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THE SECOND COHOMOLOGY SPACES $\mathcal{K}(2)$ WITH COEFFICIENTS IN THE SUPERSPACE OF WEIGHTED DENSITIES

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Over the $(1,2)$-dimensional supercircle, we investigate the second cohomology space associated with the Lie superalgebra $\mathcal{K}(2)$ of vector fields on the supercircle $S^{1|2}$ with coefficients in the space of weighted densities. We explicitly give the 2-cocycle spanning for these cohomology spaces.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra and let M a $\mathfrak{g}$-module. We associate a cochain complex known as the **Chevalley–Eilenberg differential**. The nth space of this complex is denoted by $C^n(\mathfrak{g}, M)$.

This space is trivial for $n < 0$. At the same time, if $n > 0$, then this is the space of n-linear antisymmetric mappings of $\mathfrak{g}$ into M. They are called n-cochains of $\mathfrak{g}$ with coefficients in M. The space of 0-cochains $C^0(\mathfrak{g}, M)$ reduces to M. The differential δ^n is defined by the following formula: for $c \in C^n(\mathfrak{g},\ )$, the $(n+1)$-cochain $\delta^n(c)$ evaluated on $g_1, g_2, \ldots, g_{n+1} \in \mathfrak{g}$ gives

$$\delta^n c(g_1, \ldots, g_{n+1}) = \sum_{1 \le s < t \le n+1} (-1)^{s+t-1} c\left([g_s, g_t], g_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, \ldots, g_{q+1}\right)$$

$$+ \sum_{1 \le s \le n+1} (-1)^s g_s c\left(g_1, \ldots, \hat{g}_s, \ldots, g_{n+1}\right),$$

where the notation $\hat{g}_i$ indicates that the ith term is omitted.

We now check that $\delta^{n+1} \circ \delta^n = 0$. Thus, we have a complex

$$0 \to C^0(\mathfrak{g}, M) \to \ldots \to C^{n-1}(\mathfrak{g}, M) \overset{d^{n-1}}{\to} C^n(\mathfrak{g}, M) \to \ldots.$$

By $H^n(\mathfrak{g}, M) = \ker d^n / \operatorname{Im} d^{n-1}$ we denote the quotient space. This space is called the space of n-cohomology from $\mathfrak{g}$ with coefficients in M.

We also denote:

$Z^n(\mathfrak{g}, M) = \ker \delta_n$ is the space of n-cocycles,

$B^n(\mathfrak{g}, M) = \Im \delta_{n-1}$ is the space of n-coboundaries.

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For $M = \mathbb{R}$ (or $\mathbb{C}$) considered as a trivial module, in this case, we denote the cohomology by $H^n(\mathfrak{g})$.

We now recall the classical interpretations of cohomology spaces of low degrees:

The space

$$H^0(\mathfrak{g}, M) \simeq \mathrm{Inv}_{\mathfrak{g}}(M) := \{m \in M;\ X.m = 0\ \ \forall X \in \mathfrak{g},\}.$$

The space $H^1(\mathfrak{g}, M)$ classifies derivations of $\mathfrak{g}$ with values in M modulo inner ones. This result is especially useful for $M = \mathfrak{g}$ with the adjoint representation. In this case, a derivation is a map $\varrho\colon\ \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$\varrho([X, Y]) - [\varrho(X), Y] - [X, \varrho(Y)] = 0,$$

while an inner derivation is specified by the adjoint action of some element $Z \in \mathfrak{g}$.

The space $H^2(\mathfrak{g}, M)$ classifies extensions of the Lie algebra $\mathfrak{g}$ by M, i.e., short exact sequences of Lie algebras

$$0 \to M \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

in which M is considered as an Abelian Lie algebra. We mainly consider two particular cases of this situation, which are extensively studied in what follows:

If M is a trivial $\mathfrak{g}$-module (typically, $M = \mathbb{R}$ or $\mathbb{C}$), then $H^2(\mathfrak{g}, M)$ classifies central extensions modulo trivial ones. Recall that a central extension of $\mathfrak{g}$ by $\mathbb{R}$ produces a new Lie bracket on $\hat{\mathfrak{g}} = \mathfrak{g} \oplus M$ by setting

$$[(X, \lambda), (Y, \mu)] = ([X, Y], c(X, Y)).$$

This is trivial if the cocycle $c = dl$ is a coboundary of a 1-cochain l. In this case, the map $(X, \lambda) \to (X, \lambda - l(X))$ yields a Lie isomorphism between $\hat{\mathfrak{g}}$ and $\mathfrak{g} \oplus M$ considered as a direct sum of Lie algebras.

If $M = \mathfrak{g}$ with the adjoint representation, then $H^2(\mathfrak{g}, \mathfrak{g})$ classifies infinitesimal deformations modulo trivial ones. By the definition, a (formal) series

$$(X, Y) \to \Phi_\lambda(X, Y) := [X, Y] + \lambda f_1(X, Y) + \lambda^2 f_2(X, Y) + \dots$$

is a deformation of the Lie bracket $[,]$ if Φ_λ is a Lie bracket for every λ, i.e., is an antisymmetric bilinear form in X, Y and satisfies the Jacobi identity. If we simply set

$$[X, Y]_\lambda = [X, Y] + \lambda c(X, Y),$$

where c is a 2-cochain with values in $\mathfrak{g}$ and λ is a scalar, then this bracket satisfies the Jacobi identity modulo the terms of order $O\left(\lambda^2\right)$ if and only if c is a 2-cocycle. Thus, we get what is called an infinitesimal deformation of the bracket of $\mathfrak{g}$, which is trivial if c is a coboundary. This means (as in the case of central extensions) that an adequate linear isomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ transforms the initial bracket $[,]$ into the deformed bracket $[,]_\lambda$. The infinitesimal deformation associated with a cocycle c does not always give rise to an actual deformation that coincides with the infinitesimal deformation of order 1, i.e., such that $f_1 = c$, as one can check by looking inductively on the functions $f_2, f_3, \dots$ satisfying Jacobi's identities of orders $2,\ 3, \dots$. The cohomological obstructions to prolongations of deformations are contained in $H^3(\mathfrak{g}, \mathfrak{g})$.

A natural generalization of the Virasoro algebra is given by extensions of the Lie algebra $\mathfrak{vect}(S^1)$ of vector fields on the circle by the modules $\mathcal{F}_\lambda$ of λ-densities on the circle. The problem of classifying extensions of this kind is equivalent to the problem of calculation of the cohomology $H^2\big(\mathrm{Vect}\left(S^1\right);\mathcal{F}_\lambda\big)$. In [4, 5], V. Ovsienko, C. Roger and P. Marcel computed the space $\mathrm{H}^2\big(\mathrm{Vect}\left(S^1\right);\mathcal{F}_\lambda\big)$, where $\mathrm{Vect}\left(S^1\right)$ is the algebra of smooth vector fields on the circle S^1 and $\mathcal{F}_\lambda$ is the space of λ densities. Following Ovsienko and Roger, B. Agrebaoui, I. Basdouri, and M. Boujelben [1] computed $H^2_{\mathrm{diff}}\big(\mathcal{K}(1);\mathfrak{F}^1_\lambda\big)$, where $\mathcal{K}(1)$ is the Lie superalgebra of contact vector fields on the supercircle $S^{1|1}$ with coefficients in the space of weighted densities.

In this paper, we explicitly compute $H^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)$, where $\mathcal{K}(2)$ is the lie superalgebra of contact vector fields in $S^{1|2}$ with coefficients in the spaces of weighted densities $\mathfrak{F}^2_\lambda$.

The present paper is organized as follows. In Section 2, we present some preliminary definitions and explain the notation. In Section 3, we compute the 2-cohomology space $H^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)$ and classify the extensions of a Lie superalgebra $\mathcal{K}(2)$ by $\mathfrak{F}^2_\lambda$.

2. Preliminaries

In this section, we recall some tools pertaining to the problem of cohomology, such as weighted densities, superfunctions, and contact projective vector fields on $S^{1|n}$.

2.1. Standard Contact Structure on $S^{1|n}$. Let $S^{1|n}$ be a supercircle with coordinates $(x,\theta_1,\ldots\ldots,\theta_n)$, where x is an even indeterminate and $\theta_1,\ldots,\theta_n$ are odd indeterminates: $\theta_i\theta_j = -\theta_j\theta_i$. This superspace is equipped with a standard contact structure given by the distribution $D=\langle\overline{\eta}_1,\ldots,\overline{\eta}_n\rangle$ generated by the vector fields $\overline{\eta}_i=\partial_{\theta_i}-\theta_i\partial_x$. This means that the distribution D is the kernel of the following 1-form:

$$\alpha_n = dx+\sum_{i=1}^{n}\theta_i d\theta_i.$$

2.2. Superfunctions on $S^{1|n}$. We define the geometry of the superspace $S^{1|n}$, where $n\in\mathbb{N}$, by describing its associative supercommutative superalgebra of superfunctions on $S^{1|n}$, which is denoted by $C^\infty(S^{1|n})$. This is the space of functions F of the form

$$F=\sum_{1\le i_1<\ldots<i_k\le n} f_{i_1,\ldots,i_k}(x)\theta_{i_1}\ldots\theta_{i_k},\quad \text{where}\quad f_{i_1,\ldots,i_k}\in C^\infty\left(S^1\right). \tag{2.1}$$

Clearly, the even (resp., odd) elements of $C^\infty\left(S^{1|n}\right)$ are functions given by (2.1) for which the summation is carried out only over even (resp., odd) integer k. By $p(F)$ we denote the parity of a homogeneous function F. On $C^\infty\left(S^{1|n}\right)$, we consider the following contact bracket:

$$\{F,G\}=FG'-F'G-\frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{n}\overline{\eta}_i(F)\overline{\eta}_i(G),$$

where the superscript $'$ stands for $\dfrac{\partial}{\partial x}$.

2.3. Vector Fields on $S^{1|n}$. A vector field on $S^{1|n}$ is a superderivation of the associative supercommutative superalgebra $C^\infty\left(S^{1|n}\right)$. In coordinates, it can be expressed as follows:

$$X=f\partial_x+\sum_{i=1}^{n}g_i\partial_{\theta_i},$$

where f and g_i are the elements of $C^\infty\left(S^{1|n}\right)$.

The superspace of all vector fields on $C^\infty(S^{1|n})$ is a Lie superalgebra. It is denoted by $\mathrm{Vect}\,(C^\infty(S^{1|n}))$.

2.4. Lie Superalgebra of Contact Vector Fields on $S^{1|n}$. Consider the superspace $\mathcal{K}(n)$ of contact vector fields on $S^{1|n}$. Thus, $\mathcal{K}(n)$ is the superspace of vector fields on $S^{1|n}$ with respect to the 1-form α_n. By definition, the Lie superalgebra of contact vector fields is

$$\mathcal{K}(n)=\Big\{X\in \mathrm{Vect}\big(S^{1|n}\big)\ \Big|\ \text{there exists } F_X\in C^\infty\big(S^{1|n}\big) \text{ such that } \mathfrak{L}_{X_F}(\alpha_n)=F\alpha_n\Big\}.$$

We define vector fields η_i and $\overline{\eta_i}$ as follows: $\eta_i=\partial_{\theta_i}+\theta_i\partial_x$ and $\overline{\eta}_i=\partial_{\theta_i}-\theta_i\partial_x$. Then any contact vector field on $S^{1|n}$ can be represented in the following explicit form:

$$X_F=F\partial_x-\frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{n}\overline{\eta}_i(F)\overline{\eta}_i,\quad \text{where}\quad F\in C^\infty\big(S^{1|n}\big).$$

The $\mathcal{K}(n)$ acts upon $S^{1|n}$ as follows:

$$\mathfrak{L}_{X_F}(X_G)=F\partial_x X_G+(-1)^{p(F)+1}\frac{1}{2}\sum_{i=1}^{n}\overline{\eta}_i(F)\overline{\eta_i}(G).$$

The vector field X_F has the same parity as F. The bracket in $\mathcal{K}(n)$ can be represented as follows:

$$[X_F,X_G]=X_{\{F,G\}}.$$

The Lie superalgebra $\mathfrak{osp}(2|n)$ is called the Lie superalgebra of contact projective vector fields. Thus, $\mathfrak{osp}(2|n)$ is an $(n+2|2n)$-dimensional Lie superalgebra spanned by the following contact projective vector fields:

$$\big\{X_x, X_{x^2}, X_1, 2X_{\theta_i\theta_j}, X_{\theta_i}, X_{x\theta_i},\ \ i,j=1,\dots,n\big\}.$$

2.5. Modules of Weighted Densities. We now consider the 1-parameter action of $\mathcal{K}(n)$ on $C^\infty(S^{1|n})$ specified by the rule

$$\mathfrak{L}^\lambda_{X_F}=X_F+\lambda F'.$$

We denote this $\mathcal{K}(n)$-module by $\mathfrak{F}^n_\lambda$; this is the space of all weighted densities of weight λ on $S^{1|n}$:

$$\mathfrak{F}^n_\lambda=\Big\{F\alpha_n^\lambda \mid F\in C^\infty(S^{1|n})\Big\}.$$

The superspace $\mathfrak{F}^n_\lambda$ has the $\mathcal{K}(n)$-module structure defined by the Lie derivative:

$$\mathfrak{L}^\lambda_{X_G}\big(F\alpha_n^\lambda\big)=\big(X_G+\lambda G'\big)(F)\alpha^n_\lambda,$$

where

$$G':=\frac{\partial G}{\partial x}.$$

Clearly, $\mathcal{K}(n)$ is isomorphic to $\mathfrak{F}^n_{-1}$ as a $\mathcal{K}(n)$-module and

$$\mathfrak{F}^n_\lambda \simeq \mathfrak{F}^{n-1}_\lambda \oplus \Pi\Big(\mathfrak{F}^{n-1}_{\lambda+\frac{1}{2}}\Big),$$

where Π is the change of parity function.

3. Space $\mathrm{H}^2\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)$

In the present paper, we study the differential cohomology spaces $\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)$. Indeed, we consider only cochains $(X_F, X_G) \to \Omega(F,G)\alpha^2_\lambda$, where Ω is a differential operator.

3.1. Main Theorem. The main result of this paper is the following theorem:

Theorem 3.1.

$$\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big) \simeq \begin{cases} \mathbb{K} & \textit{for} \quad \lambda = 0, \dfrac{1}{2}, 1, \dfrac{3}{2}, 2, 3, \\ 0, & \textit{otherwise}. \end{cases}$$

The nontrivial spaces $\mathrm{H}^2\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)$ *are spanned by the following* 2*-cocycles:*

$$\Omega_0(X_F, X_G) = (\bar\eta_1(F)\bar\eta_2(G) - \bar\eta_2(F)\bar\eta_1(G))\,\theta_1\theta_2,$$

$$\Omega_{\frac{1}{2}}(X_F, X_G) = \frac{1}{2}\,(\bar\eta_1\bar\eta_2(F)\bar\eta_1(G) - \bar\eta_1(F)\bar\eta_1\bar\eta_1(G))\,\theta_1\theta_2,$$

$$\Omega_1(X_F, X_G) = (F\bar\eta_1\bar\eta_2(G) - \bar\eta_1\bar\eta_2(F)G + \bar\eta_1(F)\bar\eta_2(G) + \bar\eta_2(F)\bar\eta_1(G))\,\theta_1\theta_2,$$

$$\Omega_{\frac{3}{2}}(X_F, X_G) = (\bar\eta_1\bar\eta_2(F)\bar\eta_1(G) + \bar\eta_1\bar\eta_2(F)\bar\eta_2(G) - \bar\eta_1(F)\bar\eta_1\bar\eta_2(G) - \bar\eta_2(F)\bar\eta_1\bar\eta_2(G))\,\theta_1\theta_2,$$

$$\Omega_2(X_F, X_G) = \bar\eta_1\bar\eta_2(F')\bar\eta_1\bar\eta_2(G'),$$

$$\begin{aligned}\Omega_3(X_F, X_G) = \Big(&(-1)^{|F|}\left(\bar\eta_1(F'')\bar\eta_1(G'') + \bar\eta_2(F'')\bar\eta_2(G'')\right) \\ &+ 2\left(\bar\eta_1\bar\eta_2(F')\bar\eta_1\bar\eta_2(G'') - \bar\eta_1\bar\eta_2(F'')\bar\eta_1\bar\eta_2(G')\right)\Big).\end{aligned}$$

Corollary 3.1.

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(2),\,\mathcal{K}(2)) \simeq 0. \tag{3.1}$$

3.2. Relationship Between $\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2),\mathfrak{F}^2_\lambda\big)$ and $\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(1),\mathfrak{F}^1_\lambda\big)$. Prior to proving Theorem 3.1, we present some results illustrating the relationship between the cohomology space in supercircle $S^{1|1}$ and $S^{1|2}$.

Proposition 3.1 [1].

$$H^2_{\text{diff}}\left(\mathcal{K}(1);\mathfrak{F}^1_\lambda\right)\simeq\begin{cases}\mathbb{K} & \textit{for}\quad \lambda=0,3,5,\\ \mathbb{K}^2 & \textit{for}\quad \lambda=\frac{1}{2},\frac{3}{2},\\ 0, & \textit{otherwise}.\end{cases}$$

The nontrivial spaces $H^2\big(\mathcal{K}(1);\mathfrak{F}^1_\lambda\big)$ *are spanned by the* 2*-cocycles:*

$$\omega_0(X_F,X_G)=FG'-F'G-\left(\frac{1}{4}+\frac{3}{4}(-1)^{p(F)p(G)}\right)\bar\eta_1(F)\eta_1(G),$$

$$\omega_{\frac{1}{2}}(X_F,X_G)=(-1)^{p(F)+p(G)}\left(F'\eta_1(G')-\eta_1(F')G'\right)\alpha_1^{\frac{1}{2}},$$

$$\widetilde\omega_{\frac{1}{2}}(X_F,X_G)=\left(\frac{1}{2}+\frac{1}{4}\left(1+(-1)^{p(F)p(G)}\right)\right)(-1)^{p(F)+p(G)}\left(F\eta_1(G')-\eta_1(F')G\right)\alpha_1^{\frac{1}{2}},$$

$$\omega_{\frac{3}{2}}(X_F,X_G)=\left(\bar\eta_1(F'')G-(-1)^{p(F)}F'\bar\eta_1(G'')\right)-\frac{1}{2}\theta_1\left(\eta_1(F)\eta_1(G'')+\eta_1(F'')\eta_1(G)\right)\alpha_1^{\frac{3}{2}},$$

$$\widetilde\omega_{\frac{3}{2}}(X_F,X_G)=\left(F'\bar\eta_1(G'')-\bar\eta_1(F'')G'\right)\alpha_1^{\frac{3}{2}},$$

$$\omega_3(X_F,X_G)=\left(\eta_1(F'')\bar\eta_1(G'')G'\right)\alpha_1^{3},$$

$$\begin{aligned}\omega_5(X_F,X_G)=\Big(&\Big(F^{(3)}G^{(4)}F^{(4)}G^{(3)}\Big)\\ &+\frac{3}{2}\left(\eta_1(F^{(4)})\eta_1\big(G^{(2)}\big)-\eta_1\big(F^{(2)}\big)\eta_1\big(G^{(4)}\big)\right)-4\eta_1\big(F^{(3)}\big)\eta_1\big(G^{(3)}\big)\Big)\alpha_1^{5}.\end{aligned}$$

The following lemma gives the general form of each Ω:

Lemma 3.1. *The 2-cocycle* Ω *belongs to* $\mathrm{Z}^2\big(\mathcal{K}(2),\mathfrak{F}^2_\lambda\big)$*. Up to a coboundary, the map* Ω *is given by*

$$\Omega(X_F,X_G)=\sum_{i,j,k,l}a_{i,j,k,l}\overline{\eta}_1^i\overline{\eta}_2^j(F)\overline{\eta}_1^k\overline{\eta}_2^l(G)\alpha_2^\lambda,$$

where $a_{i,j,k,l}$ *depends only on* $\theta_1,\ \theta_2$*, and the parity of* F *and* G*.*

Proof. Every differential operator Ω can be expressed in the form

$$\Omega(X_F,X_G)=\sum a_{i,j,k,l}\overline{\eta}_1^i\overline{\eta}_2^j(F)\overline{\eta}_1^k\overline{\eta}_2^l(G)\alpha_2^\lambda,$$

where the coefficients $a_{i,j,k,l}$ are arbitrary function. By using the 2-cocycle equation, we can show that

$$\frac{\partial}{\partial x}a_{i,j,k,l}=0.$$

The dependence on the parity of F and G follows from the fact that Ω is skew-symmetric:

$$a_{i,j,k,l}(F,G) = (-1)^{\varepsilon_{ij}(F,G)} a_{i,j,k,l}(F,G),$$

where

$$\varepsilon_{ij}(F,G) = ij(p(F)+1)(p(G)+1) + p(F)p(G) + 1.$$

Lemma 3.1 is proved.

Further, in order to prove Theorem 3.1, it is also necessary to compute the cohomology space vanishing on $\mathcal{K}(1)$. We are interested in cohomology space vanishing on $\mathcal{K}(1)$, i.e., we assume

$$\Omega(X,Y) = 0, \quad \text{if} \quad X, Y \in \mathcal{K}(1).$$

Therefore, the relevant cohomology space is

$$\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2), \mathcal{K}(1), \mathfrak{F}^2_\lambda\big).$$

Theorem 3.2. *The space*

$$\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(2), \mathcal{K}(1), \mathfrak{F}^2_\lambda\right) \simeq \begin{cases} \mathbb{K} & \text{for} \quad \lambda = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. Let Ω be a 2-cocycle of $\mathcal{K}(2)$ vanishing on $\mathcal{K}(1)$. The expressions for Ω are given in Lemma 3.1. By using "MATHEMATICA," we check that the 2-cocycle condition has the solution

$$\Omega(X_F, X_G) = \begin{cases} 0 & \text{for} \quad \lambda \neq 2, \\ \nu \bar{\eta}_1\bar{\eta}_2(F')\bar{\eta}_1\bar{\eta}_2(G')\alpha_2^2 & \text{for} \quad \lambda = 2, \end{cases}$$

where ν is constant. Assume that the map Ω is a trivial 2-cocycle vanishing on $\mathcal{K}(1)$. Thus, there exists an even operator $b\colon \mathcal{K}(2) \to \mathfrak{F}^2_2$ given by

$$b(X_F) = \left(\sum_k \kappa_k(x,\theta_1,\theta_2)\eta_1\eta_2\big(F^{(k)}\big) + \sum_l \mu_l(x,\theta_1,\theta_2)F^{(l)}\right)\alpha_2^\lambda,$$

where the coefficients $\kappa_k(x,\theta_1,\theta_2)$ and $\mu_l(x,\theta_1,\theta_2)$ are arbitrary, such that Ω is equal to $\delta(b)$, i.e.,

$$\begin{aligned}\Omega(X_F,\, X_G) := (-1)^{p(X_F)p(b)}\mathfrak{L}^2_{X_F}(b(X_G)) \\ - (-1)^{p(X_G)(p(X_F))}\mathfrak{L}^2_{X_G}(b(X_F)) - b([X_F, X_G]).\end{aligned} \tag{3.3}$$

Condition (3.3) implies that its coefficients are constant.

By using "MATHEMATICA," we check that condition (3.3) has no solutions. We can see that expression (3.2) never appears on the right-hand side of (3.3). This contradicts our assumption.

Theorem 3.2 is proved.

Proof of Theorem 3.1. Consider 2-cocycles $\Omega \in Z^2_{\mathrm{diff}}\left(\mathcal{K}(2);\mathfrak{F}^2_\lambda\right)$. If $\Omega_{|\mathcal{K}(1)\otimes\mathcal{K}(1)}$ is trivial, then the 2-cocycle Ω is completely described by Theorem 3.2. Thus, assume that $\Omega_{|\mathcal{K}(1)\otimes\mathcal{K}(1)}$ is nontrivial. Clearly, by analyzing Proposition 3.1, we conclude that the nontrivial space $\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(2);\mathfrak{F}^2_\lambda\right)$ can appear only for

$$\lambda \in \left\{\frac{-1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, 3, 5\right\}.$$

The $\mathcal{K}(1)$-isomorphism:

$$\mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{F}^2_\lambda\right) \simeq \mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1);\mathfrak{F}^1_\lambda\right) \oplus \mathrm{H}^2_{\mathrm{diff}}\left(\mathcal{K}(1);\prod\left(\mathfrak{F}^1_{\lambda+\frac{1}{2}}\right)\right).$$

Together with Proposition 3.1 that describes, up to a coboundary and to within a scalar factor, the restriction of any 2-cocycle Ω to $\mathcal{K}(1)$. First, we separately consider the even and odd cases. Thus, even cohomology spaces can appear only for $\lambda \in \{0, 1, 3, 5\}$. At the same time, odd cohomology spaces can appear only for

$$\lambda \in \left\{\frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right\}.$$

In each case, the restriction of Ω to $\mathcal{K}(1)$ is a linear combination of the corresponding 2-cocycles given in Proposition 3.1. First, the operators Ω labeled by semiintegers are odd and given by the formula

$$\Omega(X_F, X_G) = \sum_{i,j,k,l} a_{i,j,k,l}\overline{\eta}_1^i\overline{\eta}_2^j(F)\overline{\eta}_1^k\overline{\eta}_2^l(G)\alpha_2^\lambda,$$

where $i+j+k+l \in \{1,3\}$ and the coefficients a_{ijkl} are arbitrary functions independent on the variable x. At the same time, they depend on θ and the parity of F and G. By using "MATHEMATICA," we investigate the dimension of the space of operators satisfying the 2-cocycle condition:

$$\begin{aligned}\delta(\Omega)(X_F, X_G, X_H) := {} & (-1)^{p(F)}X_F.\Omega(X_G, X_H) - (-1)^{p(G)(1+p(F))}X_G.\Omega(X_F, X_H)\\ & + (-1)^{p(H)(1+p(G)+p(F))}X_H.\Omega(X_F, X_G) - \Omega([X_F, X_G], X_H)\\ & + (-1)^{p(G)p(H)}\Omega([X_F, X_H], X_G) - (-1)^{p(F)(p(G)+p(H))}\Omega([X_G, X_H], X_F) = 0, \quad (3.4)\end{aligned}$$

where $X_F.\Omega(X_F, X_H) = \mathfrak{L}^\lambda_{X_F}(\Omega(X_G, X_H))$ and $F, G, H \in \mathcal{C}^\infty\left(S^{1|2}\right)$.

The number of variables generating any 2-cocycle is much smaller than the number of equations obtained from the 2-cocycle condition for particular values of a_{ijkl}. We have:

For $\lambda = \dfrac{1}{2}$:

In this case, by direct computations, we can see that the 2-cocycle condition is always satisfied for the following particular values:

$$a_{1000} = 0, \qquad a_{0100} = 0, \qquad a_{0010} = 0, \qquad a_{0001} = 0,$$

$$a_{1110} = \frac{1}{2}\theta_1\theta_2, \qquad a_{1101} = 0, \qquad a_{0111} = 0, \qquad a_{1011} = -\frac{1}{2}\theta_1\theta_2.$$

We study all trivial 2-cocycles, i.e., operators of the form $\delta b,$ where b is a linear operator given by

$$b(X_F) = \big(\kappa\eta_1\eta_2(F) + \mu F\big)\alpha_2^{\lambda}.$$

As a result of direct computations, we obtain

$$\begin{aligned}\delta b(X_F, X_G) = \frac{1}{2}\Big(&\kappa\left(3f_1(x)g_1(x) + 3f_2(x)g_2(x) + g_0(x)f_0'(x) - f_0(x)g_0'(x)\right)\\
&- \kappa\theta_1\big(3f_{12}(x)g_2(x) - 3f_2(x)g_{12}(x) - 3g_1(x)f_0'(x) + 3g_0(x)f_1'(x) + f_1(x)g_0'(x)\\
&- f_0(x)g_1'(x)\big) - \kappa\theta_2\big(3f_{12}(x)g_1(x) - 3f_1(x)g_{12}(x) - 6g_2(x)f_0'(x) - g_0(x)f_2'(x)\\
&+ 6f_2(x)g_0'(x) - f_0(x)g_2'(x)\big) + \kappa\theta_1\theta_2\big(g_{12}(x)f_0'(x) + 2g_2(x)f_1'(x) - 2g_1(x)f_2'(x)\\
&+ g_0(x)f_{12}'(x) - f_{12}(x)g_0'(x) + 2f_2(x)g_1'(x) - 2f_1(x)g_2'(x) - f_0(x)g_{12}'(x)\big)\\
&+ \mu\big(3g_{12}f_0'(x) + 2g_2(x)f_1'(x) - 2g_1(x)f_2'(x) - 3f_{12}(x)g_0' + 4f_2(x)g_1'(x)\\
&- 4f_1(x)g_2'(x)\big) + \mu\theta_1\big(- g_{12}(x)f_1'(x) + 2g_1(x)f_{12}'(x) + f_2'(x)g_0'(x) - f_{12}(x)g_1'(x)\\
&- f_0'(x)g_2'(x) - 4f_1(x)g_{12}'(x) - 4g_2(x)f_0''(x) + 4f_2(x)g_0''(x)\big) + \mu\theta_2\big(- g_{12}(x)f_2'(x)\\
&+ 4g_2(x)f_{12}'(x) - f_1'(x)g_0'(x) + f_0'(x)g_1'(x) + 3f_{12}(x)g_2'(x) - 4f_1(x)g_{12}'(x)\\
&+ 2g_1(x)f_0''(x) - 2f_2(x)g_{12}''(x)\big) + \mu\theta_1\theta_2\big(- g_{12}(x)f_{12}'(x) - 2f_1'(x)g_1'(x)\\
&- 2f_2'(x)g_2'(x) + f_{12}(x)g_{12}'(x) + g_0'(x)f_0''(x) + 2g_1(x)f_1''(x) - g_2(x)f_2''(x)\\
&- f_0'(x)g_0''(x) - 2f_1(x)g_1''(x) - 2f_2(x)g_2''(x)\big)\Big).\end{aligned}$$

It is now easy to check that the equation $\Omega - \delta b = 0$ has no solutions. Thus, the 2-cocycle is nontrivial and

$$\dim \mathrm{H}^2_{\text{diff}}(\mathcal{K}(2); \mathfrak{F}^2_\lambda) = \dim \mathrm{Z}^2_{\text{diff}}\big(\mathcal{K}(2); \mathfrak{F}^2_\lambda\big).$$

Hence, the cohomology space is one-dimensional.

For $\lambda = \dfrac{3}{2}$:

In this case, by direct computations, we can show that the 2-cocycle condition is always satisfied for the following particular values:

$$a_{1000} = 0, \qquad a_{0100} = 0, \qquad a_{0010} = 0, \qquad a_{0001} = 0,$$

$$a_{1110} = \theta_1\theta_2, \qquad a_{1101} = \theta_1\theta_2, \qquad a_{0111} = -\theta_1\theta_2, \qquad a_{1011} = -\theta_1\theta_2.$$

We now study the triviality of this 2-cocycle. It is easy to see that any coboundary $\delta b(X_F, X_G)$ can be expressed in the following form:

$$\delta b(X_F,X_G)=\kappa\Big(\frac{3}{2}\,f_1(x)g_1(x)+\frac{3}{2}\,f_2(x)g_2(x)+\frac{3}{2}\,g_0(x)f_0'(x)-\frac{3}{2}\,f_0(x)g_0'(x)\Big)$$

$$+\kappa\theta_1\Big(-\frac{1}{2}\,f_{12}(x)g_2(x)+\frac{3}{2}f_2(x)g_{12}(x)+4g_1(x)f_0I(x)-\frac{3}{2}g_0(x)f_1I(x)$$

$$+4f_1(x)g_0I(x)+\frac{3}{2}\,f_0(x)g_1I(x)\Big)+\kappa\theta_2\Big(-\frac{1}{2}\,f_{12}(x)g_1(x)+\frac{3}{2}\,f_1(x)g_{12}(x)$$

$$+4g_2(x)f_0'(x)+\frac{3}{2}\,g_0(x)f_2'(x)-4f_2(x)g_0'(x)+\frac{3}{2}\,f_0(x)g_2'(x)\Big)$$

$$+\kappa\theta_1\theta_2\Big(\frac{3}{2}\,g_{12}(x)f_0'(x)+\frac{3}{2}\,g_0(x)f_{12}'(x)+\frac{3}{2}\,f_{12}(x)g_0'(x)+f_2(x)g_1'(x)$$

$$-f_1(x)g_2'(x)-\frac{3}{2}\,f_0(x)g_{12}'(x)\Big)+\mu\Big(\frac{5}{2}\,g_{12}(x)f_0'(x)+g_2(x)f_1'(x)-g_1(x)f_2'(x)$$

$$-\frac{5}{2}\,f_{12}(x)g_0'(x)+2f_2(x)g_1'(x)-2f_1(x)g_2'(x)\Big)+\mu\theta_2\,\Big(-\frac{3}{2}\,g_{12}(x)f_2'(x)$$

$$+2g_2(x)f_{12}'(x)-\frac{3}{2}\,f_1'(x)g_0'(x)+\frac{3}{2}\,f_0'(x)g_1'(x)+\frac{5}{2}\,f_{12}(x)g_2'(x)-f_2(x)g_{12}'(x)$$

$$+g_1(x)f_0''(x)-f_1(x)g_0''(x)\Big)+\mu\theta_1\Big(-\frac{3}{2}\,g_{12}(x)f_1'(x)+g_1(x)f_{12}'(x)$$

$$+\frac{3}{2}\,f_2'(x)g_0'(x)+\frac{3}{2}\,f_{12}(x)g_1'(x)-\frac{3}{2}\,f_0'(x)g_2'(x)-2f_1(x)g_{12}'(x)-2g_2(x)f_0''(x)$$

$$+2f_2(x)g_0''(x)\Big)+\mu\theta_1\theta_2\,\Big(\frac{1}{2}\,g_{12}(x)f_{12}'(x)-3f_1'(x)g_1'(x)-3f_2'(x)g_2'(x)$$

$$-\frac{1}{2}\,f_{12}(x)g_{12}'(x)-\frac{1}{2}\,g_0'(x)f_0''(x)-g_1(x)f_1''(x)-g_2(x)f_2''(x)+\frac{1}{2}\,f_0'(x)g_0''(x)$$

$$-f_1(x)g_1''(x)-f_2(x)g_2''(x)\Big).$$

Hence, in the same way as earlier, we conclude that the equation $\Omega-\delta b=0$ has no solutions. Thus, the 2-cocycle is nontrivial and

$$\dim \mathrm{H}^2_{\rm diff}\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big)=\dim \mathrm{Z}^2_{\rm diff}\,\big(\mathcal{K}(2);\mathfrak{F}^2_\lambda\big).$$

We deduce that the cohomology space is one-dimensional.

For $\lambda \in \left\{ \frac{-1}{2}, \frac{5}{2}, \frac{9}{2} \right\}$, equation (3.4) does not have solutions. Thus,

$$\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2), \mathfrak{F}^2_\lambda\big) \simeq 0.$$

By applying the 2-cocycle equation to Ω and using "MATHEMATICA," we deduce the expressions for Ω. To be more precise, we get

$$\Omega = \begin{cases} \frac{1}{2}\left(\bar{\eta}_1\bar{\eta}_2(F)\bar{\eta}_1(G) - \bar{\eta}_1(F)\bar{\eta}_1\bar{\eta}_1(G)\right)\theta_1\theta_2 & \text{if} \quad \lambda = \frac{1}{2}, \\ \left(\bar{\eta}_1\bar{\eta}_2(F)\bar{\eta}_1(G) + \bar{\eta}_1\bar{\eta}_2(F)\bar{\eta}_2(G) - \bar{\eta}_1(F)\bar{\eta}_1\bar{\eta}_2(G) - \bar{\eta}_2(F)\bar{\eta}_1\bar{\eta}_2(G)\right)\theta_1\theta_2 & \text{if} \quad \lambda = \frac{3}{2}. \end{cases}$$

In this case, the proof is the same as in the case of odd 2-cocycles. The operators Ω labeled by integers are even and given by

$$\Omega(X_F, X_G) = \sum_{i,j,k,l} a_{i,j,k,l}\overline{\eta}_1^i\overline{\eta}_2^j(F)\overline{\eta}_1^k\overline{\eta}_2^l(G)\alpha_2^\lambda,$$

where $i+j+k+l \in \{0,2,4\}$ and the coefficients a_{ijkl} are arbitrary functions independent on the variable x. However, they depend on θ and the parity of F and G.

Using "MATHEMATICA", we conclude that this map satisfies the 2-cocycle equation

$$\begin{aligned} \delta(\Omega)(X_F, X_G, X_H) := {} & X_F.\Omega(X_G, X_H) - (-1)^{p(G)p(F)}X_G.\Omega(X_F, X_H) \\ & + (-1)^{p(H)(p(G)+p(F))}X_H.\Omega(X_F, X_G) \\ & - \Omega([X_F, X_G], X_H) + (-1)^{p(G)p(H)}\Omega([X_F, X_H], X_G) \\ & - (-1)^{p(F)(p(G)+p(H))}\Omega([X_G, X_H], X_F) = 0, \end{aligned} \tag{3.5}$$

where $F, G, H \in \mathcal{C}^\infty\big(S^{1|2}\big)$.

For $\lambda = 0$:

In this case, by direct computations, we can see that the 2-cocycle condition is always satisfied for the following particular values:

$$a_{0000} = 0, \qquad a_{1100} = 0, \qquad a_{0011} = 0, \qquad a_{1001} = \theta_1\theta_2,$$

$$a_{0110} = -\theta_1\theta_2, \qquad a_{1010} = 0, \qquad a_{0101} = 0, \qquad a_{0111} = 0, \qquad a_{1111} = 0.$$

On the other hand, we can see that the coboundary $\delta b(X_F, X_G)$ can be expressed as follows:

$$\begin{aligned} \delta b(X_F, X_G) = {} & \kappa\Big(-\frac{1}{2}f_1(x)g_1(x) - \frac{1}{2}f_2(x)g_2(x)\Big) \\ & + \kappa\theta_1\Big(\frac{1}{2}f_{12}(x)g_2(x) + \frac{1}{2}f_2(x)g_{12}(x) - \frac{1}{2}g_1(x)f_0'(x) + \frac{1}{2}f_1(x)g_0'(x)\Big) \end{aligned}$$

$$+\kappa\theta_2\Big(\frac{1}{2}\,f_{12}(x)g_1(x)+\frac{1}{2}\,f_1(x)g_{12}(x)-\frac{1}{2}\,g_2(x)f_0'(x)+\frac{1}{2}\,f_2(x)g_0'(x)\Big)$$

$$+\kappa\theta_1\theta_2\Big(-\frac{1}{2}\,g_2(x)f_1'(x)+\frac{1}{2}\,g_1(x)f_2'(x)+\frac{1}{2}\,f_2(x)g_1'(x)-\frac{1}{2}\,f_1(x)g_2'(x)\Big)$$

$$+\mu\left(g_{12}(x)f_0'(x)-f_{12}(x)g_0'(x)+f_2(x)g_1'(x)-f_1(x)g_2'(x)\right)$$

$$+\mu\theta_2\left(-g_2(x)f_{12}'(x)+f_{12}(x)g_2'(x)+2f_2(x)g_{12}'(x)+2g_1(x)f_0''(x)\right.$$

$$\left.-2f_1(x)g_0''(x)\right)+\mu\theta_1\left(-g_1(x)f_{12}'(x)-f_{12}(x)g_1'(x)+f_1(x)g_{12}'(x)\right.$$

$$\left.-f_1'(x)g_{12}(x)-g_2(x)f_0''(x)+f_2(x)g_0''(x)\right)+\mu\theta_1\theta_2\left(-g_{12}(x)f_{12}'(x)\right.$$

$$+f_1'(x)g_1'(x)+2f_2'(x)g_2'(x)+f_{12}(x)g_{12}'(x)+g_0'(x)f_0''(x)+2g_1(x)f_1''(x)$$

$$\left.+2g_2(x)f_2''(x)-f_0'(x)g_0''(x)+2f_1(x)g_1''(x)+2f_2(x)g_2''(x)\right).$$

Thus, the cohomology space is one-dimensional because the equation $\Omega-\delta b=0$ has no solutions. Hence, the 2-cocycle is nontrivial and

$$\dim \mathrm{H}^2_{\text{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_0\big)=\dim \mathrm{Z}^2_{\text{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_0\big)=1.$$

For $\lambda=1$:

In this case, by direct computations, we can see that the 2-cocycle condition is always satisfied for the following particular values:

$$a_{0000}=0,\qquad a_{1100}=-\theta_1\theta_2,\qquad a_{0011}=\theta_1\theta_2,\qquad a_{1001}=\theta_1\theta_2,$$

$$a_{0110}=\theta_1\theta_2,\qquad a_{1010}=0,\qquad a_{0101}=0,\qquad a_{1111}=0.$$

Further, by direct computations, we get

$$\delta b(X_F,X_G)=\kappa\Big(-\frac{1}{2}\,f_1(x)g_1(x)-\frac{1}{2}\,f_2(x)g_2(x)+g_0(x)f_0'(x)-f_0(x)g_0'(x)\Big)$$

$$+\kappa\theta_1\Big(\frac{1}{2}\,f_{12}(x)g_2(x)+\frac{1}{2}\,f_2(x)g_{12}(x)+\frac{1}{2}\,g_1(x)f_0'(x)+g_0(x)f_1'(x)$$

$$-\frac{1}{2}\,f_1(x)g_0'(x)-f_0(x)g_1'(x)\Big)+\kappa\theta_2\Big(\frac{1}{2}\,f_{12}(x)g_1(x)+\frac{1}{2}\,f_1(x)g_{12}(x)$$

$$+\frac{1}{2}\,g_2(x)f_0'(x)+g_0(x)f_2'(x)-\frac{1}{2}\,f_2(x)g_0'(x)-f_0(x)g_2'(x)\Big)$$

$$+\kappa\theta_1\theta_2\Big(g_{12}(x)f_0'(x)+\frac{1}{2}\,g_2(x)f_1'(x)-\frac{1}{2}\,g_1(x)f_2'(x)+g_0(x)f_{12}'(x)$$

$$-f_{12}(x)g_0'(x)+\frac{3}{2}\,f_2(x)\,g_1'(x)-\frac{3}{2}\,f_1(x)g_2'(x)-f_0(x)g_12'(x)\Big)$$

$$+\mu\big(2g_{12}(x)f_0'(x)-2f_{12}(x)g_0'(x)+f_2(x)g_1'(x)-f_1(x)g_2'(x)\big)$$

$$+\mu\theta_2\big(g_{12}(x)f_2'(x)-g_2(x)f_{12}'(x)-f_1'(x)g_0'(x)+f_0'(x)g_1'(x)$$

$$+2f_2(x)g_{12}'(x)+2g_1(x)f_0''(x)-2f_1(x)g_0''(x)\big)+\mu\theta_1\big(g_{12}(x)f_1'(x)$$

$$-g_1(x)f_{12}'(x)+f_2'(x)g_0'(x)-2f_{12}(x)g_1'(x)-f_0'(x)g_2'(x)+f_1(x)g_{12}'(x)$$

$$-g_2(x)f_0''(x)+f_2(x)g_0''(x)\big)+\mu\theta_1\theta_2\,\big(2f_1'(x)g_2'(x)+2f_2'(x)g_1'(x)$$

$$+2g_1(x)f_1''(x)+2g_2(x)f_2''(x)+2f_1(x)g_1''(x)+2f_2(x)g_2''(x)\big).$$

Hence, we conclude that the cohomology space is one-dimensional because the equation $\Omega-\delta b=0$ does not have solutions. Therefore, the 2-cocycle is nontrivial and

$$\dim \mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_1\big)=\dim \mathrm{Z}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_1\big).$$

For $\lambda=3$, equation (3.5) has a single solution Ω. It is now easy to check that the equation $\Omega-\delta b=0$ has no solutions. Hence, the 2-cocycle is nontrivial and

$$\dim \mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_3\big)=\dim \mathrm{Z}^2_{\mathrm{diff}}\big(\mathcal{K}(2);\mathfrak{F}^2_3\big)=1.$$

For $\lambda=5$, equation (3.5) has no solutions. Thus,

$$\mathrm{H}^2_{\mathrm{diff}}\big(\mathcal{K}(2),\mathfrak{F}^2_5\big)\simeq 0.$$

By using "MATHEMATICA," in the case where the condition of 2-cocycle has solutions, we deduce the expressions of Ω. To be more precise, we get

$$\Omega=\begin{cases}\left(\bar\eta_1(F)\bar\eta_2(G)-\bar\eta_2(F)\bar\eta_1(G)\right)\theta_1\theta_2 & \text{for}\quad \lambda=0,\\ \left(F\bar\eta_1\bar\eta_2(G)-\bar\eta_1\bar\eta_2(F)G+\bar\eta_1(F)\bar\eta_2(G)+\bar\eta_2(F)\bar\eta_1(G)\right)\theta_1\theta_2 & \text{for}\quad \lambda=1,\\ \big((-1)^{|F|}(M(F,G))+2(N(F,G))\big) & \text{for}\quad \lambda=3,\end{cases}$$

where

$$M(F,G)=\bar\eta_1(F'')\bar\eta_1(G'')+\bar\eta_2(F'')\bar\eta_2(G''),$$

$$N(F,G) = \bar{\eta}_1\bar{\eta}_2(F')\bar{\eta}_1\bar{\eta}_2(G'') - \bar{\eta}_1\bar{\eta}_2(F'')\bar{\eta}_1\bar{\eta}_2(G').$$

Theorem 3.1 is proved.

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