DOI 10.1007/s11253-020-01824-8
Ukrainian Mathematical Journal, Vol. 72, No. 5, October, 2020 (Ukrainian Original Vol. 72, No. 5, May, 2020)

ON A POLETSKII-TYPE INEQUALITY FOR MAPPINGS OF THE RIEMANNIAN SURFACES

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We establish upper estimates for the distortion of the modulus of families of curves under mappings from the Sobolev class whose dilatation is locally integrable. As a consequence, we prove theorems on the local and boundary behaviors of these mappings.

1. Introduction

The present paper is devoted to the investigation of mappings with bounded and finite distortion, which are now extensively studied (see, e.g., [1–3]). In particular, Riemannian surfaces of the hyperbolic type were considered in [4, 5]. Note that the estimates for distortions of the modulus under mappings are of primary importance (see, e.g., [1], Sec. 2.3, [2], Sec. 4.1, [3], Definition 13.1 and [6], Theorem 3.1). Indeed, these estimates enable one to study some fundamental properties of these mappings (see [3], Theorems 17.13 and 17.15, [6], Theorem 4.2, and [7], Theorems 3.6 and 3.7). The main aim of the present paper is to establish an upper estimate for the distortion of the modulus of families of curves under mappings and to study their boundary behavior on the basis of this estimate (see the last section). In this connection, it is necessary to mention the classical Poletskii inequality obtained in [8] (Theorem 1) for mappings of the Euclidean space with bounded characteristic (see also [2], Theorems 8.1 and 8.5). In the present paper, we consider mappings of the Riemannian surfaces in the case where the characteristic of quasiconformality can be unbounded.

We now recall some definitions. We define a *Riemannian surface* as a two-dimensional manifold with countable base in which the mapping of transition between the corresponding maps is conformal [4]. Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$ considered in what follows are surfaces of the *hyperbolic type*, i.e., conformally equivalent to the quotient spaces $\mathbb{D}/G$ and $\mathbb{D}/G_*$, respectively, where $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ and G and G_* are certain discontinuous groups of linear-fractional automorphisms of a unit disk without fixed points. By the Klein–Poincaré theorem on factorization (see [9], Theorem 6.I), we can identify the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$ with the corresponding quotient spaces $\mathbb{D}/G$ and $\mathbb{D}/G_*$ of the indicated form. Hence, we assume that $\mathbb{S}=D/G$ and $\mathbb{S}_*=D/G_*$, where G and G_* are groups of linear-fractional automorphisms of the unit disk without fixed points whose action is discontinuous in $\mathbb{D}$. Recall that G *discontinuously* acts in $\mathbb{D}$ if every point $x\in\mathbb{D}$ has a neighborhood U such that $g(U)\cap U=\varnothing$ for all $g\in G,\ g\neq I$, except finitely many elements g, where I is the identity mapping. We also say that G does not have *fixed points* in $\mathbb{D}$ if, for every $a\in\mathbb{D}$, the equality $g(a)=a$ is true only in the case where $g=I$.

Recall that each element p_0 of the quotient space $\mathbb{D}/G$ is an *orbit* of the point $z_0\in\mathbb{D}$, i.e.,

$$p=\{z\in\mathbb{D}\colon\ z=g(z_0),\ g\in G\}.$$

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 72, No. 5, pp. 705–720, May, 2020. Original article submitted January 3, 2020.

 0041-5995/20/7205–0816

Everywhere in what follows, in the unit disk $\mathbb{D}$, we use the so-called *hyperbolic metric*

$$h(z_1, z_2) = \log \frac{1+t}{1-t}, \quad t = \frac{|z_1 - z_2|}{|1 - z_1\overline{z_2}|}, \tag{1}$$

the *hyperbolic area* of the set $S \subset \mathbb{D}$, and the length of the curve $\gamma\colon [a,b] \to \mathbb{D}$ given, respectively, by the following formulas:

$$v(S) = \int\limits_S \frac{4\,dm(z)}{(1-|z|^2)^2}, \quad z = x + iy, \qquad \text{and} \qquad s_h(\gamma) := \sup_\pi \sum_{k=0}^{n-1} h(\gamma(t_k), \gamma(t_{k+1})), \tag{2}$$

where h is given by (1) and the supremum is taken over all partitions

$$\pi = \{a = t_0 \le t_1 \le t_2 \le \ldots \le t_n = b\}$$

$\{$see [4], relations (2.4) and (2.5)$\}$. We can directly show that the hyperbolic metric, length, and area are invariant under linear-fractional mappings of the unit disk onto itself.

For a point $y_0 \in \mathbb{D}$ and a number $r \ge 0$, we define a *hyperbolic disk* $B_h(y_0, r)$ and a *hyperbolic circle* $S_h(y_0, r)$ by the formulas

$$B_h(y_0, r) := \big\{y \in \mathbb{D}\colon\ h(y_0, y) < r\big\} \qquad \text{and} \qquad S_h(y_0, r) := \big\{y \in \mathbb{D}\colon\ h(y_0, y) = r\big\},$$

respectively. The Riemannian surfaces can be metrized as follows: If $p_1, p_2 \in \mathbb{D}/G$, then we set

$$\widetilde{h}(p_1, p_2) := \inf_{g_1, g_2 \in G} h(g_1(z_1), g_2(z_2)), \tag{3}$$

where

$$p_i = G_{z_i} = \big\{\xi \in \mathbb{D}\colon\ \exists g \in G\colon\ \xi = g(z_i)\big\}, \qquad i = 1, 2.$$

In the last case, we say that the set G_{z_i} is an *orbit* of the point z_i and p_1 and p_2 are *orbits* of the points z_1 and z_2, respectively. Further,

$$\widetilde{B}(p_0, r) := \big\{p \in \mathbb{S}\colon\ \widetilde{h}(p_0, p) < r\big\} \qquad \text{and} \qquad \widetilde{S}(p_0, r) := \big\{p \in \mathbb{S}\colon\ \widetilde{h}(p_0, p) = r\big\}$$

are, respectively, the disk and the circle centered at the point p_0 of the surface $\mathbb{S}$. Here and in what follows, $B(z_0, r)$ and $S(z_0, r)$ denote the disk and the circle centered at the point $z_0 \in \mathbb{C}$ of the plane, respectively.

To simplify our investigations, we introduce a so-called *fundamental set* F. It is defined as a subset of $\mathbb{D}$ that contains one (and only one) point of the orbit $z \in G_{z_0}$ (see [10], Chap. 9, Sec. 9.1). A *fundamental domain* D_0 is defined as a domain in $\mathbb{D}$ such that $v(\partial D_0) = 0$ with the following property: $D_0 \subset F \subset \overline{D_0}$ (see the same citation). As the most interesting example of fundamental domain, we can mention a *Dirichlet polygon:*

$$D_\zeta = \bigcap_{g \in G, g \ne I} H_g(\zeta),$$

where

$$H_g(\zeta) = \{z \in \mathbb{D} \colon h(z, \zeta) < h(z, g(\zeta))\}$$

{see [4], relation (2.6)}.

Let π be the natural projection of $\mathbb{D}$ onto $\mathbb{D}/G$. Then π is an analytic function conformal on D_0 {see also Proposition 9.2.2 in [10] and the comment after relation (2.11) in [4]}. In addition, we note that there exists a one-to-one correspondence between the points of F and $\mathbb{D}/G$ and, hence, between the points of F and $\mathbb{S}$. In particular, for a measurable set $E \subset \mathbb{D}/G$, we set

$$\widetilde{v}(E) := v\big(\pi^{-1}(E)\big), \tag{4}$$

where v is the hyperbolic measure in the unit disk with an elementary area

$$dv(z) = \frac{4\, dm(z)}{(1 - |z|^2)^2}$$

and m is the plane Lebesgue measure. Here and in what follows, we say the set $E \subset \mathbb{D}/G$ is measurable if $\pi^{-1}(E)$ is measurable in $\mathbb{D}$ with respect to the Lebesgue measure. Similarly, we can define the Borel set $E \subset \mathbb{D}/G$.

Let D and D_* be domains on the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. By $\widetilde{h}$ we denote a metric on the Riemannian surface $\mathbb{S}$ and by $\widetilde{h_*}$ we denote a metric on the Riemannian surface $\mathbb{S}_*$. Elements of length and volume on the surfaces $\mathbb{S}$ and $\mathbb{S}_*$ are denoted by $ds_{\widetilde{h}}$, $d\widetilde{v}$ and $ds_{\widetilde{h_*}}$, $d\widetilde{v_*}$, respectively. A mapping $f \colon D \to D_*$ is called *discrete* if the preimage $f^{-1}(y)$ of every point $y \in D_*$ consists solely of isolated points and *open* if the image of any open set $U \subset D$ is an open set in D_*. For the definitions of mappings of the Sobolev class $W^{1,1}_{\rm loc}$ on the Riemannian surface, see, e.g., [4]. Further, for mappings $f \colon D \to D_*$ from the class $W^{1,1}_{\rm loc}$ in local coordinates, we have

$$f_{\overline{z}} = (f_x + i f_y)/2 \quad \text{and} \quad f_z = (f_x - i f_y)/2, \quad z = x + iy.$$

In addition, the *norm* and *Jacobian* of the mapping f in local coordinates are given by the expressions

$$\|f'(z)\| = |f_z| + |f_{\overline{z}}| \qquad \text{and} \qquad J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2,$$

respectively. We say that $f \in W^{1,2}_{\rm loc}(D)$ if $f \in W^{1,2}_{\rm loc}$ and, in addition, in local coordinates, $\|f'(z)\| \in L^2_{\rm loc}(D)$. In what follows, we always assume that an almost everywhere differentiable mapping f has a nonnegative Jacobian almost everywhere in the local coordinates. The *dilatation of order* p for the mapping f at a point z is given by the formula

$$K_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \quad \text{for} \quad J_f(z) \neq 0,$$

$$K_f(z) = 1 \quad \text{for} \quad \|f'(z)\| = 0, \qquad \text{and} \qquad K_f(z) = \infty, \quad \textit{otherwise}.$$

We can easily show that the quantity $K_f(z)$ is independent of local coordinates. As a rule, a curve γ on the Riemannian surface $\mathbb{S}$ is defined as a continuous mapping $\gamma \colon I \to \mathbb{S}$, where I is either the finite segment $[a, b]$, or the interval (a, b), or one of the half intervals $[a, b)$ or $(a, b]$ of the numerical straight line. According to [11] (Sec. 7.1) (see also [3], Theorem 2.4), an arbitrary rectifiable curve $\gamma \colon I \to \mathbb{C}$ (respectively, $\gamma \colon I \to \mathbb{S}$) admits a parametrization $\gamma(t)=(\gamma^0\circ l_\gamma)(t)$, where l_γ denotes the length of the curve γ on the segment $[a,t]$. Depending on the context, this length can be understood either in the Euclidean sense, or in the hyperbolic sense, or in the sense of Riemannian surface. In this case, the curve $\gamma^0\colon [0,l(\gamma)]\to\mathbb{C}$ $\big($respectively, $\gamma^0\colon [0,l(\gamma)]\to\mathbb{S}\big)$ is unique and called a *normal representation* of the curve γ. For a locally rectifiable curve $\gamma\colon [a,b]\to\mathbb{D}$, we set

$$\int\limits_{\alpha}\rho(x)\,ds_h(x)=\int\limits_{0}^{l(\gamma)}\rho(\alpha^0(s))\,ds.$$

Similarly, for a locally rectifiable curve $\gamma\colon [a,b]\to\mathbb{D}/G$, we denote

$$\int\limits_{\alpha}\rho(p)\,ds_{\widetilde{h}}(p)=\int\limits_{0}^{l(\gamma)}\rho(\alpha^0(s))\,ds.$$

Let Γ be a family of curves in $\mathbb{S}$. A Borel function $\rho\colon \mathbb{S}\to[0,\infty]$ is called *admissible* for the family Γ of curves γ if

$$\int\limits_{\gamma}\rho(p)\,ds_{\widetilde{h}}(p)\geq 1$$

for any (locally rectifiable) curve $\gamma\in\Gamma$. In the concise form, we can write this as follows: $\rho\in\operatorname{adm}\Gamma$. The *modulus* of the family Γ is defined as

$$M(\Gamma):=\inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{S}}\rho^2(p)\,d\widetilde{v}(p).$$

Let $\Delta\subset\mathbb{R}$ be an open interval of the real axis and let $\gamma\colon \Delta\to\mathbb{S}$ be a locally rectifiable curve. In this case, it is clear that there exists a unique nondecreasing function of length $l_\gamma\colon \Delta\to\Delta_\gamma\subset\mathbb{R}$ satisfying the condition $l_\gamma(t_0)=0,\ t_0\in\Delta$, and such that the value $l_\gamma(t)$ is equal to the length of a subcurve $\gamma\,|_{[t_0,t]}$ of the curve γ for $t>t_0$ and to the length of the subcurve $\gamma\,|_{[t,t_0]}$ taken with the negative sign for $t<t_0,\ t\in\Delta$. Let $g\colon |\gamma|\to\mathbb{S}_*$ be a continuous mapping, where $|\gamma|=\gamma(\Delta)\subset\mathbb{S}$. Assume that the curve $\widetilde{\gamma}=g\circ\gamma$ is also locally rectifiable. Then it is clear that there exists a unique nondecreasing function $L_{\gamma,g}\colon \Delta_\gamma\to\Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}(l_\gamma(t))=l_{\widetilde{\gamma}}(t)$ for all $t\in\Delta$. If the curve γ is given on the segment $[a,b]$ or on the half interval $[a,b)$, then we assume that $a=t_0$. The curve γ is called the (*complete*) *lifting of the curve* $\widetilde{\gamma}$ *under the mapping* $f\colon D\to\mathbb{S}_*$ if $\widetilde{\gamma}=f\circ\gamma$.

The following definition can be also found in [2] (Sec. 8.4) or [3] (Definition 5.2): We say that a mapping $f\colon D\to\mathbb{R}^n$ belongs to the class ACP in the domain D (*is absolutely continuous on almost all curves* in the domain D and we write $f\in ACP$) if, for almost all curves γ in the domain D, the curve $\widetilde{\gamma}=f\circ\gamma$ is locally rectifiable and, in addition, the function of length $L_{\gamma,f}$ introduced above is absolutely continuous on all segments lying in Δ_γ. Here and in what follows, a property P holds on almost all curves if the modulus of the family of curves for which this property is violated is equal to zero.

Assume that $f\colon D\to\mathbb{S}_*$ is such that none of the curves $\alpha\subset D$ is mapped into a point under the mapping f. Then the function $L^{-1}_{\gamma,f}$ can be correctly defined. In this case, we say that f has the *property* ACP^{-1} in the domain $D\subset\mathbb{S}$ (and write $f\in ACP^{-1}$) if, for almost all curves $\widetilde{\gamma}\in f(D)$, every lifting γ of the curve $\widetilde{\gamma}$ under the mapping $f,\ f\circ\gamma=\widetilde{\gamma}$, is a locally rectifiable curve and, in addition, the inverse function $L^{-1}_{\gamma,f}$ is absolutely continuous on all segments lying in $\Delta_{\widetilde{\gamma}}$ for almost all curves $\widetilde{\gamma}$ in $f(D)$ and all liftings γ of the curve $\widetilde{\gamma} = f \circ \gamma$. Note that if f is a homeomorphism such that $f^{-1} \in W^{1,2}_{\rm loc}\big(f(D)\big)$, then it always belongs to the class ACP^{-1} (see [3], Theorem 28.2). We say that the mapping f possesses the *Luzin N-property* if $\widetilde{v_*}\big(f(E)\big) = 0$ for any $E \subset D$ such that $\widetilde{v}(E) = 0$. Similarly, we say that the mapping f has the *Luzin N^{-1}-property* if $\widetilde{v}\big(f^{-1}(E_*)\big) = 0$ for any $E_* \subset D_*$ such that $\widetilde{v_*}(E_*) = 0$. The following assertion is true (see also [4], Lemma 3.1):

Theorem 1. *Suppose that D and D_* are domains on the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and moreover, $\overline{D}$ and $\overline{D_*}$ are compact sets. Assume that f is an almost everywhere differentiable mapping of the domain D onto D_* that belongs to the class ACP^{-1} and possesses the Luzin N- and N^{-1}-properties. Then, for each family of (locally rectifiable) curves Γ in the domain D and each admissible function $\rho \in \operatorname{adm}\Gamma$, the following inequality is true:*

$$M(f(\Gamma)) \le \int\limits_D K_f(p)\rho^2(p)\, d\widetilde{v}(p). \tag{5}$$

In view of Theorem 28.2 in [3] and Corollary B in [12], we get the following corollary:

Corollary 1. *Let D and D_* be domains of the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and let, in addition, $\overline{D}$ and $\overline{D_*}$ be compact sets. Also let f be a mapping of the domain D onto D_* such that $f \in W^{1,2}_{\rm loc}(D)$ and $f^{-1} \in W^{1,2}_{\rm loc}\big(f(D)\big)$. Then relation (5) is true.*

2. Preliminary Remarks

Prior to the formulation of auxiliary statements and proving our main results, we make some important remarks. Assume that F is a fundamental set and D_0 is a fundamental domain (see the remarks made in the introduction). For $z_1, z_2 \in F$, we set

$$d(z_1, z_2) := \widetilde{h}\big(\pi(z_1), \pi(z_2)\big),$$

where $\widetilde{h}$ is given in (3). Note that, by definition, $d(z_1, z_2) \le h(z_1, z_2)$. We show that, for any compact set $A \subset \mathbb{D}$, there exists $\delta = \delta(A) > 0$ such that

$$d(z_1, z_2) = h(z_1, z_2) \quad \forall z_1, z_2 \in A, \quad h(z_1, z_2) < \delta. \tag{6}$$

Assume the contrary. Then, for any $k \in \mathbb{N}$, there exist complex numbers $x_k, z_k \in A$ such that $h(z_k, x_k) < 1/k$ and, moreover, $d(z_k, x_k) < h(z_k, x_k)$. Thus, by the definition of the metric d and invariance of the metric h under linear-fractional mappings of a unit disk onto itself, one can find $g_k \in G$ such that

$$d(z_k, x_k) \le h\big(z_k, g_k(x_k)\big) < h(z_k, x_k) < 1/k, \qquad g_k \in G, \quad k = 1, 2, \ldots . \tag{7}$$

Since A is a compact set in $\mathbb{D}$, we can assume that $x_k, z_k \to x_0 \in \mathbb{D}$ as $k \to \infty$. Thus, by the triangle inequality, it follows from (7) that

$$h\big(g_k(x_k), x_0\big) \le h\big(g_k(x_k), z_k\big) + h(z_k, x_0) \to 0$$

as $k \to \infty$ and, hence, $h(x_k, g_k^{-1}(x_0)) \to 0$ as $k \to \infty$ because the metric h is invariant under the linear-fractional mapping. However, by the triangle inequality, we also find

$$h\big(g_k^{-1}(x_0), x_0\big) \le h\big(g_k^{-1}(x_0), x_k\big) + h(x_k, x_0) \to 0 \quad \text{as} \quad k \to \infty,$$

which contradicts the discontinuity of the group G in $\mathbb{D}$ because the condition $g(U) \cap U = \varnothing$ is not satisfied for an arbitrarily small neighborhood U of the point x_0 and in the case of infinitely many elements $g \in G$. This proves (6).

The following lemma is true:

Lemma 1. *Suppose that* $0 < 2r_0 < 1$. *Then there exists a constant* $C_1 = C_1(r_0) > 0$ *such that*

$$C_1 h(z_1, z_2) \le |z_1 - z_2| \le h(z_1, z_2) \quad \forall z_1, z_2 \in B(0, r_0). \tag{8}$$

Moreover, the right inequality in (8) is true for all $z_1, z_2 \in \mathbb{D}$.

Proof. Note that, by the triangle inequality, $0 < |z_1 - z_2| < 2r_0$. Therefore, $r := |z_1 - z_2|$ varies from 0 to $2r_0 < 1$. Recall that

$$h(z_1, z_2) = \log \frac{1 + \dfrac{|z_1 - z_2|}{|1 - z_1 \overline{z_2}|}}{1 - \dfrac{|z_1 - z_2|}{|1 - z_1 \overline{z_2}|}}.$$

Denoting $r = |z_1 - z_2|$, we can write

$$h(z_1, z_2) \ge \log \frac{1 + r/2}{1 - r/2}.$$

We also note that

$$h(z_1, z_2) \ge \log \frac{1 + r/2}{1 - r/2} \ge r, \quad r \in (0, 1). \tag{9}$$

Indeed, the function

$$\varphi(r) = \log \frac{1 + r/2}{1 - r/2} - r$$

increases in $r \in [0, 1]$, which can be verified by taking the derivative. Thus, its minimum is attained for $r = 0$, i.e., $\varphi(r) \ge 0$ for all $r \in (0, 1)$ and inequality (9) holds.

We now establish the left inequality in (8). To this end, we note that

$$h(z_1, z_2) \le \log \frac{1 - r_0^2 + r}{1 - r_0^2 - r}, \qquad \log \frac{1 - r_0^2 + r}{1 - r_0^2 - r} \sim \frac{2}{1 - r_0^2} r \quad \text{as} \quad r \to 0.$$

Thus, for some $0 < r_1 < r_0$ and $M = M(r_0)$, we get

$$h(z_1, z_2) \le \log \frac{1 - r_0^2 + r}{1 - r_0^2 - r} \le Mr, \quad r \in (0, r_1).$$

For $r \in [0, 2r_0]$, the function $1 - r_0^2 - r$ is strictly positive in r. Hence, the function

$$\frac{1}{r} \log \frac{1 - r_0^2 + r}{1 - r_0^2 - r}$$

is continuous on $r \in [r_1, 2r_0]$ and, therefore, it is bounded for the same r with a certain constant $\widetilde{C}$. Setting

$$C_1^{-1} := \max\big\{M, \widetilde{C}\big\},$$

we get

$$h(z_1, z_2) \le \log \frac{1 - r_0^2 + r}{1 - r_0^2 - r} \le C_1^{-1} r = C_1^{-1} |z_1 - z_2| \quad \forall z_1, z_2 \in B(0, r_0).$$

Lemma 1 is proved.

We introduce the following important statement that generalizes Theorem 1.3(5) in [3]:

Lemma 2. *Suppose that the curve $\alpha\colon [a, b] \to \mathbb{D}$ is rectifiable in a sense of the hyperbolic length s_h in (2) and, moreover, $s_h = s_h(t)$ denotes the hyperbolic length of the curve α determined on the segment $[a, t]$, $a \le t \le b$. Then $\alpha'(t)$ and $s_h'(t)$ exist for almost all $t \in [a, b]$ and, in addition,*

$$\frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2} = s_h'(t) \tag{10}$$

for almost all $t \in [a, b]$.

Proof. The function $s_h = s_h(t)$ is monotone and, hence, almost everywhere differentiable. Furthermore, since $\alpha(t)$ is rectifiable, there exists $0 < r_0 < 1$ such that $\alpha(t) \in B(0, r_0)$ for all $t \in [a, b]$. Thus, by Lemma 1, the curve α is also rectifiable in the Euclidean sense. Therefore, it has a bounded variation and, hence, is also differentiable almost everywhere.

To establish equality (10), we follow the scheme used in the proof of Theorem 1.3(5) in [3]. First, by the definition of hyperbolic length of a curve in (2), we can write

$$\frac{h\big(\alpha(t), \alpha(t_0)\big)}{|t - t_0|} \le \frac{\big|s_h(t) - s_h(t_0)\big|}{|t - t_0|}. \tag{11}$$

Multiplying the numerator and denominator of relation (11) by $\big|\alpha(t) - \alpha(t_0)\big|$, we find

$$\frac{\big|\alpha(t) - \alpha(t_0)\big|}{\big|\alpha(t) - \alpha(t_0)\big|} \frac{h\big(\alpha(t), \alpha(t_0)\big)}{|t - t_0|} \le \frac{\big|s_h(t) - s_h(t_0)\big|}{|t - t_0|}. \tag{12}$$

We study the behavior of the function

$$\varphi(t) = \frac{h(\alpha(t), \alpha(t_0))}{\big|\alpha(t) - \alpha(t_0)\big|} \quad \text{as} \quad t \to t_0.$$

Since

$$\log\frac{1+x}{1-x}\sim 2x \quad \text{as} \quad x\to 0,$$

we get

$$\varphi(t)=\log\left(\frac{1+\dfrac{|\alpha(t)-\alpha(t_0)|}{|1-\alpha(t)\overline{\alpha(t_0)}|}}{1-\dfrac{|\alpha(t)-\alpha(t_0)|}{|1-\alpha(t)\overline{\alpha(t_0)}|}}\right)\frac{1}{|\alpha(t)-\alpha(t_0)|}\sim\frac{2|\alpha(t)-\alpha(t_0)|}{|1-\alpha(t)\overline{\alpha(t_0)}|}\frac{1}{|\alpha(t)-\alpha(t_0)|}$$

as $t\to t_0$. Then

$$\varphi(t)\to\frac{2}{1-|\alpha(t_0)|^2} \quad \text{as} \quad t\to t_0.$$

In this case, passing to the limit in (12) as $t\to t_0,$ we obtain

$$\frac{2|\alpha'(t_0)|}{1-|\alpha(t_0)|^2}\le s_h'(t_0) \tag{13}$$

for almost all $t_0\in[a,b]$.

To complete the proof, it remains to establish the inequality opposite to (13). By A we denote the set of all points of the segment $[a,b]$ for which $\alpha'(t)$ and $s_h'(t)$ exist and, moreover,

$$\frac{2|\alpha'(t_0)|}{1-|\alpha(t_0)|^2}< s_h'(t_0).$$

Let A_k be the set of all points $t\in A$ such that, for any $a\le p\le t\le q\le b,\ 0<q-p<1/k,$ the inequality

$$\frac{s_h(q)-s_h(p)}{q-p}\ge\frac{h\big(\alpha(q),\alpha(p)\big)}{q-p}+1/k$$

holds. By using the definitions of the sets A and $A_k,$ we can show that

$$A=\bigcup_{k=1}^{\infty}A_k$$

(see the proof of Theorem 1.3(5) in [3]). To complete the proof, it suffices to show that $m_1(A_k)=0$ for any $k=1,2,\ldots,$ where m_1 is the Lebesgue measure on $\mathbb{R}^1$.

By $l(\alpha)$ we denote the length of the curve α. For any $\varepsilon>0,$ we split the segment $[a,b]$ by points $a\le t_0\le t_1\le t_2\le\ldots\le t_m=b$ so that

$$l(\alpha)\le\sum_{k=1}^{m}h\big(\alpha(t_k),\alpha(t_{k-1})\big)+\varepsilon/k$$

and $t_j - t_{j-1} < 1/k$ for all $j = 1, 2, \ldots, m$. If $[t_{j-1}, t_j] \cap A_k \neq \varnothing$, then, by the definition of the sets A_k, we find

$$s_h(t_j) - s_h(t_{j-1}) \geq h\big(\alpha(t_j), \alpha(t_{j-1})\big) + (t_j - t_{j-1})/k.$$

Denoting $\Delta_j := [t_{j-1}, t_j]$, we get

$$\begin{aligned} m_1(A_k) &\leq \sum_{\Delta_j \cap A_k \neq \varnothing} m_1(\Delta_j) \\ &\leq k \sum_{j=1}^{m} \big(s_h(t_j) - s_h(t_{j-1}) - h(\alpha(t_j), \alpha(t_{j-1}))\big) \\ &\leq k \left(l(\alpha) - \sum_{j=1}^{m} h(\alpha(t_j), \alpha(t_{j-1})) \right) \leq \varepsilon. \end{aligned}$$

The last relation proves the equality $m_1(A_k) = 0$. Hence, in view of the fact that

$$A = \bigcup_{k=1}^{\infty} A_k,$$

we obtain $m_1(A) = 0$, Q.E.D.

Lemma 2 is proved.

Reasoning as in the proof of item (4) of Theorem 1.3 from [3], we can show that the curve $\gamma\colon I \to \mathbb{D}$ is absolutely continuous if and only if its function of length $s_h(t)$ is absolutely continuous. Thus, if γ is absolutely continuous, then, as a result of the change of variables, we find

$$\int_{\gamma} \rho(x)\, ds_h(x) = \int_{0}^{l(\gamma)} \rho(\gamma^0(s))\, ds = \int_{a}^{b} \rho(\gamma(t)) s_h'(t)\, dt.$$

Hence, we obtain the following statement from Lemma 2:

Corollary 2. *Let $\alpha\colon [a, b] \to \mathbb{D}$ be an absolutely continuous curve and let $\rho\colon \mathbb{D} \to \mathbb{R}$ be a nonnegative Borel function. Then*

$$\int_{\alpha} \rho(x)\, ds_h(x) = \int_{a}^{b} \frac{2\rho(\alpha(t))|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt \tag{14}$$

and, in particular,

$$l(\alpha) = \int_{a}^{b} \frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt. \tag{15}$$

Let $p_0 \in \mathbb{S}$ and let $z_0 \in \mathbb{D}$ be such that $\pi(z_0) = p_0$, where π is a natural projection of $\mathbb{D}$ onto $\mathbb{D}/G$. By D_0 we denote a Dirichlet polygon centered at the point z_0 and set $\varphi := \pi^{-1}$. Note that the mapping φ is a homeomorphism of $(\mathbb{S}, \widetilde{h})$ onto (F, d), where $\widetilde{h}$ is a metric on the surface $\mathbb{S}$, d is the metric defined above on the fundamental set F, and $D_0 \subset F \subset \overline{D_0}$. Without loss of generality, we can assume that $z_0 = 0$. Indeed, otherwise, consider an auxiliary mapping $g_0(z) = (z - z_0)/(1 - z\overline{z_0})$ without fixed points inside the unit disk. We choose a compact neighborhood $V \subset \mathbb{D}$ of the point $0 \in F \subset \mathbb{D}$ such that $d(x, z) = h(x, z)$ for all $x, z \in V$, which is possible in view of condition (6). Moreover, we choose V such that $V \subset B(0, r_0)$ for some $0 < r_0 < 1$. We set $U := \pi(V)$. In this case, U is called a *normal neighborhood of the point* p_0.

By using Lemmas 8.2 and 8.3 from [2], passing to a covering of the Riemannian surface by finitely or countably many normal neighborhoods, and using the countable semiadditivity of the measure $\widetilde{h}$, we arrive at the following statement:

Proposition 1. *Suppose that the mapping $f\colon D \to \mathbb{S}_*$ is almost everywhere differentiable in the local coordinates and, in addition, possesses the Luzin N- and N^{-1}-properties. Then there exists an at most countable sequence of compact sets $C_k^* \subset D$ such that $\widetilde{v}(B) = 0$, where $B = D \setminus \bigcup\limits_{k=1}^{\infty} C_k^*$ and $f|_{C_k^*}$ is one-to-one and bi-Lipschitz in the local coordinates for each $k = 1, 2, \ldots$. Moreover, the mapping f is differentiable for all $x \in C_k^*$ and the condition $J_f(x) \neq 0$ is satisfied.*

For an arbitrary set $B \subset \mathbb{S}$, we denote

$$l_\gamma(B) = \mathrm{mes}_1\big\{s \in [0, l(\gamma)]\colon \gamma(s) \in B\big\},$$

where, as a rule, mes_1 denotes a linear Lebesgue measure in $\mathbb{R}$ and $l(\gamma)$ is the length of γ. In a similar way, we can define a quantity $l_\gamma(B)$ for the dashed curve γ, i.e., for

$$\gamma\colon \bigcup_{i=1}^{\infty}(a_i, b_i) \to \mathbb{S},$$

where $a_i < b_i$ for all $i \in \mathbb{N}$ and $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ for all $i \neq j$.

We now prove the following assertion (see also [3], Theorem 33.1):

Lemma 3. *Suppose that $B_0 \subset \mathbb{S}$ is a set of $\widetilde{v}$-measure zero. Then, for almost all curves γ in $\mathbb{S}$,*

$$l_\gamma(B_0) = 0.$$

Proof. In view of the regularity of Lebesgue measure, there exists a Borel set $B \subset \mathbb{S}$ such that $B_0 \subset B$ and $\widetilde{v}(B_0) = \widetilde{B} = 0$, where $\widetilde{v}$ is the measure on the surface $\mathbb{S}$ given by relation (4). Consider a covering of the surface $\mathbb{S}$ by all possible balls of the form $\widetilde{B}(x_0, r_0)$, where $r_0 = r(x_0) > 0$, such that $\widetilde{B}(x_0, r_0)$ lies in a certain normal neighborhood U of the point x_0. Since $\mathbb{S}$ is a space with countable base, by the Lindelöf theorem [13], we can select a sequence of points x_i, $i = 1, 2, \ldots$, and radii of the balls corresponding to these points $r_i = r_i(x_i)$, $i = 1, 2, \ldots$, such that

$$\mathbb{S} = \bigcup_{i=1}^{\infty} \widetilde{B}(x_i, r_i), \qquad \overline{\widetilde{B}}(x_i, r_i) \subset U_i.$$

By $\varphi_i = \pi_i^{-1}$ we denote a mapping corresponding to the definition of the normal neighborhood U_i (see the comments made prior to Proposition 1). Let g_i be the characteristic function of the set $\varphi_i(B \cap U_i)$. For the sake of convenience, we denote $\varphi_i(\gamma) := \varphi_i\big(\gamma \cap \widetilde{B}(x_i, r_i)\big)$. By Theorem 3.2.5, for $m = 1$ [14], we get

$$\int\limits_{\varphi_i(\gamma)} g_i(z)|dz| = \mathcal{H}^1\big(\varphi_i(B \cap |\gamma|)\big), \tag{16}$$

where $\gamma\colon [a, b] \to \mathbb{S}$ is an arbitrary locally rectifiable curve, $|\gamma|$ is the support of the curve γ in $\mathbb{S}$, and $|dz|$ is an element of Euclidean length. Reasoning as in the proof of Theorem 33.1 in [3], we set

$$\rho(p) = \begin{cases} \infty, & p \in B, \\ 0, & p \notin B. \end{cases}$$

Note that ρ is a Borel set. Let Γ_i be a subfamily of all curves from Γ for which

$$\mathcal{H}^1\big(\varphi_i(B \cap |\gamma| \cap \widetilde{B}(x_i, r_i))\big) > 0.$$

In view of (16), for any $\gamma \in \Gamma_i$, we obtain

$$\int\limits_{\gamma\cap\widetilde{B}(x_i,r_i)} \rho(p)\, ds_{\widetilde{h}}(p) = \int\limits_{\varphi_i(\gamma)} \rho(\pi_i(y))\, ds_h(y)$$

$$= 2\int\limits_{\varphi_i(\gamma)} \frac{\rho(\pi_i(y))}{1 - |y|^2}|dy| = 2\int\limits_{\varphi_i(\gamma)} \frac{g_i(y)\rho(\pi_i(y))}{1 - |y|^2}|dy| = \infty,$$

where $\gamma \cap \widetilde{B} = \gamma|_{S_i}$ is the dashed curve and

$$S_i = \big\{s \in [0, l(\gamma)]\colon \gamma(s) \in \widetilde{B}(x_i, r_i)\big\}.$$

Then $\rho \in \operatorname{adm}\Gamma_i$. Hence,

$$M(\Gamma_i) \le \int\limits_{\mathbb{S}} \rho^2(p)\, d\widetilde{v}(p) = 0. \tag{17}$$

Note that

$$\Gamma > \bigcup_{i=1}^{\infty} \Gamma_i.$$

Therefore, relation (17) implies that

$$M(\Gamma) \le \sum_{i=1}^{\infty} M(\Gamma_i) = 0.$$

Lemma 3 is proved.

Let $f\colon D \to \mathbb{C}$ (or $f\colon D \to \mathbb{S}$) be a mapping for which the image of any curve in D is not degenerated into a point. Assume that I_0 is an interval, $\beta\colon I_0 \to \mathbb{C}$ (or $\beta\colon I_0 \to \mathbb{S}$) is a rectifiable curve, and $\alpha\colon I \to D$ is a curve such that $f \circ \alpha \subset \beta$. If the function of length $l_\beta\colon I_0 \to [0, l(\beta)]$ is constant on some interval $J \subset I$, then β is constant on J and, in view of the assumption for f, the curve α is also constant on J. This implies that there exists a unique curve $\alpha^*\colon l_\beta(I) \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that α^* is an *f-representation of the curve α with respect to β*.

3. Proof of Theorem 1

Let B_0 and C_k^*, $k = 1, 2, \ldots$, be the sets corresponding to the notation introduced in Proposition 1. Setting

$$B_1 = C_1^*, \qquad B_2 = C_2^* \setminus B_1, \ldots, \qquad \text{and} \qquad B_k = C_k^* \setminus \bigcup_{l=1}^{k-1} B_l,$$

we obtain a countable covering of the domain D by the sets B_k, $k = 0, 1, 2, \ldots$, that are mutually disjoint and such that $\widetilde{v}(B_0) = 0$ and

$$B_0 = D \setminus \bigcup_{k=1}^{\infty} B_k.$$

Since, under the condition, the mapping f possesses the N-property in D, we get

$$\widetilde{v_*}(f(B_0)) = 0.$$

Since $\overline{D}$ and $\overline{D_*}$ are compact sets, there exist finite coverings U_i, $1 \le i \le I_0$, and V_n, $1 \le n \le N_0$, such that

$$\overline{D} \subset \bigcup_{i=1}^{I_0} U_i \qquad \text{and} \qquad \overline{D_*} \subset \bigcup_{n=1}^{N_0} V_n,$$

where U_i and V_n are normal neighborhoods of some points $x_i \in \mathbb{S}$ and $y_n \in \mathbb{S}_*$. We can choose these coverings so that $\widetilde{v}(\partial U_i) = \widetilde{v_*}(\partial V_n) = 0$ for each U_i, $1 \le i \le I_0$, and V_n, $1 \le n \le N_0$. In particular, there exist conformal mappings $\varphi_i\colon U_i \to B(0, r_i)$, $0 < r_i < 1$, and $\psi_n\colon V_n \to B(0, R_n)$, $0 < R_n < 1$, such that the length and area in U_i and V_n are computed by using the maps φ_i and ψ_n according to relations (2) and (15). We set

$$R_0 := \max_{1 \le n \le N_0} R_n, \qquad r_0 := \max_{1 \le i \le I_0} r_i,$$

and

$$U_1' = U_1,\ U_2' = U_2 \setminus \overline{U_1},\ U_3' = U_3 \setminus \left(\overline{U_1} \cup \overline{U_2}\right),\ \ldots,\ U_{I_0}' = U_{I_0} \setminus \left(\overline{U_1} \cup \overline{U_2} \ldots \overline{U_{I_0 - 1}}\right).$$

Note that, by definition, $U_i' \subset U_i$ for $1 \le i \le I_0$ and $U_i' \cap U_j' = \varnothing$ for $i \ne j$. Furthermore,

$$D \subset \left(\bigcup_{i=1}^{I_0} U_i'\right) \bigcup B_0^*,$$

where U_i' are open and $\widetilde{v}(B_0^*) = 0$.

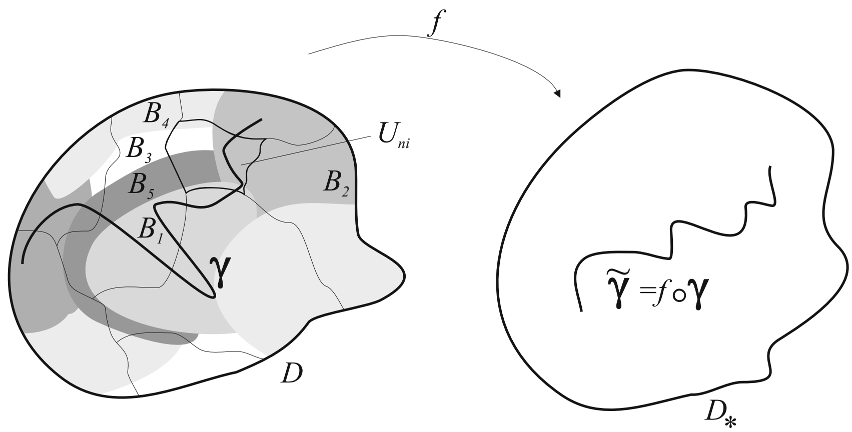

Fig. 1

Similarly, we set

$$V_1' = V_1,\ \ V_2' = V_2 \setminus \overline{V_1},\ \ V_3' = V_3 \setminus \big(\overline{V_1} \cup \overline{V_2}\big),\ \ \ldots,\ \ V_{N_0}' = V_{N_0} \setminus \big(\overline{V_1} \cup \overline{V_2} \ldots \overline{V_{N_0-1}}\big).$$

By definition, $V_n' \subset V_n$ for $1 \le n \le N_0$ and $V_n' \cap V_j' = \varnothing$ for $n \ne j$. Furthermore,

$$D_* \subset \left(\bigcup_{n=1}^{N_0} V_n'\right) \bigcup B_0^{**},$$

where V_n' are open and $\widetilde{v}_*(B_0^{**}) = 0$.

Moreover, we set $U_{n,i} = f^{-1}(V_n') \cap U_i'$. Note that, by construction and in view of the continuity of f, the sets $U_{n,i}$ are open. In addition, according to the N^{-1}-property, we find $\widetilde{v}\big(f^{-1}(B_0^{**})\big) = 0$. Thus,

$$D \subset \left(\bigcup_{\substack{1\le i\le I_0\\ 1\le n\le N_0}} U_{n,i}\right) \bigcup f^{-1}(B_0^{**}) \bigcup B_0^{*} \tag{18}$$

(see Fig. 1). Note that the equality $U_{n_1 i_1} = U_{n_2 i_2}$ is possible only for $n_1 = n_2$ and $i_1 = i_2$. Indeed, let $p \in U_{n_1 i_1} \cap U_{n_2 i_2}$. Thus, in particular, $p \in U_{i_1}' \cap U_{i_2}'$, which is possible only for $i_1 = i_2$ because $U_i' \cap U_j' = \varnothing$ for $i \ne j$. Further, in view of the condition $p \in U_{n_1 i_1} \cap U_{n_2 i_2}$, we get

$$f(p) \in V_{n_1}' \cap V_{n_2}',$$

which is impossible for $n_1 \ne n_2$ because $V_i' \cap V_j' = \varnothing$ for $i \ne j$. Thus, it is established that $i_1 = i_2$ and $n_1 = n_2$ simultaneously, Q.E.D.

We set

$$f_{n,i}(p) := \big(\psi_n \circ f \circ \varphi_i^{-1}\big)(\varphi_i(p)), \quad p \in U_{n,i}.$$

Let $\rho \in \operatorname{adm} \Gamma$ and let

$$\widetilde{\rho}(p_*) = \chi_{f(D\setminus B_0)} \sup_{p \in f^{-1}(p_*) \cap D \setminus B_0} \rho^*(p),$$

where

$$\rho^*(p) = \begin{cases} \rho(p)/l\big(f'_{n,i}(\varphi_i(p))\big), & p \in U_{n,i} \setminus B_0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\widetilde{\rho}(p_*) = \sup_{\substack{k\in\mathbb{N}, 1\le i\le I_0 \\ 1\le n\le N_0}} \rho_{k,i,n}(p_*),$$

where

$$\rho_{k,i,n}(p_*) = \begin{cases} \rho^*\big(f^{-1}_{k,i,n}(p_*)\big), & \text{for } p_* \in f(B_k \cap U_{n,i}), \\ 0, & \text{otherwise,} \end{cases}$$

where the mapping $f_{k,i,n} = f\,\big|_{B_k\cap U_{n,i}}$, $k = 1, 2, \ldots$, is injective. This implies that $\widetilde{\rho}$ is a Borel function (see [14], Sec. 2.3.2).

We first consider the case where $\widetilde{\gamma}$ is a closed rectifiable curve of the family $f(\Gamma)$. Then $\widetilde{\gamma}\colon [a, b] \to \mathbb{S}_*$ and $\widetilde{\gamma} = f \circ \gamma$, where $\gamma \in \Gamma$. Let $\widetilde{\gamma}^0$ be a normal representation of the curve $\widetilde{\gamma}$ and let $\gamma^*\colon \big[0, l(\widetilde{\gamma})\big] \to D$ be an f-representation with respect to $\widetilde{\gamma}$, i.e., $f(\gamma^*(s)) = \widetilde{\gamma}^0(s)$ for $s \in \big[0, l(\widetilde{\gamma})\big]$. Note that the set

$$S_{n,i} = \big\{s \in \big[0, l(\widetilde{\gamma})\big] : \gamma^*(s) \in U_{n,i}\big\}$$

is open in $\mathbb{R}$ as the preimage of the open set U_{n_i} under the continuous mapping γ^*. Thus, $\widetilde{\gamma}|_{S_{n,i}}$ is an at most countable set of open arcs such that the length of each of these arcs is computed in the coordinates (V'_n, ψ_n) with the use of the hyperbolic metric (see the remarks made in the introduction). Denote $\widetilde{\gamma}_{n,i} := \widetilde{\gamma}|_{S_{n,i}}$. According to the obtained results,

$$\widetilde{\gamma}_{n,i} = \bigcup_{l=1}^{\infty} \widetilde{\gamma}^l_{n,i},$$

where $\widetilde{\gamma}^l_{n,i}$ is an open arc. Since we take a closed curve $\widetilde{\gamma}$, exactly two indicated arcs can be semiopen. However, we do not interpret intervals of the form $[a, c)$ and $(c, b]$ as open sets with respect to the interval $[a, b]$. Since f has the N-property, we conclude that $\widetilde{v}_*(B_0^{**} \cup f(B_0^*)) = 0$. By Lemma 3,

$$\big[0, l(\widetilde{\gamma})\big] = \bigcup_{\substack{1\le i\le I_0 \\ 1\le n\le N_0}} S_{n,i} \cup B_*,$$

where B_* has a zero linear measure. In this case,

$$\int\limits_{\widetilde{\gamma}} \widetilde{\rho}(p_*)\, ds_{\widetilde{h_*}}(p_*) = \sum_{\substack{1\le i\le I_0 \\ 1\le n\le N_0}} \int\limits_{S_{n,i}} \widetilde{\rho}\big(\widetilde{\gamma}^0(s)\big)\, ds \tag{19}$$

for almost all curves $\widetilde{\gamma} \in f(\Gamma)$. Since $\widetilde{v}_*(f(B_0)) = 0$, by Lemma 3, we get $\widetilde{\gamma}^0(s) \notin f(B_0)$ for almost all $s \in \big[0, l(\widetilde{\gamma})\big]$ and almost all curves $\widetilde{\gamma} \in f(\Gamma)$. Thus, for almost all curves $\widetilde{\gamma}$ and all γ such that $\widetilde{\gamma} = f \circ \gamma$, we can write

$$\int\limits_{S_{n,i}} \widetilde{\rho}(\widetilde{\gamma}^0(s))\,ds = \int\limits_{S_{n,i}} \sup\limits_{p\in f^{-1}(\widetilde{\gamma}^0(s))\cap D\setminus B_0} \rho^*(p)\,ds$$

$$\geq \int\limits_{S_{n,i}} \frac{\rho(\gamma^*(s))}{l(f'_{n,i}(\varphi_i(\gamma^*(s))))}\,ds. \tag{20}$$

Since $\widetilde{\gamma}$ is rectifiable, $\widetilde{\gamma}^0$ is also rectifiable and, in particular, $\widetilde{\gamma}^0(s)$ is almost everywhere differentiable (see Lemma 1). We now show that γ^* is absolutely continuous for almost all curves $\widetilde{\gamma}$. Indeed, γ^* is rectifiable for almost all $\widetilde{\gamma}$ because $f\in ACP^{-1}$. Let $L^{-1}_{\gamma,f}$ be the function from the definition of the ACP^{-1}-property. Then

$$\gamma^*\circ l_{\widetilde{\gamma}}(t)=\gamma(t)=\gamma^0\circ l_\gamma(t)=\gamma^0\circ L^{-1}_{\gamma,f}\bigl(l_{\widetilde{\gamma}}(t)\bigr). \tag{21}$$

Denoting $s:=l_{\widetilde{\gamma}}(t)$, we get

$$\gamma^*(s)=\gamma^0\circ L^{-1}_{\gamma,f}(s). \tag{22}$$

Hence, γ^* is absolutely continuous in the local coordinates because, by the condition, $L^{-1}_{\gamma,f}(s)$ is absolutely continuous and

$$\widetilde{h}\bigl(\gamma^0(s_1),\gamma^0(s_2)\bigr)\leq |s_1-s_2|$$

for all $s_1,s_2\in[0,l(\gamma)]$. Here, we have also used the fact that $\widetilde{h}\bigl(\gamma^0(s_1),\gamma^0(s_2)\bigr)$ locally coincides with

$$h\bigl(\varphi(\gamma^0(s_1)),\varphi(\gamma^0(s_2))\bigr)$$

in the corresponding local coordinates (U,φ) and, in addition, by Lemma 1,

$$\bigl|\varphi(\gamma^0(s_1))-\varphi(\gamma^0(s_2))\bigr|\leq h\bigl(\varphi(\gamma^0(s_1)),\varphi(\gamma^0(s_2))\bigr).$$

Since $\widetilde{\gamma}^0(s)\notin f(B_0)$ for almost all $s\in\bigl[0,l(\widetilde{\gamma})\bigr]$ and almost all curves $\widetilde{\gamma}$, we conclude that $\gamma^*(s)\notin B_0$ for almost all $s\in\bigl[0,l(\widetilde{\gamma})\bigr]$. Hence, $\bigl(f_{n,i}\bigl(\varphi_i(\gamma^*(s))\bigr)\bigr)'$ and $(\varphi_i(\gamma^*(s)))'$ exist for almost all $s\in\bigl[0,l(\widetilde{\gamma})\bigr]\cap S_{n,i}$ and every $1\leq i\leq I_0$ and $1\leq n\leq N_0$. Recall that

$$\widetilde{\gamma}_{n,i}=\bigcup\limits_{l=1}^{\infty}\widetilde{\gamma}^l_{n,i},$$

where $\widetilde{\gamma}^l_{n,i}:=\widetilde{\gamma}|_{\Delta^l_{n,i}}$, $\Delta^l_{n,i}=(\alpha^l_{n,i},\beta^l_{n,i})$, or $\Delta^l_{n,i}=[\alpha^l_{n,i},\beta^l_{n,i})$, or $\Delta^l_{n,i}=(\alpha^l_{n,i},\beta^l_{n,i}]$. Note that

$$l_{\widetilde{\gamma}}(s)=\alpha^l_{n,i}+s_h(s)\quad \forall s\in\Delta^l_{n,i},\quad l=1,2,\ldots, \tag{23}$$

where $s_h(s)$ denotes the hyperbolic length of the curve $\psi_n\bigl(\widetilde{\gamma}_{\Delta^l_{n,i}}\bigr)$ on $[\alpha^l_{n,i},s]$ and, moreover, $s_h(s)\equiv s$.

By Lemma 2, for almost all $s \in \Delta^l_{n,i}$, it follows from (23) that

$$\left|\frac{d}{ds}\big(f_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big)\right| = \frac{1-\big|f_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big|^2}{2} \le \frac{1}{2}. \tag{24}$$

On the other hand, according to the rule of differentiation of a composite function, for almost all $s \in \Delta^l_{n,i}$, we find

$$\begin{aligned}\left|\frac{d}{ds}\big(f_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big)\right| &= \big|f'_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big(\varphi_i(\gamma^*(s))\big)'\big| \\ &= \left|f'_{n,i}\big(\varphi_i(\gamma^*(s))\big)\frac{\big(\varphi_i(\gamma^*(s))\big)'}{\big|\big(\varphi_i(\gamma^*(s))\big)'\big|}\right|\big|\big(\varphi_i(\gamma^*(s))\big)'\big| \\ &\ge l\big(f'_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big)\big|\big(\varphi_i(\gamma^*(s))\big)'\big|. \end{aligned} \tag{25}$$

Combining (24) and (25), for almost all $s \in S_{n,i}$, we get

$$\frac{\rho(\gamma^*(s))}{l\big(f'_{n,i}\big(\varphi_i(\gamma^*(s))\big)\big)} \ge 2\rho(\gamma^*(s))\big|\big(\varphi_i(\gamma^*(s))\big)'\big|. \tag{26}$$

Let γ^0 be the normal representation of the curve γ. Since $f \in ACP^{-1}$, we conclude that $L^{-1}_{\gamma,f}$ has the N-property with respect to a one-dimensional linear measure for almost all $\widetilde{\gamma} = f \circ \gamma$ (see [14], Theorem 2.10.13). Hence, by relation (21), we obtain

$$\gamma^0(s_0) \notin f^{-1}(B_0^{**}) \bigcup B_0^*$$

for almost all $s_0 \in [0, l(\gamma)]$ and almost all curves $\widetilde{\gamma} = f \circ \gamma$. Denote

$$Q_{n,i} = \big\{s_0 \in [0, l(\gamma)] : \, s_0 \in U_{n,i}\big\}.$$

Thus, by the absolute continuity of the curve $\gamma^*(s)$ and Corollary 2 [see relations (14)], (18), and (22), we can write

$$\begin{aligned} 1 &\le \int\limits_{\gamma} \rho(p)\, ds_{\widetilde{h}}(p) \\ &= \sum_{\substack{1\le i\le I_0 \\ 1\le n\le N_0}} \int\limits_{Q_{n,i}} \rho(\gamma^0(s_0))\, ds_0 \\ &= \sum_{\substack{1\le i\le I_0 \\ 1\le n\le N_0}} \int\limits_{S_{n,i}} \frac{2\rho(\gamma^*(s))\big|\big(\varphi_i(\gamma^*(s))\big)'\big|}{1-\big|\varphi_i(\gamma^*(s))\big|^2}\, ds \\ &\le \frac{2}{1-r_0^2} \sum_{\substack{1\le i\le I_0 \\ 1\le n\le N_0}} \int\limits_{S_{n,i}} \rho(\gamma^*(s))\big|\big(\varphi_i(\gamma^*(s))\big)'\big|\, ds. \end{aligned} \tag{27}$$

Combining (19), (20), (26), and (27), we conclude that

$$\frac{1}{1-r_0^2}\int\limits_{\widetilde{\gamma}}\widetilde{\rho}(p_*)\,ds_{\widetilde{h_*}}(p_*)\geq 1$$

for almost all closed curves $\widetilde{\gamma}\in f(\Gamma)$. To obtain the result for an arbitrary curve $\widetilde{\gamma}$, we take the supremum of the expression

$$\frac{1}{1-r_0^2}\int\limits_{\widetilde{\gamma}'}\widetilde{\rho}(p)\,ds_{\widetilde{h_*}}(p_*)\geq 1$$

over all closed subcurves $\widetilde{\gamma}'$ of the curve $\widetilde{\gamma}$.

Hence,

$$\frac{1}{1-r_0^2}\,\widetilde{\rho}\in\operatorname{adm} f(\Gamma)$$

and

$$M\left(f\left(\Gamma\right)\right)\leq\frac{1}{\left(1-r_0^2\right)^2}\int\limits_{D_*}\widetilde{\rho}^{\,2}(p_*)\,dv_*(p_*). \tag{28}$$

According to Theorem 3.2.5 [14], for $m=2$, we get

$$\int\limits_{U_{n,i}\cap B_k}K_f(p)\rho^2(p)\,d\widetilde{v}(p)$$

$$=4\int\limits_{\varphi_i(U_{n,i}\cap B_k)}\frac{\left\|(\psi_n\circ f\circ\varphi_i^{-1})'(x)\right\|^2}{\det\left\{(\psi_n\circ f\circ\varphi_i^{-1})'(x)\right\}(1-|x|^2)^2}\rho^2\left(\varphi_i^{-1}(x)\right)dm(x)$$

$$\geq 4\int\limits_{\varphi_i(U_{n,i}\cap B_k)}\frac{\left\|(\psi_n\circ f\circ\varphi_i^{-1})'(x)\right\|^2}{\det\left\{(\psi_n\circ f\circ\varphi_i^{-1})'(x)\right\}}\rho^2\left(\varphi_i^{-1}(x)\right)dm(x)$$

$$=4\int\limits_{\psi_n(f((U_{n,i}\cap B_k)))}\frac{\rho^2\left((f_k^{-1}\circ\psi_n^{-1})(y)\right)}{\left\{l\left(f'_{n,i}\left((\varphi_i\circ f_k^{-1}\circ\psi_n^{-1})(y)\right)\right)\right\}^2}\,dm(y)$$

$$\geq\left(1-R_0^2\right)^2\int\limits_{f(D)}\rho^2_{k,i,n}(p_*)\,d\widetilde{v}_*(p_*). \tag{29}$$

Finally, by the Lebesgue theorem (see [15], Theorem I.12.3), in view of (28) and (29), we obtain

$$\int\limits_{D}K_f(p)\rho^2(p)\,d\widetilde{v}(p)=\sum\limits_{\substack{1\leq i\leq I_0,\ 1\leq n\leq N_0\\ 1\leq k<\infty}}\ \int\limits_{U_{n,i}\cap B_k}K_f(p)\rho^2(p)\,d\widetilde{v}(p)$$

$$\geq \left(1-R_0^2\right)^2 \int\limits_{f(D)} \sum_{\substack{1\leq i\leq I_0,\ 1\leq n\leq N_0\\ 1\leq k<\infty}} \rho_{k,i,n}^2(p_*)\,d\widetilde{v}_*(p_*)$$

$$\geq \left(1-R_0^2\right)^2 \int\limits_{f(D)} \sup_{\substack{1\leq i\leq I_0,\ 1\leq n\leq N_0\\ 1\leq k<\infty}} \rho_{k,i,n}^2(p_*)\,d\widetilde{v}_*(p_*)$$

$$= \left(1-R_0^2\right)^2 \int\limits_{f(D)} \widetilde{\rho}^2(p_*)\,d\widetilde{v}_*(p_*) \geq \left(1-R_0^2\right)^2\left(1-r_0^2\right)^2 M(f(\Gamma)).$$

The final relation

$$M(f(\Gamma)) \leq c \int\limits_D K_f(p)\rho^2(p)\,d\widetilde{v}(p) \tag{30}$$

is true for any $\rho \in \operatorname{adm}\Gamma$, where

$$c := \frac{1}{\left(1-R_0^2\right)^2\left(1-r_0^2\right)^2}.$$

Passing to the limit as r_0 and R_0 tend to zero in relation (30), we arrive at relation (5).

Theorem 1 is proved.

4. Boundary Behavior of the Mappings

In this section, we apply Theorem 1 to the problem of boundary behavior of the mappings. Let D be a domain in $\mathbb{S}$ and let $E,\ F \subset D$ be arbitrary sets. Further, by $\Gamma(E,F,D)$ we denote a family of all curves $\gamma\colon [a,b]\to D$ connecting E and F in D, i.e., $\gamma(a)\in E,\ \gamma(b)\in F$, and $\gamma(t)\in D$ for $t\in[a,\,b]$. We say that the boundary ∂D of the domain D is *strongly attainable at the point* $p_0\in\partial D$ if, for any neighborhood U of the point p_0, there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of this point, and a number $\delta>0$ such that, for any continua E and F intersecting both ∂U and ∂V, the inequality $M\big(\Gamma(E,F,D)\big)\geq\delta$ is true. We also say that the boundary ∂D is *strongly attainable* if it is strongly attainable at each its point. For the set $E\subset\mathbb{S}$, as a rule, we can write

$$C(f,E) = \big\{p_*\in\mathbb{S}_*\colon \exists p_k\in D,\ \ p\in E\colon p_k\to p,\ \ f(p_k)\to p_*,\ \ k\to\infty\big\}.$$

Following [16] (Sec. 2) (see also [2], Sec. 6.1, Chap. 6), we say that a function $\varphi\colon D\to\mathbb{R}$ has a *finite mean oscillation* at a point $p_0\in D$ (we write $\varphi\in FMO(p_0)$) if

$$\limsup_{\varepsilon\to 0}\frac{1}{\widetilde{v}\big(\widetilde{B}(p_0,\varepsilon)\big)}\int\limits_{\widetilde{B}(p_0,\varepsilon)}\big|\varphi(p)-\overline{\varphi}_\varepsilon\big|\,d\widetilde{v}(p)<\infty,$$

where

$$\overline{\varphi}_\varepsilon = \frac{1}{\widetilde{v}(\widetilde{B}(p_0,\varepsilon))}\int\limits_{\widetilde{B}(p_0,\varepsilon)}\varphi(p)\,d\widetilde{v}(p).$$

The following assertion is true:

Theorem 2. *Suppose that D and D_* are domains of the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively, and moreover, $\overline{D}$ and $\overline{D_*}$ are compact sets. Assume that f is an open discrete almost everywhere differentiable mapping of the domain D onto D_* that belongs to the class ACP^{-1} and possesses the Luzin N- and N^{-1}-properties. Suppose that the domain D is locally linearly connected at the point $b \in \partial D$, $C(f, \partial D) \subset \partial D'$ and $\partial D'$ is strongly attainable at at least one of its points $p_* \in C(f, b)$. If $Q \in FMO(b)$, then $C(f, b) = \{p_*\}$.*

Proof. By Theorem 1, the mapping f satisfies relation (5) for any family of curves Γ in the domain D. In particular, for any two continua $C_0 \subset \overline{\widetilde{B}(b, r_1)}$ and $C_1 \subset \mathbb{S} \setminus \widetilde{B}(b, r_2)$, the condition

$$M\big(f(\Gamma(C_1, C_0, D))\big) \le \int\limits_{A \cap D} Q(p)\rho^2(p)\, d\widetilde{v}(p) \quad \forall \rho \in \operatorname{adm} \Gamma(C_1, C_0, D),$$

$$A = A(b, r_1, r_2) = \big\{p \in \mathbb{S} \colon r_1 < \widetilde{h}(p, b) < r_2\big\}, \quad 0 < r_1 < r_2 < \infty,$$

is satisfied.

Let $\eta \colon (r_1, r_2) \to [0, \infty]$ be an arbitrary Lebesgue-measurable function satisfying the condition

$$\int\limits_{r_1}^{r_2} \eta(t)\, dt \ge 1.$$

We set $\rho(p) = \eta\big(\widetilde{h}(p, p_0)\big)$. Then, for any (locally rectifiable) curve $\gamma \in \Gamma(C_1, C_0, D)$, the condition

$$\int\limits_{\gamma} \rho(p)\, ds_{\widetilde{h}}(p) \ge 1$$

is satisfied in view of [2] (Proposition 13.4). In this case,

$$M\big(f(\Gamma(C_1, C_0, D))\big) \le \int\limits_{A \cap D} Q(p)\eta\big(\widetilde{h}(p, p_0)\big) d\widetilde{v}(p).$$

Note that each curve $\beta \colon [a, b) \to D_*$ has the maximum f-lifting with origin at the point $p \in f^{-1}(\beta(a))$ in the domain D (see [17], Lemma 2.1). In addition, the Riemannian surfaces are locally regular in a sense of Ahlfors (see, e.g., [10], Theorem 7.2.2). In this case, the required conclusion follows from Theorem 5 in [18].

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