

$(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^3$,

$|x| := (x,x)^{1/2},\ x \in \mathbb{R}^3$;

$a := \frac{4}{15}(2\pi G)^{3/2} n \langle m^{3/2} \rangle$ is the fluctuation coefficient, where $\langle m^{3/2} \rangle$ is the mean value of $m^{3/2}$ for the analyzed distribution of stars in the star system,

and

$\mathbb{F}$ is the operator of Fourier transformation.

The indicated Holtsmark distribution belongs to the class of Lévy distributions

$$\mathcal{L}_\alpha(x) = \mathbb{F}^{-1}\big[e^{-b|\xi|^\alpha}\big](x), \quad x \in \mathbb{R}^3,$$

of symmetric stable random processes [3]. The fact that $\mathcal{L}_\alpha(\cdot)$ is a probability distribution function only for $\alpha \in (0; 2]$ was established by Lévy in [4].

The classical Gauss $(\alpha = 2)$ and Cauchy $(\alpha = 1)$ distributions are also important representatives of this class. Numerous examples of applications of the Holtsmark, Cauchy, Gauss, and Pareto distributions in astronomy, nuclear physics, economics, sociology, industrial, and military fields can be found in the contemporary literature [5–10]. Each of these applications is characterized by the stochastic features of the Lévy distributions for one or another value of α.

Parallel with individual characteristics, the analyzed symmetric and stable Lévy random processes have the common nature. In the present paper, it is shown that each Lévy process of this kind with $\alpha \in (0; 2)$ can be interpreted as the process of local influence of moving objects in a system of masses whose interaction is described by a certain power law $(\cdot)^{-\beta}$. In particular, the classical Holtsmark process $(\alpha = 3/2)$ is associated with the interaction for which $\beta = 2$ (Newton gravitation). Moreover, for the Cauchy process, the interaction is characterized by the exponent $\beta = 3$. We also consider the Holtsmark problem in the general statement and deduce a pseudodifferential equation with the Riesz operator of fractional differentiation and the Green function of the Cauchy problem for which the corresponding nonstationary Holtsmark distribution exists. This equation enables one to study the Holtsmark processes in domains with edges by the methods of the theory of boundary-value problems for pseudodifferential equations.

2. Fractal Holtsmark Distributions

Consider a star system in which the interaction of masses is described by the Riesz potential [11], i.e., the gravitational force between any two stars with masses M and m is described by the formula

$$F = G\frac{Mm}{|r|^{\beta}}r^{\circ}, \quad \beta > 0, \tag{1}$$

where G is the corresponding gravitational constant and r is the vector of distance between these stars. We develop the Holtsmark idea and find the nonstationary distribution $W_{\beta}(F(t))$ of the force $F(t)$ acting upon the unit mass of a star Z at time t and caused by the gravity of stars from its close surrounding.

We also assume that the distribution of stars fluctuates in the vicinity of Z and that the distribution of stars of different masses m in the star system obeys a certain known empirical law. Moreover, at any time t, the fluctuations of density of the stars satisfy the condition of constancy of their mean value per unit volume

$$n(r;m;t) \equiv n(t).$$

Assume that the analyzed star Z is located at the origin of coordinates of the system and that its spherical neighborhood of radius R contains $N(t)$ stars at time t. Thus, as indicated above,

$$F(t) = G\sum_{j=1}^{N(t)} \frac{m_j}{|r_j|^{\beta+1}} r_j \equiv \sum_{j=1}^{N(t)} F_j$$

and

$$N(t) = \frac{4}{3}\pi R^3 n(t). \tag{2}$$

First, for fixed t, we consider the distribution $W_{\beta,N(t)}\big(F(t)\big)$ at the center of a spherical neighborhood of the radius R that contains $N(t)$ stars of the system and find the probability $W_{\beta,N(t)}\big(F_{\circ}(t)\big)dF_{\circ}(t)$ of the event that the quantity $F(t)$ belongs to the cube

$$\big[F_{\circ}(t); F_{\circ}(t) + dF_{\circ}(t)\big] \subset \mathbb{R}^3.$$

By using the well-known method of characteristic functions, we obtain

$$W_{\beta,N(t)}\big(F_{\circ}(t)\big) = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} e^{-i(\xi,F_{\circ}(t))} A_{N(t)}(\xi)\,d\xi,$$

where

$$A_{N(t)}(\xi) := \prod_{j=1}^{N(t)} \int\limits_{0}^{+\infty} \left(\int\limits_{\mathbb{K}_R(0)} e^{i(\xi,F_j)} \tau_j(r_j;m_j;t)\,dr_j \right) dm_j.$$

Here, $\mathbb{K}_R(0)$ is a ball of radius R centered at the origin and $\tau_j(r_j;m_j;t)$ is the distribution of probability of the event that, at time t, the jth star has the mass m_j and occupies the position r_j. If we have only fluctuations satisfying the condition of space constancy of the mean density, then

$$\tau_j(r_j; m_j; t) = \frac{3\tau(m;t)}{4\pi R^3},$$

where $\tau(m;t)$ is the frequency of appearance of stars with different masses at time t.

This yields the representation

$$A_{N(t)}(\xi) = \left(\frac{3}{4\pi R^3} \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{K}_R(0)} e^{i(\xi,\eta)} \tau(m;t) dr \right) dm \right)^{N(t)},$$

where

$$\eta := Gmr/|r|^{\beta+1}. \tag{3}$$

Let $R \to +\infty$ and $N(t) \to +\infty$. Then, according to (2), we get

$$W_\beta(F(t)) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} e^{-i(\xi,F(t))} A(\xi;t)\, d\xi, \tag{4}$$

where

$$A(\xi;t) := \lim_{R\to\infty} \left[\frac{3}{4\pi R^3} \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{K}_R(0)} e^{i(\xi,\eta)} \tau(m;t) dr \right) dm \right]^{4\pi R^3 n(t)/3}.$$

Since, for any t,

$$\frac{3}{4\pi R^3} \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{K}_R(0)} \tau(m;t) dr \right) dm = 1,$$

we conclude that

$$A(\xi;t) = \lim_{R\to\infty} \left[1 - \frac{3}{4\pi R^3} \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{K}_R(0)} (1 - e^{i(\xi,\eta)}) \tau(m;t) dr \right) dm \right]^{4\pi R^3 n(t)/3}. \tag{5}$$

Further, in view of the absolute convergence of the integral with respect to r in this relation over the entire space $\mathbb{R}^3$ for $\beta > \frac{3}{2}$, we can rewrite equality (5) in the form

$$A(\xi;t) = \lim_{R\to\infty} \left[1 - \frac{3}{4\pi R^3} \int\limits_0^{+\infty} \left(\int\limits_{\mathbb{R}^3} (1 - e^{i(\xi,\eta)}) \tau(m;t) dr \right) dm \right]^{4\pi R^3 n(t)/3}.$$

This yields the relation

$$A(\xi;t)=e^{-n(t)B_\beta(\xi;t)}, \tag{6}$$

where

$$B_\beta(\xi;t):=\int_0^{+\infty}\left(\int_{\mathbb{R}^3}(1-e^{i(\xi,\eta)})\tau(m;t)dr\right)dm.$$

In the inner integral of this equality, we perform the change of the variable of integration r by the variable η according to relation (3) and, hence, pass to the spherical coordinate system whose z-axis is directed along the vector ξ. As a result, we obtain

$$B_\beta(\xi;t)=\frac{4\pi(G|\xi|)^{3/\beta}\langle m^{3/\beta}\rangle}{\beta}\int_0^{+\infty}(\rho-\sin\rho)\rho^{-2-3/\beta}d\rho.$$

Note that the integral in the last equality converges only for $\beta>\dfrac{3}{2}$. Integrating this integral by parts, we arrive at the relation

$$B_\beta(\xi;t)=\frac{4\beta\pi\mathrm{I}(\beta)}{3(\beta+3)}(G|\xi|)^{3/\beta}\langle m^{3/\beta}\rangle, \tag{7}$$

$$t\geq 0,\quad \xi\in\mathbb{R}^3,\quad \beta>3/2,$$

where

$$\mathrm{I}(\beta):=\begin{cases}\dfrac{\beta}{3-\beta}\Gamma(2-3/\beta)\cos\dfrac{(2-3/\beta)\pi}{2}, & \dfrac{3}{2}<\beta<3,\\[2ex] \dfrac{\pi}{2}, & \beta=3,\\[2ex] \Gamma(1-3/\beta)\sin\dfrac{(1-3/\beta)\pi}{2}, & \beta>3\end{cases}$$

(here, $\Gamma(\cdot)$ is the Euler gamma-function).

Combining equalities (4), (6), and (7), we finally get

$$W_\beta(F(t))=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{-i(\xi,F(t))}e^{-a_\beta(t)|\xi|^{3/\beta}}d\xi,$$

where

$$a_\beta(t):=\frac{4\beta\pi\mathrm{I}(\beta)}{3(\beta+3)}G^{3/\beta}n(t)\langle m^{3/\beta}\rangle.$$

Thus, the following assertion is true:

Theorem 1. *Under the assumptions made above, for any* $\beta > 3/2$, *the function*

$$W_\beta(F(t)) = \mathbb{F}^{-1}\Big[e^{-a_\beta(t)|\xi|^{3/\beta}}\Big](F;t) \tag{8}$$

is the probability distribution of the force $F(t)$ *of local influence of moving objects in the system interacting according to the power law (1).*

Denote

$$\mathcal{H}_\gamma(F;t) := W_\beta(F(t)), \quad \text{where} \quad \gamma := 2/\beta.$$

The function $\mathcal{H}_\gamma(\cdot;\cdot)$ is called the Holtsmark distribution of order γ of the fluctuations of nonstationary gravitational fields. The classical situation considered by Holtsmark corresponds to $\beta = 2$; in this case, the order of distribution $\gamma = 1$. In view of the inequality $\beta > 3/2$, this is the unique case in which the value of γ is integer. The other possible values of γ have nonzero fractional parts, namely, $\gamma \in (0;4/3)$.

3. Relationship with the Pseudodifferential Equation

It is preferable to reduce the problem of investigation of the fluctuations of local interaction of moving objects, especially in bounded domains with certain boundary conditions, to the solution of the corresponding boundary-value problem for differential or pseudodifferential equations. This would enable us to apply the developed computational tools of the theory of boundary-value problems and use some well-known results from this theory. To this end, it is necessary to deduce the corresponding differential equation, which adequately describes the analyzed process. We try to derive this equation under certain conditions by using the distribution function $\mathcal{H}_\gamma$. First, we discuss the properties of this function.

Assume that the coefficient $a_\beta(\cdot)$ is a positive function continuously differentiable on the segment $(0;T]$. In view of [12, 13], we immediately conclude that, for all $\gamma \in (0;4/3)$, the function $\mathcal{H}_\gamma(x;t)$ is differentiable with respect to t on the set $\mathbb{R}^3 \times (0;T]$ and infinitely differentiable with respect to the variable x. Moreover, the derivatives of this function can be estimated as follows:

$$\begin{gathered} |\partial_x^k \mathcal{H}_\gamma(x;t)| \le c_1 t\big(t^{\frac{2}{3\gamma}} + |x|\big)^{-(3+|k|+\frac{3\gamma}{2})}, \\ |\partial_t \partial_x^k \mathcal{H}_\gamma(x;t)| \le c_2 t^{\frac{2}{\gamma}-1}\big(t^{\frac{2}{3\gamma}} + |x|\big)^{-(3+|k|+\frac{3\gamma}{2})} \end{gathered} \tag{9}$$

with some positive constants c_1 and c_2.

Estimate (9) guarantees that $\mathcal{H}_\gamma(\cdot;t)$ belongs to $L_1(\mathbb{R}^3)$ for any fixed $t \in (0;T]$, which, in turn, guarantees the existence of the Fourier transform of the function $\mathcal{H}_\gamma(\cdot;t)$ and the validity of the equality

$$\mathbb{F}[\mathcal{H}_\gamma(x;t)](\xi;t) = e^{-a_\beta(t)|\xi|^{\frac{3\gamma}{2}}}, \quad t \in (0;T], \quad \xi \in \mathbb{R}^3. \tag{10}$$

By the classical methods, we conclude that the equality

$$\partial_t \mathcal{H}_\gamma(x;t) = -\frac{a'_\beta(t)}{(2\pi)^3} \int\limits_{\mathbb{R}^3} |\xi|^{\frac{3\gamma}{2}} e^{-i(x,\xi) - a_\beta(t)|\xi|^{\frac{3\gamma}{2}}}\, d\xi, \quad t \in (0;T], \quad \xi \in \mathbb{R}^3,$$

is true. By using this inequality and (10), we find

$$\partial_t \mathcal{H}_\gamma(x;t) = -a'_\beta(t)\mathbb{F}^{-1}\big[|\xi|^{\frac{3\gamma}{2}}\mathbb{F}[\mathcal{H}_\gamma](\xi;t)\big](x;t), \quad t \in (0;T], \quad \xi \in \mathbb{R}^3.$$

Thus, the Holtsmark distribution $\mathcal{H}_\gamma$ is a solution of the equation

$$\partial_t u(x;t) + a'_\beta(t)A_\nu u(\xi;t) = 0, \quad t \in (0;T], \quad \xi \in \mathbb{R}^3, \tag{11}$$

with the Riesz operator A_ν of fractional differentiation of order $\nu := \dfrac{3\gamma}{2}$, see [14].

We now clarify the problem of existence of the limit value of distribution $\mathcal{H}_\gamma(\cdot;t)$ at the point $t=0$.

First, we consider the case $a_\beta(0) \neq 0$. By using equality (10) and the well-known formula for the Fourier transform of the convolution of elements from the Lebesgue class $L_1(\mathbb{R}^3)$, we get

$$\mathcal{H}_\gamma(x;t) = \big(G_\nu * \hat{\mathcal{H}}_\gamma\big)(x;t), \quad t \in (0;T], \quad \xi \in \mathbb{R}^3,$$

where $\hat{\mathcal{H}}_\gamma(\cdot) := \mathcal{H}_\gamma(\cdot;0)$ is the corresponding stationary Holtsmark distribution and

$$G_\nu(\cdot;t) := \mathbb{F}^{-1}\Big[e^{-\int_0^t a'_\beta(\tau)d\tau|\xi|^\nu}\Big](\cdot;t).$$

For each function $\varphi(\cdot)$ continuous and bounded in $\mathbb{R}^3$, the following limit relation [13] is true:

$$\big(G_\nu * \varphi\big)(\cdot;t) \underset{t\to+0}{\to} \varphi(\cdot). \tag{12}$$

Hence, in view of the infinite differentiability and boundedness of the function $\hat{\mathcal{H}}_\gamma(\cdot)$ on $\mathbb{R}^3$, we arrive at the relation

$$\mathcal{H}_\gamma(\cdot;t) \underset{t\to+0}{\to} \hat{\mathcal{H}}_\gamma(\cdot). \tag{13}$$

Thus, the distribution $\mathcal{H}_\gamma(x;t)$ is the classical solution of the Cauchy problem (11), (13).

Now let $a_\beta(0) = 0$. Then the following equality directly follows from (10):

$$\mathcal{H}_\gamma(\cdot;t) = G_\nu(\cdot;t), \quad t \in (0;T]. \tag{14}$$

Note that relation (12) characterizes the property of "δ-similarity" of the function $G_\nu(\cdot;t)$ in the space S' of Schwartz distributions [15]:

$$G_\nu(\cdot;t) \underset{t\to+0}{\to} \delta(\cdot) \tag{15}$$

(here, $\delta(\cdot)$ is the Dirac delta function). Hence, for $a_\beta(0) = 0$, the Holtsmark distribution $\mathcal{H}_\gamma(\cdot;t)$ is a solution of the Cauchy problem (11), (15) satisfying Eq. (11) (in the ordinary sense) and the initial condition (15) (in the sense of weak convergence in the space S'). This solution G_ν is called the Green function of the Cauchy problem for Eq. (11).

We summarize these results in the form of the following assertion:

Theorem 2. *Suppose that* $\beta > 3/2$ *and* $a_\beta(\cdot)$ *is a positive function continuously differentiable on the segment* $(0;T]$. *Then, for* $a_\beta(0) \neq 0$, *the corresponding Holtsmark distribution* $\mathcal{H}_{2/\beta}(\cdot;t)$ *on the set* $\mathbb{R}^3 \times (0;T]$ *is a classical solution of the Cauchy problem (11), (13). In the case where* $a_\beta(0) = 0$, $\mathcal{H}_{2/\beta}(\cdot;t)$ *is the Green function of this problem.*

Remark. Equality (14) clarifies the meaning of the Green function of the Cauchy problem for Eq. (11). Indeed, G_ν is the initial Holtsmark distribution of the local influence exerted upon the analyzed object by its moving surrounding, which characterizes this process from the very beginning, i.e., from the time of appearance of the elements of local influence in the immediate surrounding of the object.

The investigations of the Green function of the Cauchy problem for pseudodifferential equations of the form (11) were originated by Éidel'man and Drin' in the early 1980s [16, 17]. They proposed a method for the construction and investigation of the function G_ν based on the Fourier transformation and obtained the following estimates:

$$\left|\partial_x^k G_\nu(x;t)\right| \leq c_1 t(t^{1/\nu} + |x|)^{-(n+|k|+[\nu])}, \qquad k \in \mathbb{Z}_+^n, \quad t \in (0;T], \quad x \in \mathbb{R}^n \tag{16}$$

(here, $[\cdot]$ is the integer part of a number). However, this method imposes restrictions on the order ν of the pseudodifferential equation: $\nu > 1$.

Fedoryuk [18] determined the exact asymptotic behavior of the Green function $G_\nu(\cdot;t)$ in the vicinity of infinitely remote points

$$G_\nu(\cdot;t) \sim |\cdot|^{-n-\nu}, \quad t > 0. \tag{17}$$

Later, Schneider [19] efficiently applied the Mellin transformation to express the function $G_\nu(\cdot;t)$ via special Fox H-functions and, as a result, established asymptotics (17). Note that asymptotics (17) for the pseudodifferential equation (11) with $a_\beta(t) = t$ and $\nu \in (0;1]$ was described by Blumenthal and Getoor [20] much earlier than in [18].

In [21], Kochubei proposed a new approach to the investigation of properties of the function $G_\nu(\cdot;t)$ based on the use of the theory of generalized functions and harmonic analysis. For the first time, he established estimates (16) in which $[\nu]$ was replaced by ν in the case where the dimension of the space variable is greater than 1 and $\nu \geq 1$.

In [12, 13], we developed the idea proposed in [21] and extended estimate (16) to the case $\nu > 0$.

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