DOI 10.1007/s11253-020-01826-6
Ukrainian Mathematical Journal, Vol. 72, No. 5, October, 2020 (Ukrainian Original Vol. 72, No. 5, May, 2020)

WEAKLY PERIODIC GROUND STATES FOR THE λ-MODEL

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For the λ-model on a Cayley tree of order $k \geq 2,$ we describe the set of periodic and weakly periodic ground states corresponding to normal divisors of index 2 of the group representation of Cayley tree.

1. Introduction

It is known that the phase diagram of Gibbs measures for a given Hamiltonian is close to the phase diagram of ground isolated (stable) states of this Hamiltonian. At low temperatures, a periodic Gibbs measure is associated with a periodic ground state (see [1, 3, 15, 21]). This leads to a natural problem of description of periodic and weakly periodic ground states. The periodic ground states for the Potts model with competing interactions on a Cayley tree of order $k = 2$ were studied in [10, 19]. A weakly periodic Gibbs measure for the Potts model was investigated in [15]. For the Potts model, weakly periodic ground states for the normal divisor of index 2 were studied in [13]. For the Ising model with competing interactions,weakly periodic ground states were described in [7, 8].

The Potts model in the q-state is one of the most well studied models in statistical mechanics due to the high theoretical interest and practical applications [1, 2]. The Potts model [5] was introduced as a generalization of the Ising model to the case of more than two components. This model is used to study various problems in statistical physics (see, e.g., [6]). In [14], some explicit formulas were obtained for the free energies and entropies in the Potts model on a Cayley tree. Note that the Potts model is now one of the most important models in statistical mechanics. Thus, it is quite natural to consider more general models than the Potts model. In this connection, a so-called λ-model (i.e., a model with nearest neighbors in which the interactions depend on the function λ) was proposed in [17, 18]. This model includes the Potts model as a special case obtained if the λ-function is taken in the form $\lambda(x, y) = J\delta_{x,y},$ where δ is the Kronecker symbol. The indicated λ-model includes various possible interactions that cannot be described by the Potts models, including, in particular, the Widom–Rowlinson model with interaction described by the following function (see [20]):

$$\lambda(x,y) = J\delta_{xy} + \frac{\ln(\mu)}{\beta}|x-y|, \qquad x,y \in \{1,2,3\}, \quad \mu > 0, \quad \beta > 0.$$

This model differs from the Potts model for $\mu \neq 1.$ Moreover, its phase diagram is richer than the phase diagram of the Potts model. This example shows that the model analyzed in the present work is more general than the Widom–Rowlinson model and has a complex structure of ground states [16].

In the present paper, we consider the λ-model on a Cayley tree of order $k \geq 2.$ The aim of the present paper is to describe periodic and weakly periodic ground states of this model.

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 72, No. 5, pp. 667–678, May, 2020. Original article submitted October 12, 2019.

The obtained results make it possible, in particular, to analyze the structure of ground states for the Widom–Rowlinson model. Moreover, we can study free energy and the phase diagram of the investigated λ-model (see [14]).

It is worth noting that our results differ from the results obtained in [16] because only 1- and 2-periodic ground states were described in [16], whereas in the present paper, we find more general periodic and weakly periodic ground states for the λ-model on an arbitrary Cayley tree.

2. Definitions and Known Facts

Let $\tau^k = (V, L),\ k \geq 1$, be a Cayley tree of order $k \geq 1$, where V is the set of vertices and L is the set of edges τ^k.

If $x, y \in V$ are the endpoints of a certain edge $l \in L$, then x and y are called *nearest neighbors* and we write $l = \langle x, y\rangle$.

The distance $d(x,y),\ x, y \in V$, on a Cayley tree is given by the formula

$$d(x,y) = \min\big\{d \mid \exists x = x_0, x_1, \ldots, x_{d-1},\ \ x_d = y \in V,\ \text{ where } \langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle\big\}.$$

Assume that G_k is a group representation of a Cayley tree (see, e.g., [7–12, 15]), i.e., G_k is the free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \ldots, a_{k+1}$ such that $a_i^2 = e,\ i = 1, 2, \ldots, k+1$, where $e \in G_k$ is the identity element.

By $S(x)$ we denote the set of "direct derivatives" of the point $x \in G_k$. Moreover, by $S_1(x)$ we denote the set of all nearest neighbors of the point $x \in G_k$, i.e., $S_1(x) = \{y \in G_k \colon \langle x, y\rangle\}$ and $x_\downarrow = S_1(x) \setminus S(x)$.

We consider a model in which the spin takes values from the set $\Phi = \{1, 2, \ldots, q\},\ q \geq 2$. The configuration σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$. The set of all configurations coincides with $\Omega = \Phi^V$.

We define a G_k^*-periodic configuration $\sigma(x)$ invariant under the subgroup $G_k^* \subset G_k$ of finite index, i.e., $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$. For a given periodic configuration, the index of subgroup is called a period of configuration. Let

$$G_k/G_k^* = \{H_1, \ldots, H_r\}$$

be the quotient group, where G_k^* is a normal divisor with index $r \geq 1$. A configuration $\{\sigma(x), x \in V\}$ is called G_k^*-weakly periodic if $\sigma(x) = \sigma_{ij}$ for all $x_\downarrow \in H_i$ and $x \in H_j$ for all $x \in G_k$, i.e., the value of configuration at x depends not on x but on the number of the class of belonging of x and $x_\downarrow$.

The Hamiltonian of the λ-model has the form

$$H(\sigma) = \sum_{\substack{\langle x,y\rangle \\ x,y \in V}} \lambda\big(\sigma(x), \sigma(y)\big), \tag{1}$$

$$\lambda(i,j) = \begin{cases} \overline{a} & \text{for} \quad |i-j| = 2, \\ \overline{b} & \text{for} \quad |i-j| = 1, \\ \overline{c} & \text{for} \quad i = j, \end{cases}$$

where $\overline{a}, \overline{b}, \overline{c} \in R$.

Remark 1. If $\overline{a} = \overline{b} = 0$ and $\overline{c} = J$, then the λ-model is reduced to the Potts model.

3. Ground States

In this section, we study and describe all possible ground states for the λ-model.

For a pair of configurations σ and φ that coincide almost everywhere, i.e., everywhere except finitely many points, we consider the relative Hamiltonian $H(\sigma,\varphi)$ equal to the difference between the energies of the configurations σ and φ, i.e.,

$$H(\sigma,\varphi)=\sum_{\substack{\langle x,y\rangle\\ x,y\in V}}\big(\lambda(\sigma(x),\sigma(y))-\lambda(\varphi(x),\varphi(y))\big). \tag{2}$$

Let M be a set of unit balls with vertices in V. The restriction of a configuration σ to the ball $b\in M$ is called a *bounded configuration* σ_b. We define the energy of configuration σ_b on b as follows:

$$U(\sigma_b)=\frac{1}{2}\sum_{\substack{\langle x,y\rangle\\ x,y\in b}}\lambda\big(\sigma(x),\sigma(y)\big). \tag{3}$$

The following lemma is true [7, 10]:

Lemma 1. *The relative Hamiltonian (2) has the form*

$$H(\sigma,\varphi)=\sum_{b\in M}\big(U(\sigma_b)-U(\varphi_b)\big).$$

In the present paper, we consider the case $q=3$, i.e., $|\Phi|=3$. By c_b we denote the center of the unit ball b. Also let

$$B_t=\{x\in S_1(c_b):\ \varphi_b(x)=t\}\quad\text{for all}\quad t\in\Phi.$$

Let $\varphi_b(c_b)=d,\ |B_d|=i$, and $|B_f|=n$. Then

$$|B_g|=k+1-i-n,$$

where $d\neq f,\ d\neq g,\ f\neq g$, and $d,f,g\in\Phi$. The following lemma can be easily proved:

Lemma 2. *For every configuration φ_b, the following assertion is true:*

$$U(\varphi_b)\in\big\{U_{i,n}:\ i=0,1,\ldots,k+1,\ \ n=0,1,\ldots,k+1-i\big\},$$

where

$$U_{i,n}=\frac{i\overline{a}+n\overline{b}+(k+1-i-n)\overline{c}}{2}. \tag{4}$$

Definition 1. *A configuration φ is called a ground state for the Hamiltonian H if*

$$U(\varphi_b)=\min\big\{U_{i,n}:\ i=0,1,\ldots,k+1,\ \ n=0,1,\ldots,k+1-i\big\}$$

for any $b\in M$.

Denote

$$C_{i,n} = \{\varphi_b \colon U(\varphi_b) = U_{i,n}\}$$

and

$$A_{\xi,\eta} = \Big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon U_{\xi,\eta} = \min\big\{U_{i,n} \colon i = 0, 1, \dots, k+1, \;\; n = 0, 1, \dots, k+1-i\big\}\Big\}. \tag{5}$$

In the case $k = 2,$ it is easy to see that

$$U(\sigma_b) \in \{U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{1,0}, U_{1,1}, U_{1,2}, U_{2,0}, U_{2,1}, U_{3,0}\} \quad \text{for any} \quad \sigma_b,$$

where

$$U_{0,0} = \frac{3\overline{c}}{2}, \qquad U_{0,1} = \frac{\overline{b} + 2\overline{c}}{2}, \qquad U_{0,2} = \frac{2\overline{b} + \overline{c}}{2}, \qquad U_{0,3} = \frac{3\overline{b}}{2},$$

$$U_{1,0} = \frac{\overline{a} + 2\overline{c}}{2}, \qquad U_{1,1} = \frac{\overline{a} + \overline{b} + \overline{c}}{2}, \qquad U_{1,2} = \frac{\overline{a} + 2\overline{b}}{2},$$

$$U_{2,0} = \frac{2\overline{a} + \overline{c}}{2}, \qquad U_{2,1} = \frac{2\overline{a} + \overline{b}}{2}, \qquad U_{3,0} = \frac{3\overline{a}}{2}.$$

By using (5), we get

$$A_{0,0} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{c} \le \overline{b} \le \overline{a}\big\} \cup \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{c} \le \overline{a} \le \overline{b}\big\},$$

$$A_{0,1} = A_{0,2} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{b} = \overline{c} \le \overline{a}\big\},$$

$$A_{0,3} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{b} \le \overline{c} \le \overline{a}\big\} \cup \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{b} \le \overline{a} \le \overline{c}\big\},$$

$$A_{1,0} = A_{2,0} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{a} = \overline{c} \le \overline{b}\big\},$$

$$A_{1,1} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{a} = \overline{b} = \overline{c}\big\}, \qquad A_{1,2} = A_{2,1} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{a} = \overline{b} \le \overline{c}\big\},$$

$$A_{3,0} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{a} \le \overline{b} \le \overline{c}\big\} \cup \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{a} \le \overline{c} \le \overline{b}\big\},$$

and

$$R^3 = \bigcup_{i,n} A_{i,n}.$$

For $k \ge 3,$ in a similar way, we find $A_{i,n},\; i = 0, 1, 2, \dots, k+1,\; n = 0, 1, \dots, k+1-i$:

$$A_{0,0} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{c} \le \overline{b} \le \overline{a}\big\} \cup \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{c} \le \overline{a} \le \overline{b}\big\},$$

$$A_{0,1} = A_{0,2} = A_{0,3} = \ldots = A_{0,k} = \big\{(\overline{a}, \overline{b}, \overline{c}) \in R^3 \colon \overline{b} = \overline{c} \le \overline{a}\big\},$$

$$A_{0,k+1} = \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{b} \le \overline{c} \le \overline{a}\} \cup \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{b} \le \overline{a} \le \overline{c}\},$$

$$A_{1,0} = A_{2,0} = \ldots = A_{k,0} = \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{a} = \overline{c} \le \overline{b}\},$$

$$A_{1,1} = A_{1,2} = \ldots = A_{1,k-1} = A_{2,1} = \ldots = A_{2,k-2}$$

$$= \ldots = A_{k-1,1} = \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{a} = \overline{b} = \overline{c}\},$$

$$A_{1,k} = A_{2,k-1} = A_{3,k-2} = \ldots = A_{k,1} = \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{a} = \overline{b} \le \overline{c}\},$$

$$A_{k+1,0} = \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{a} \le \overline{b} \le \overline{c}\} \cup \{(\overline{a}, \overline{b}, \overline{c}) \in R^3 : \overline{a} \le \overline{c} \le \overline{b}\},$$

and

$$R^3 = \bigcup_{i,n} A_{i,n}.$$

4. Periodic Ground States

In this section, we describe periodic ground states of the λ-model.
Let $A \subset \{1, 2, \ldots, k+1\}$,

$$H_A = \Big\{x \in G_k : \sum\nolimits_{j \in A} w_j(x) \text{ is even}\Big\}, \qquad \text{and} \qquad G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\},$$

where $w_j(x)$ is the number of a_j in the word x. Note that H_A is a normal divisor of index 2 in G_k (see [15]). For $A = \{1, 2, \ldots, k+1\}$, the normal divisor H_A coincides with $G_k^{(2)}$.

Consider a quotient group $G_k / H_A = \{H_0, H_1\}$, where $H_0 = H_A$ and $H_1 = G_k \setminus H_A$.

The H_A-periodic configurations have the form

$$\sigma(x) = \begin{cases} \sigma_1 & \text{for} \quad x \in H_0, \\ \sigma_2 & \text{for} \quad x \in H_1, \end{cases}$$

where $\sigma_i \in \Phi,\ i = 1, 2$.

Theorem 1. *Let $k \ge 3$ and $|A| = 1$. Then the following assertions are true:*

(i) if $|\sigma_1 - \sigma_2| = 0$, then the corresponding configurations σ on the set $A_{0,0}$ are periodic ground states;

(ii) if $|\sigma_1 - \sigma_2| = 1$, then the corresponding configurations σ on the set $A_{0,1}$ are periodic ground states;

(iii) if $|\sigma_1 - \sigma_2| = 2$, then the corresponding configurations σ on the set $A_{1,0}$ are periodic ground states.

Proof. We prove the first assertion. Consider

$$\varphi(x) = \begin{cases} i & \text{for} \quad x \in H_0, \\ i & \text{for} \quad x \in H_1, \end{cases}$$

where $i \in \Phi$. Then $\varphi_b \in C_{0,0}$ for any $b \in M$. Therefore, $U(\varphi_b) = U_{0,0}$ for any $\varphi_b \in M$, i.e., the corresponding configurations φ on the set $A_{0,0}$ are periodic ground states.

We now prove the second assertion. Consider

$$\varphi(x) = \begin{cases} i & \text{for} \quad x \in H_0, \\ j & \text{for} \quad x \in H_1, \end{cases}$$

where $|i-j| = 1$ and $i, j \in \Phi$.

The following cases are possible:

(i) $c_b \in H_0$, then $\varphi_b(c_b) = i$, $|B_i| = k$, $|B_j| = 1$, and hence, $\varphi_b \in C_{0,1}$;

(ii) $c_b \in H_1$, then $\varphi_b(c_b) = j$, $|B_i| = 1$, $|B_j| = k$, and hence, $\varphi_b \in C_{0,1}$.

These cases imply that $U(\varphi_b) = U_{0,1}$ for any $\varphi_b \in M$. Therefore, the periodic configuration $\varphi(x)$ is a periodic ground state on the set $A_{0,1}$.

We now prove the third assertion. Consider

$$\varphi(x) = \begin{cases} i & \text{for} \quad x \in H_0, \\ j & \text{for} \quad x \in H_1, \end{cases}$$

where $|i-j| = 2$ and $i, j \in \Phi$.

The following cases are possible:

(i) $c_b \in H_0$; then $\varphi_b(c_b) = i$, $|B_i| = k$, $|B_j| = 1$, and therefore, $\varphi_b \in C_{1,0}$;

(ii) $c_b \in H_1$; then $\varphi_b(c_b) = j$, $|B_i| = 1$, $|B_j| = k$, and therefore, $\varphi_b \in C_{1,0}$.

It follows from these cases that $U(\varphi_b) = U_{1,0}$ for any $\varphi_b \in M$. Hence, the periodic configuration $\varphi(x)$ is a periodic ground state on the set $A_{1,0}$.

Theorem 1 is proved.

Remark 2. The periodic ground states from part (i) of Theorem 1 are translation invariant.

5. Weakly Periodic Ground States

In this section, we study weakly periodic ground states that are not periodic. Note that weakly periodic ground states were introduced in [7].

The H_A-weakly periodic configurations have the form

$$\varphi(x) = \begin{cases} a_{00} & \text{for} \quad x_\downarrow \in H_0, \quad x \in H_0, \\ a_{01} & \text{for} \quad x_\downarrow \in H_0, \quad x \in H_1, \\ a_{10} & \text{for} \quad x_\downarrow \in H_1, \quad x \in H_0, \\ a_{11} & \text{for} \quad x_\downarrow \in H_1, \quad x \in H_1, \end{cases}$$

where $a_{ij} \in \Phi$, $i, j \in \{0, 1\}$.

Further, for the sake of convenience, we represent a H_A-weakly periodic configuration $\varphi(x), x \in G_k$ in the form $\varphi = (a_{00}, a_{01}, a_{10}, a_{11})$.

Let

$$\varphi_1(x) = (n, n, n, m), \quad \varphi_2(x) = (n, n, m, n), \quad \varphi_3(x) = (n, m, n, n),$$

$$\varphi_4(x) = (m, n, n, n), \quad \varphi_5(x) = (n, n, m, m), \quad \text{and} \quad \varphi_6(x) = (n, m, m, n), \quad \text{where} \quad n, m \in \Phi,$$

$$\varphi_7(x) = (1, 2, 2, 3), \quad \varphi_8(x) = (2, 1, 3, 2), \quad \varphi_9(x) = (3, 2, 2, 1), \quad \text{and} \quad \varphi_{10}(x) = (2, 3, 1, 2).$$

Note that weakly periodic ground states depend on the choice of the normal divisor.

Remark 3. All configurations on the set $A_{1,1}$ are ground states. Hence, $\bar{A} = R^3 \backslash A_{1,1}$.

Theorem 2. *For $k = 2$ and $|A| = 2$, the following assertions are true:*

(1.1) if $|n - m| = 1$, then H_A-weakly periodic configurations $\varphi_i(x)$, $i = 1, 2, \ldots, 10$, on the set $A_{0,1}$ are H_A-weakly periodic ground states, which are not periodic or translation-invariant;

(1.2) if $|n - m| = 2$, then H_A-weakly periodic configurations $\varphi_i(x)$, $i = 1, 2, \ldots, 6$, on the set $A_{1,0}$ are H_A-weakly periodic ground states, which are not periodic or translation-invariant;

(2) any H_A-weakly periodic configurations on the set $\bar{A}$, except translation-invariant, periodic, and mentioned in items 1.1 and 1.2, are not H_A-weakly periodic ground states.

Proof. We first prove Assertion 1.1. For $|n - m| = 1$, we consider the weakly periodic configuration $\varphi_1(x)$. Let $c_b \in H_0$. The following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n$, $|B_n| = 3$, $|B_m| = 0$, and therefore, $\varphi_{1b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = n$, $|B_n| = 2$, $|B_m| = 1$, and therefore, $\varphi_{1b} \in C_{0,1}$.

Let $c_b \in H_1$. The following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n$, $|B_n| = 1$, $|B_m| = 2$, and therefore, $\varphi_{1b} \in C_{0,2}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = m$, $|B_n| = 2$, $|B_m| = 1$, and therefore, $\varphi_{1b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = m$, $|B_n| = 1$, $|B_m| = 2$, and therefore, $\varphi_{1b} \in C_{0,1}$.

Hence, on the set

$$A_{0,0} \cap A_{0,1} \cap A_{0,2} = A_{0,1},$$

the configuration $\varphi_1(x)$ is the weakly periodic ground state.

The other cases of Assertion 1.1 are proved similarly.

We now prove Assertion 1.2. For $|n - m| = 2$, we consider the weakly periodic configuration $\varphi_1(x)$. Let $c_b \in H_0$. The following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n$, $|B_n| = 3$, $|B_m| = 0$, and therefore, $\varphi_{1b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = n$, $|B_n| = 2$, $|B_m| = 1$, and therefore, $\varphi_{1b} \in C_{1,0}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n,\ |B_n| = 1,\ |B_m| = 2$, and therefore, $\varphi_{1b} \in C_{2,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 2,\ |B_m| = 1$, and therefore, $\varphi_{1b} \in C_{2,0}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 1,\ |B_m| = 2$, and therefore, $\varphi_{1b} \in C_{1,0}$.

Hence, on the set

$$A_{0,0} \cap A_{1,0} \cap A_{2,0} = A_{1,0},$$

the configuration $\varphi_1(x)$ is a weakly periodic ground state.

The other cases of Assertion 1.2 are proved similarly.

Further, we proceed to the proof of Assertion 2. We now consider H_A-weakly periodic configurations that are not translation-invariant configurations, periodic configurations, or configurations described in items 1.1 and 1.2. Reasoning similarly, we can easily prove that, on the set $\bar{A}$, they are not H_A-weakly periodic ground states.

Theorem 2 is proved.

Remark 4. For $k = 2,\ |A| = k + 1$, the normal divisor of H_A coincides with $G_k^{(2)}$. In a special case, the accumulated results are reduced to the results obtained in [22].

We now consider the case where $k \geq 3$ and $|A| = 1$.

Theorem 3. *For $k \geq 3$ and $|A| = 1$, the following assertions are true:*

(1.1) if $|n - m| = 1$, then the H_A-weakly periodic configurations $\varphi_i(x),\ i = 1, 2, \dots, 10$, on the set $A_{0,1}$ are H_A-weakly periodic ground states that are not periodic or translation-invariant;

(1.2) if $|n - m| = 2$, then the H_A-weakly periodic configurations $\varphi_i(x),\ i = 1, 2, \dots, 6$, on the set $A_{1,0}$ are H_A-weakly periodic ground states that are not periodic or translation-invariant;

(2) any H_A-weakly periodic configurations on the set $\bar{A}$ that are not translation-invariant configurations, periodic configurations, or configurations mentioned in items 1.1 and 1.2 are not H_A-weakly periodic ground states.

Proof. We first prove Assertion 1.1. For $|n - m| = 1$, we consider the weakly periodic configuration $\varphi_1(x)$. Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n,\ |B_n| = k + 1,\ |B_m| = 0$, and therefore, $\varphi_{1b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{1b} \in C_{0,1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{1b} \in C_{0,k}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 2,\ |B_m| = k - 1$, and therefore, $\varphi_{1b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{1b} \in C_{0,1}$.

Hence, on the set

$$A_{0,0} \cap A_{0,1} \cap A_{0,k} \cap A_{0,2} = A_{0,1},$$

the configuration $\varphi_1(x)$ is a weakly periodic ground state.

Consider a weakly periodic configuration $\varphi_2(x)$.
Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{2b}(c_{b\downarrow}) = n$; then $\varphi_{2b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{2b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{2b}(c_{b\downarrow}) = m$; then $\varphi_{2b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{2b} \in C_{0,1}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{2b}(c_{b\downarrow}) = n$; then $\varphi_{2b}(c_b) = m,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{2b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{2b}(c_{b\downarrow}) = n$; then $\varphi_{2b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{2b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{2b}(c_{b\downarrow}) = n$; then $\varphi_{2b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{2b} \in C_{0,1}$.

Hence, on the set

$$A_{0,0} \cap A_{0,1} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_2(x)$ is a weakly periodic ground state.

Consider a weakly periodic configuration $\varphi_3(x)$.
Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{3b}(c_{b\downarrow}) = n$; then $\varphi_{3b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{3b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{3b}(c_{b\downarrow}) = n$; then $\varphi_{3b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{3b} \in C_{0,0}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{3b}(c_{b\downarrow}) = n$; then $\varphi_{3b}(c_b) = m,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{3b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{3b}(c_{b\downarrow}) = m$, then $\varphi_{3b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{3b} \in C_{0,1}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{3b}(c_{b\downarrow}) = n$; then $\varphi_{3b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{3b} \in C_{0,0}$.

Hence, on the set

$$A_{0,1} \cap A_{0,0} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_3(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_4(x)$.
Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{4b}(c_{b\downarrow}) = m$; then $\varphi_{4b}(c_b) = m,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{4b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{4b}(c_{b\downarrow}) = n$; then $\varphi_{4b}(c_b) = m,\ |B_n| = 2,\ |B_m| = k-1$, and therefore, $\varphi_{4b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{4b}(c_{b\downarrow}) = n$; then $\varphi_{4b}(c_b) = n,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{4b} \in C_{0,k}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{4b}(c_{b\downarrow}) = m$; then $\varphi_{4b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{4b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{4b}(c_{b\downarrow}) = n$, then $\varphi_{4b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{4b} \in C_{0,0}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k} \cap A_{0,0} = A_{0,1},$$

the configuration $\varphi_4(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_5(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{5b}(c_{b\downarrow}) = n$; then $\varphi_{5b}(c_b) = n,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{5b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{5b}(c_{b\downarrow}) = m$; then $\varphi_{5b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{5b} \in C_{0,1}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{5b}(c_{b\downarrow}) = m$; then $\varphi_{5b}(c_b) = m,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{5b} \in C_{0,k}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{5b}(c_{b\downarrow}) = n$; then $\varphi_{5b}(c_b) = n,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{5b} \in C_{0,k}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{5b}(c_{b\downarrow}) = n$; then $\varphi_{5b}(c_b) = m,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{5b} \in C_{0,1}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{5b}(c_{b\downarrow}) = m$; then $\varphi_{5b}(c_b) = m,\ |B_n| = 0,\ |B_m| = k+1$, and therefore, $\varphi_{5b} \in C_{0,0}$.

Hence, on the set

$$A_{0,0} \cap A_{0,1} \cap A_{0,k} = A_{0,1},$$

the configuration $\varphi_5(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_6(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{6b}(c_{b\downarrow}) = n$; then $\varphi_{6b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{6b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{6b}(c_{b\downarrow}) = m$; then $\varphi_{6b}(c_b) = n,\ |B_n| = k-1,\ |B_m| = 2$, and therefore, $\varphi_{6b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{6b}(c_{b\downarrow}) = n$; then $\varphi_{6b}(c_b) = m,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{6b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{6b}(c_{b\downarrow}) = n$; then $\varphi_{6b}(c_b) = m,\ |B_n| = k+1,\ |B_m| = 0$, and therefore, $\varphi_{6b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{6b}(c_{b\downarrow}) = m$; then $\varphi_{6b}(c_b) = n,\ |B_n| = k-1,\ |B_m| = 2$, and therefore, $\varphi_{6b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{6b}(c_{b\downarrow}) = n$; then $\varphi_{6b}(c_b) = n,\ |B_n| = k,\ |B_m| = 1$, and therefore, $\varphi_{6b} \in C_{0,1}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_6(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_7(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{7b}(c_{b\downarrow}) = 1$; then $\varphi_{7b}(c_b) = 1,\ |B_1| = k,\ |B_2| = 1,\ |B_3| = 0$, and therefore, $\varphi_{7b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{7b}(c_{b\downarrow}) = 2$; then $\varphi_{7b}(c_b) = 1,\ |B_1| = k-1,\ |B_2| = 2,\ |B_3| = 0$, and therefore, $\varphi_{7b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{7b}(c_{b\downarrow}) = 3$; then $\varphi_{7b}(c_b) = 2$, $|B_1| = k$, $|B_2| = 0$, $|B_3| = 1$, and therefore, $\varphi_{7b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{7b}(c_{b\downarrow}) = 1$; then $\varphi_{7b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = 0$, $|B_3| = k$, and therefore, $\varphi_{7b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{7b}(c_{b\downarrow}) = 2$; then $\varphi_{7b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = 2$, $|B_3| = k-1$, and therefore, $\varphi_{7b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{7b}(c_{b\downarrow}) = 3$; then $\varphi_{7b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = 1$, $|B_3| = k$, and therefore, $\varphi_{7b} \in C_{0,1}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_7(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_8(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{8b}(c_{b\downarrow}) = 2$; then $\varphi_{8b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k$, $|B_3| = 0$, and therefore, $\varphi_{8b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{8b}(c_{b\downarrow}) = 3$; then $\varphi_{8b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k-1$, $|B_3| = 1$, and therefore, $\varphi_{8b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{8b}(c_{b\downarrow}) = 2$; then $\varphi_{8b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = k+1$, $|B_3| = 0$, and therefore, $\varphi_{8b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{8b}(c_{b\downarrow}) = 2$; then $\varphi_{8b}(c_b) = 1$, $|B_1| = 0$, $|B_2| = k+1$, $|B_3| = 0$, and therefore, $\varphi_{8b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{8b}(c_{b\downarrow}) = 1$; then $\varphi_{8b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k-1$, $|B_3| = 1$, and therefore, $\varphi_{8b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{8b}(c_{b\downarrow}) = 2$; then $\varphi_{8b}(c_b) = 2$, $|B_1| = 0$, $|B_2| = k$, $|B_3| = 1$, and therefore, $\varphi_{8b} \in C_{0,1}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_8(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_9(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{9b}(c_{b\downarrow}) = 3$; then $\varphi_{9b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = 1$, $|B_3| = k$, and therefore, $\varphi_{9b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{9b}(c_{b\downarrow}) = 2$; then $\varphi_{9b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = 2$, $|B_3| = k-1$, and therefore, $\varphi_{9b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{9b}(c_{b\downarrow}) = 1$; then $\varphi_{9b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = 0$, $|B_3| = k$, and therefore, $\varphi_{9b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{9b}(c_{b\downarrow}) = 3$; then $\varphi_{9b}(c_b) = 2$, $|B_1| = k$, $|B_2| = 0$, $|B_3| = 1$, and therefore, $\varphi_{9b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{9b}(c_{b\downarrow}) = 2$; then $\varphi_{9b}(c_b) = 1$, $|B_1| = k-1$, $|B_2| = 2$, $|B_3| = 0$, and therefore, $\varphi_{9b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{9b}(c_{b\downarrow}) = 1$, then $\varphi_{9b}(c_b) = 1$, $|B_1| = k$, $|B_2| = 1$, $|B_3| = 0$, and therefore, $\varphi_{9b} \in C_{0,1}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_9(x)$ is a weakly periodic ground state.

Consider the weakly periodic configuration $\varphi_{10}(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{10b}(c_{b\downarrow}) = 2$; then $\varphi_{10b}(c_b) = 2$, $|B_1| = 0$, $|B_2| = k$, $|B_3| = 1$, and therefore, $\varphi_{10b} \in C_{0,1}$;

(b) $c_{b\downarrow} \in H_0$ and $\varphi_{10b}(c_{b\downarrow}) = 1$; then $\varphi_{10b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k-1$, $|B_3| = 1$, and therefore, $\varphi_{10b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{10b}(c_{b\downarrow}) = 2$; then $\varphi_{10b}(c_b) = 1$, $|B_1| = 0$, $|B_2| = k+1$, $|B_3| = 0$, and therefore, $\varphi_{10b} \in C_{0,k+1}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{10b}(c_{b\downarrow}) = 2$; then $\varphi_{10b}(c_b) = 3$, $|B_1| = 0$, $|B_2| = k+1$, $|B_3| = 0$, and therefore, $\varphi_{10b} \in C_{0,k+1}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{10b}(c_{b\downarrow}) = 3$; then $\varphi_{10b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k-1$, $|B_3| = 1$, and therefore, $\varphi_{10b} \in C_{0,2}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{10b}(c_{b\downarrow}) = 2$; then $\varphi_{10b}(c_b) = 2$, $|B_1| = 1$, $|B_2| = k$, $|B_3| = 0$, and therefore, $\varphi_{10b} \in C_{0,1}$.

Hence, on the set

$$A_{0,1} \cap A_{0,2} \cap A_{0,k+1} = A_{0,1},$$

the configuration $\varphi_{10}(x)$ is a weakly periodic ground state.

We now prove Assertion 1.2. For $|n-m| = 2$, we consider the weakly periodic configuration $\varphi_1(x)$.

Let $c_b \in H_0$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n$, $|B_n| = k+1$, $|B_m| = 0$, and therefore, $\varphi_{1b} \in C_{0,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = n$, $|B_n| = k$, $|B_m| = 1$, and therefore, $\varphi_{1b} \in C_{1,0}$.

Let $c_b \in H_1$. Then the following cases are possible:

(a) $c_{b\downarrow} \in H_0$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = n,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{1b} \in C_{k,0}$;

(b) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = n$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 2,\ |B_m| = k-1$, and therefore, $\varphi_{1b} \in C_{2,0}$;

(c) $c_{b\downarrow} \in H_1$ and $\varphi_{1b}(c_{b\downarrow}) = m$; then $\varphi_{1b}(c_b) = m,\ |B_n| = 1,\ |B_m| = k$, and therefore, $\varphi_{1b} \in C_{1,0}$.

Hence, on the set

$$A_{0,0} \cap A_{1,0} \cap A_{k,0} \cap A_{2,0} = A_{1,0},$$

the configuration $\varphi_1(x)$ is a weakly periodic ground state.

The other cases of Assertion 1.2 are proved similarly.

We now proceed to the proof of Assertion 2. We consider H_A-weakly periodic configurations other than translation-invariant configurations, periodic configurations, or configurations described in items 1.1 and 1.2. Reasoning similarly, we can easily prove that they are not H_A-weakly periodic ground states on the set $\bar{A}$.

Theorem 3 is proved.

Theorem 4. *For $k \geq 3$ and $|A| = k$, the following assertions are true:*

(1.1) if $|n-m| = 1$, then the H_A-weakly periodic configurations $\varphi_i(x),\ i = 1, 2, \ldots, 10$, on the set $A_{0,1}$ are H_A-weakly periodic ground states that are not periodic or translation-invariant;

(1.2) if $|n-m| = 2$, then the H_A-weakly periodic configurations $\varphi_i(x),\ i = 1, 2, \ldots, 6$, on the set $A_{1,0}$ are H_A-weakly periodic ground states that are not periodic or translation-invariant;

(2) any H_A-weakly periodic configurations other than translation-invariant configurations, periodic configurations , and configurations indicated in items 1.1 and 1.2 are not H_A-weakly periodic ground states on the set $\bar{A}$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.

The authors express their deep gratitude to Prof. U. A. Rozikov for useful discussions stimulating the improvement of the present paper.

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