DOI 10.1007/s11253-020-01810-0
Ukrainian Mathematical Journal, Vol. 72, No. 5, October, 2020 (Ukrainian Original Vol. 72, No. 5, May, 2020)

GEODESIC COMPLETENESS OF THE LEFT-INVARIANT METRICS ON $\mathbb{R}H^n$

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We give the full classification of left-invariant metrics of an arbitrary signature on the Lie group corresponding to the real hyperbolic space. It is shown that all metrics have constant sectional curvature and that they are geodesically complete only in the Riemannian case.

It is well known that the real hyperbolic space $\mathbb{R}H^n$ with standard Riemannian metric of constant negative curvature has the structure of a Lie group such that the metric is left-invariant. Milnor [8] considered a special class of solvable Lie groups with the property that the commutator $[x, y]$ is a linear combination of x and y for any two elements from the corresponding Lie algebra. Moreover, he showed that an algebra of this kind is isomorphic to the Lie algebra of $\mathbb{R}H^n$ and that every left-invariant Riemannian metric on this group has a constant negative sectional curvature. In the Lorentz case, Wolf [13] showed that this group admits a flat metric. At the same time, Nomizu [9] proved that, for any $K \in \mathbb{R}$, there exists a left-invariant metric with sectional curvature K. In [10], Nomizu considered these metrics in detail. They were called Lorentz–Poincaré metrics.

In the present paper, we classify all left-invariant metrics of arbitrary signature on the Lie group corresponding to the hyperbolic space. This problem was also considered in [6]. According to Arnold [1], the geodesic of an arbitrary left invariant metric on a Lie group G can be seen as a motion of a "generalized rigid body" in a configuration space G. In the Riemannian case, all geodesics are complete but Gudeiri [4] gave an example of Lorentzian metric on a four-dimensional Lie group with incomplete geodesics. Lauret [7] classified all Riemannian left-invariant metrics on four-dimensional nilpotent Lie groups, while the authors have generalized this result to an arbitrary signature [2]. Calvaruso [3] classified Lorentzian left-invariant metrics on the four-dimensional Lie groups that are Einstein or Ricci-parallel. In [11], the author classified the Riemannian and Lorentzian left-invariant metrics on the Heisenberg–Lie group.

In the preliminary section, we introduce basic notation and explain its meaning from the viewpoint of classification of the left-invariant metrics.

In Theorem 2.2, we present the classification of the left-invariant metrics of arbitrary signature on the Lie group $\mathbb{R}H^n$. It is shown that, the only metrics in the Riemannian case are standard metrics of constant curvature $K < 0$ of the hyperbolic space. In the Lorentz case, every metric from our classification is isometric either to the flat metric obtained by Wolf or to the metric of constant curvature $K \neq 0$ obtained by Nomizu. Moreover, in Theorem 2.3 we prove that every nonflat metric on $\mathbb{R}H^n$ is a metric of constant sectional curvature.

The geodesic completeness of metrics is considered in Section 3. It is shown that all geodesic curves are complete only in the Riemannian case (Theorems 3.1 and 3.2).

In Section 4, we present an isometric imbedding of $\mathbb{R}H^n$ into the space forms of the curvature K.

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Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 72, No. 5, pp. 611–619, May, 2020. Original article submitted April 7, 2017; revision submitted December 18, 2019.

 0041-5995/20/7205–0702

1. Preliminaries

The group structure in a half-space model of a real hyperbolic space

$$\mathbb{R}H^n = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n |\, x_n > 0\big\}$$

is given by

$$(x_1, \ldots, x_{n-1}, x_n)(y_1, \ldots, y_{n-1}, y_n) = \big(x_1 + x_n y_1, \ldots, x_{n-1} + x_n y_{n-1}, x_n y_n\big). \tag{1}$$

Denote by $\{e_1, \ldots, e_n\}$ the corresponding basis of the Lie algebra $\mathfrak{r}_n = \mathrm{Lie}\, \mathbb{R}H^n$. This is a semidirect product $\mathbb{R}e_n \ltimes \mathfrak{n}$, where $\mathfrak{n} = \mathcal{L}(e_1, \ldots, e_{n-1})$ is an Abelian ideal and $\mathrm{ad}(e_n)\,|_{\mathfrak{n}} = \mathrm{id}$, i.e., the nonnull commutators are

$$[e_n, e_k] = e_k, \quad k < n.$$

We also denote by $\mathcal{S}(\mathfrak{r}_n)$ the set of nonequivalent inner products of an algebra $\mathfrak{r}_n$. For a fixed basis of the Lie algebra $\mathfrak{r}_n$, the set $\mathcal{S}(\mathfrak{r}_n)$ is identified with symmetric matrices S of arbitrary signature modulo the following action of the automorphism group:

$$S \mapsto F^T S F, \quad F \in \mathrm{Aut}\,(\mathfrak{r}_n). \tag{2}$$

Here, $\mathrm{Aut}\,(\mathfrak{r}_n)$ denotes the group of automorphisms of the Lie algebra $\mathfrak{r}_n$ defined as follows:

$$\mathrm{Aut}\,(\mathfrak{r}_n) := \big\{F\colon\, \mathfrak{r}_n \to \mathfrak{r}_n \,|\, F \text{ linear, bijective, } [Fx, Fy] = F[x,y],\ \ x, y \in \mathfrak{r}_n\big\}.$$

It is easy to check the following lemma:

Lemma 1.1. *The group $\mathrm{Aut}(\mathfrak{r}_n)$ of automorphisms of the Lie algebra $\mathfrak{r}_n$ in a basis $\{e_1, e_2, \ldots, e_n\}$ consists of real matrices of the form*

$$\mathrm{Aut}\,(\mathfrak{r}_n) = \left\{ \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \middle|\, A \in GL(n-1, \mathbb{R}),\ \ a \in \mathbb{R}^{n-1} \right\} \cong \mathrm{Aff}_{n-1}(\mathbb{R}), \tag{3}$$

i.e., it is isomorphic to a group of affine transformations of $\mathbb{R}^{n-1}$.

Let $x \in \mathbb{R}H^n$ with $x = (x_1, \ldots, x_n)$. For the left translations L_x, the differential dL_x at every point $y \in \mathbb{R}H^n$ is given by

$$(dL_x)\left(\frac{\partial}{\partial x_k}(y)\right) = x_n \frac{\partial}{\partial x_k} L_x(y).$$

Therefore, the left-invariant vector fields $X_1^L, \ldots, X_n^L$ are given by

$$X_k^L(x) = x_n \frac{\partial}{\partial x_k}, \quad k \le n. \tag{4}$$

2. Left-Invariant Metrics

By $I_{p,r}$ we denote a diagonal matrix $\mathrm{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_{p+r})$, where $\epsilon_k = -1$, $1 \le k \le p$, and $\epsilon_k = 1$, $p+1 \le k \le p+r$.

Theorem 2.1. *The set $\mathcal{S}(\mathfrak{r}_n)$ of nonequivalent inner products of arbitrary signature on the algebra $\mathfrak{r}_n$ is represented by the following matrices:*

$$S_\lambda = \begin{pmatrix} I_{p,r} & 0 \\ 0 & \lambda \end{pmatrix}, \qquad p+r = n-1, \quad \lambda \neq 0,$$

$$S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p,r} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad p+r = n-2.$$

Proof. Let $F \in \mathrm{Aut}\,(\mathfrak{r}_n)$. By $\bar{S}$ we denote an arbitrary symmetric matrix representing the inner product q in the basis $\{e_1, \dots, e_n\}$. In the same basis, F is represented by matrix (3). We are looking for a new basis in which $F^T \bar{S} F$ has the simplest form.

If we represent $\bar{S}$ in the following form:

$$\bar{S} = \begin{pmatrix} S & v \\ v^T & s \end{pmatrix},$$

where $S = S^T$ is an $(n-1) \times (n-1)$ matrix, $v \in \mathbb{R}^{n-1}$, and $s \in \mathbb{R}$, then

$$F^T \bar{S} F = \begin{pmatrix} A^T S A & A^T(Sa + v) \\ (a^T S + v^T)A & a^T S a + v^T a + a^T v + s \end{pmatrix}.$$

We now distinguish between the following two cases:

Case 1. S is a regular matrix of signature (p, r). Since S is symmetric, there exists $A \in GL(n-1, \mathbb{R})$ such that

$$A^T S A = I_{p,r}, \qquad p+r = n-1.$$

Setting $a = -S^{-1}v$, we conclude that the corresponding inner product is S_λ with

$$\lambda = s - v^T S^{-1} v.$$

Since $\bar{S}$ is a nonsingular matrix, the relation $\lambda \neq 0$ must hold.

Case 2. If S is not regular, without loss of generality, we can assume that S has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \tilde{S} \end{pmatrix},$$

where $\tilde{S}$ is a regular matrix of signature $(p, r),\ p + r = n - 2$. Then there exists a regular matrix $\widetilde{A}$ such that

$$A = \begin{pmatrix} \dfrac{1}{w} & 0 \\ 0 & \widetilde{A} \end{pmatrix} \qquad \text{and} \qquad A^T S A = \begin{pmatrix} 0 & 0 \\ 0 & I_{p,r} \end{pmatrix}.$$

For the vector $v = (w, \bar{v}^T)^T,\ w \neq 0,\ \bar{v} \in \mathbb{R}^{n-2}$, we set

$$a = (a_1, \bar{a}^T)^T,$$

$$a_1 = \frac{\bar{v}^T \tilde{S}^{-1} \bar{v} - s}{2w} \in \mathbb{R}, \qquad \bar{a} = -\tilde{S}^{-1} \bar{v} \in \mathbb{R}^{n-2},$$

to obtain the inner product S_0.

Theorem 2.1 is proved.

The inner product q on $\mathfrak{r}_n$ induces a left-invariant metric g on the corresponding Lie group. For the global coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}H^n$, by using the left-invariant vector fields (4), we can find a coordinate description of the metrics defined in the previous theorem.

Theorem 2.2. *Each left-invariant metric on the group $\mathbb{R}H^n$ is isometric, to within an automorphism of $\mathbb{R}H^n$, to one of the following:*

$$g_\lambda = \frac{1}{x_n^2} \left(-dx_1^2 - \ldots - dx_p^2 + dx_{p+1}^2 + \ldots + dx_{n-1}^2 + \lambda dx_n^2 \right), \quad \lambda \neq 0,$$

$$g_0 = \frac{1}{x_n^2} \left(-dx_2^2 - \ldots - dx_{p+1}^2 + dx_{p+2}^2 + \ldots + dx_{n-1}^2 + 2 dx_1 dx_n \right).$$

In [13], Wolf showed that $\mathbb{R}H^n$ admits flat metrics. At the same time, we know that, in the four-dimensional case, according to Jensen's classification [5], it also admits Einstein metrics. Later, Milnor [8] showed that every left-invariant positive-definite metric on $\mathbb{R}H^n$ has a negative sectional curvature, while Nomizu [9] proved that, for every $K \in \mathbb{R}$, one can find a left-invariant Lorentz metric on $\mathbb{R}H^n$ with K as constant sectional curvature. However, we are able to prove more.

Theorem 2.3. *All left-invariant metrics of arbitrary signature on $\mathbb{R}H^n$ have constant sectional curvature.*

Proof. In order to prove the theorem, it is necessary to calculate the curvature tensor. We use the identification of the left-invariant vector fields X_k^L with their values in the unit element $X_k^L(e) = e_k$.

Recall that the curvature operators $R(e_i, e_j)$ belong to the algebra $\mathrm{so}(q)$ preserving the inner product q, i.e.,

$$\mathrm{so}(q) := \left\{ A \in gl(\mathfrak{r}_n) \;\middle|\; AS + SA^T = 0 \right\},$$

where S denotes the matrix of q. This algebra can be identified with the space $\Lambda^2 \mathfrak{r}_n$ of bivectors whose action on $\mathfrak{r}_n$ is given by

$$(x \wedge y) z := q(y, z) x - q(x, z) y, \quad x, y, z \in \mathfrak{r}_n.$$

By using standard calculations, for the metric g_λ, we conclude that the connection is given by the nonzero expressions

$$\nabla_{e_i} e_i = \frac{\epsilon_i}{\lambda} e_n, \quad \epsilon_i \in \{-1, 1\}, \qquad \nabla_{e_i} e_n = -e_i, \quad i < n, \tag{5}$$

and the curvature operators are defined by

$$R(e_i, e_j) = -\frac{1}{\lambda} e_i \wedge e_j.$$

From the previous reasoning, it is clear that the sectional curvature is constant

$$K = -\frac{1}{\lambda}.$$

For the metric g_0 all components of the curvature tensor R vanish; hence, the metric is flat and $K = 0$. Theorem 2.3 is proved.

3. Geodesics

Every C^1 curve $c(t)$ on the Lie group G, up to the left translations, induces a curve

$$\gamma(t) = L^{-1}_{c(t)*} \dot{c}(t) \tag{6}$$

on the corresponding Lie algebra $\gg$. The curves of $\gg$ associated with geodesics are solutions of the equations

$$\dot{x} = \operatorname{ad}^*_x x, \tag{7}$$

where ad^*_x stands for the adjoint of ad_x relative to the inner product on $\gg$.

First, we consider the Einstein metric g_λ on $\mathbb{R}H^n$. We fix a basis $\{e_1, \ldots, e_n\} \in \mathfrak{r}_n$. Thus, as a result of easy computations, we get

$$\operatorname{ad}^*_{e_k} e_k = -\frac{\epsilon_k}{\lambda} e_n, \qquad \operatorname{ad}^*_{e_n} e_k = e_k, \quad k < n.$$

In local coordinates $(x_1, \ldots, x_n)$, for

$$\gamma(t) = \sum_{k=1}^{n} x_k(t) e_k,$$

by using the equation (7), we obtain the system

$$\dot{x}_k = x_k x_n, \quad k < n, \qquad \dot{x}_n = -\frac{1}{\lambda} \sum_{j=1}^{n-1} \epsilon_j x_j^2. \tag{8}$$

Let $C_1, \ldots, C_n \in \mathbb{C}$. We denote

$$C_{n+1}^2 = -\frac{1}{\lambda} \sum\nolimits_{k=1}^{n-1} \epsilon_k C_k^2.$$

Thus, the solutions $\gamma(t) = (x_1(t), \ldots, x_n(t))$ of system (8) are given by

$$x_k(t) = \frac{C_k}{\cos(C_{n+1}t + C_n)}, \quad k < n,$$
$$x_n(t) = C_{n+1}\tan(C_{n+1}t + C_n), \quad \text{for} \quad C_{n+1}^2 \neq 0, 1, \tag{9}$$

$$x_k(t) = \frac{C_k}{t + C_n}, \quad k < n, \qquad x_n(t) = -\frac{1}{t + C_n}, \quad \text{for} \quad C_{n+1}^2 = 1, \tag{10}$$

$$x_k(t) = C_k e^{C_n t}, \quad k < n, \qquad x_n(t) = C_n, \quad \text{for} \quad C_{n+1} = 0. \tag{11}$$

Note that the constants $C_1, \ldots, C_n$ must be real in case of solutions (10) and (11). For solution (9), they can be either real or complex but, at the same time, they must satisfy additional constraints, which are explained in detail in the proof of the following theorem.

Theorem 3.1. *The left-invariant metric g_λ on $\mathbb{R}H^n$ is geodesically complete if and only if it is positive definite.*

Proof. First, we note that, in view of the left-invariance, we can consider only the curves $\gamma(t)$ in the Lie algebra $\mathfrak{r}_n$ defined by (6).

A geodesic curve whose tangent vector is e_n corresponds to solution (11) with $C_k = 0,\ k < n$, and $C_n = 1$. These are vertical lines with ends in the hyperplane $x_n = 0$, and they are complete in each signature.

Let $v = \gamma(0) \neq e_n$ be a tangent vector of a geodesic curve. Note that it follows from (5) that the two-dimensional plane $\alpha = \mathcal{L}(v, e_n)$ is totally geodesic. Therefore, it is sufficient to discuss the induced signature in this plane.

It is not difficult to show that $|v|^2 = -\lambda C_{n+1}^2$ for solution (9), $|v|^2 = 0$ for (10), and $|v|^2 = \lambda C_n^2$ in the last case (11). If the plane α is nondegenerate, then we consider solutions (9) and (10). At the same time, solution (11) occurs only if α is degenerate.

Case 1. Suppose that the plane α is Riemannian. Then $C_{n+1}^2 < 0$, i.e.,

$$C_{n+1} = iD, \quad D \in \mathbb{R}.$$

In order to determine the constant C_n, it is necessary to consider the Gram determinant associated with the plane α. It is easy to see that

$$G = -\lambda^2 C_{n+1}^2 (1 + \tan^2 C_n). \tag{12}$$

In the Riemannian case, G must be positive. This yields $C_n = iC,\ C \in \mathbb{R}$. Note that, in order to obtain real solutions, all other constants $C_k,\ k < n$, must be real. The trigonometric functions in (9) become hyperbolic functions, namely, $\cosh$ and $-\tanh$, and, hence, the curves are complete. It is easy to check that the corresponding geodesic curves on the Lie group $\mathbb{R}H^n$ are half ellipses with centers in the hyperplane $x_n = 0$. The plane α is isometric to the standard hyperbolic plane.

Case 2. Suppose that the plane α is Lorentzian.

If $|v|^2 \neq 0$, then the solution is given by (9), and we distinguish between two cases.

If the vectors v and e_n are of the same character, i.e., of the same signature, then $C_{n+1}^2 < 0$. Therefore, we must take

$$C_{n+1} = iD, \quad D \in \mathbb{R}.$$

Here, the Gram determinant (12) must be negative and, therefore,

$$C_n = iC + \frac{\pi}{2}, \quad C \in \mathbb{R}.$$

Moreover, all the constants $C_k,\ k < n,$ must be pure imaginary. The trigonometric functions in (9) turn into hyperbolic functions $\sinh$ and $-\coth$.

If the vectors v and e_n are of the different character, then $C_{n+1}^2 > 0$, and we get solution (9) in which all constants are real. In both cases, the geodesics are incomplete. The corresponding geodesic curves on $\mathbb{R}H^n$ are branches of hyperbolas satisfying $x_n > 0$.

If $|v|^2 = 0$, then the corresponding curves are given by (10). In $\mathbb{R}H^n$, these are the straight lines with ends in the hyperplane $x_n = 0$, and they are geodesically incomplete.

Case 3. Suppose that the plane α is degenerate. The vector $u = v - C_n e_n$ is a null vector orthogonal to all vectors from α. The corresponding solutions are complete geodesics given by (11). These are parabolas on $\mathbb{R}H^n$.

It is possible to conclude that the metric is complete if and only if, for every tangent vector v, the corresponding plane α is Riemannian.

Theorem 3.1 is proved.

Similarly, for the flat metric g_0 in local coordinates, we get a system

$$\dot{x}_1 = -x_1 x_n - \sum_{j=2}^{n-1} \epsilon_j x_j^2, \qquad \dot{x}_k = x_k x_n, \quad 1 < k \leq n.$$

The solutions to this system are given by the formulas

$$x_1(t) = C_1(t + C_n) + \frac{C_0}{2(t + C_n)},$$

$$x_k(t) = \frac{C_k}{t + C_n}, \quad 2 \leq k < n, \qquad x_n(t) = -\frac{1}{t + C_n},$$

$$x_1(t) = C_1 - tC_0, \qquad x_k(t) = C_k, \quad 2 \leq k < n, \quad x_n(t) = 0,$$

with

$$C_0 = \sum_{k=2}^{n-1} \epsilon_k C_k^2$$

and $C_k \in \mathbb{R},\ k \leq n.$

Hence, the following theorem holds:

Theorem 3.2. *The pseudo-Riemannian metric g_0 on the Lie group $\mathbb{R}H^n$ is geodesically incomplete.*

4. Isometric Imbedding into the Space Forms

Denote by $\mathbb{R}^n_p$ the space $\mathbb{R}^n$ with a pseudo-Riemannian metric

$$g(X,Y) = -\sum_{k=1}^{p} x_k y_k + \sum_{k=p+1}^{n} x_k y_k$$

for every $X, Y \in \mathbb{R}^n$.

Let $S^n_p \subseteq \mathbb{R}^{n+1}_p$ be the de-Sitter space

$$S^n_p = \left\{ u = (u_0, \ldots, u_n) \;\middle|\; -u_0^2 - \ldots - u_{p-1}^2 + u_p^2 + \ldots + u_n^2 = -\lambda, \;\; \lambda < 0 \right\}.$$

This is a hypersurface in $\mathbb{R}^{n+1}_p$ with its induced metric of signature $(p, n-p)$ of constant sectional curvature $K = -\lambda^{-1} > 0$.

Similarly, by $H^n_p \subseteq \mathbb{R}^{n+1}_{p+1}$ we denote the anti-de-Sitter space

$$H^n_p = \left\{ u = (u_0, \ldots, u_n) \;\middle|\; -u_0^2 - \ldots - u_p^2 + u_{p+1}^2 + \ldots + u_n^2 = -\lambda, \;\; \lambda > 0 \right\}$$

with its induced metric of signature $(p, n-p)$ and constant sectional curvature $K = -\lambda^{-1} < 0$.

We define $\tilde{S}^n_p$ and $\tilde{H}^n_p$ as connected and simply connected manifolds corresponding to S^n_p and H^n_p, respectively.

According to Wolf [12], every complete connected pseudo-Riemannian manifold of signature $(p, n-p)$ and constant sectional curvature K has a universal pseudo-Riemannian covering $\tilde{S}^n_p$ for $K > 0$, $\tilde{H}^n_p$ for $K < 0$, and $\mathbb{R}^n_p$ for $K = 0$. Our metrics g_λ and g_0 have constant sectional curvature and, despite the fact that they are not always complete, we are interested in finding a local isometry into the space forms.

Theorem 4.1. $(\mathbb{R}H^n, g_\lambda)$ *of signature* $(p, n-p)$ *is isometric to a part of* S^n_p *(for* $\lambda < 0$*) and* H^n_p *(for* $\lambda > 0$*) determined by the condition* $u_0 + u_n > 0$.

Proof. Suppose that $\lambda < 0$. Then the metric g_λ has the form

$$g_\lambda = \frac{1}{x_n^2}\left(-dx_1^2 - \ldots - dx_{p-1}^2 + dx_p^2 + \ldots + dx_{n-1}^2 + \lambda dx_n^2\right)$$

$$= \frac{1}{x_n^2}\left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right), \quad \lambda < 0.$$

We define an isometric imbedding $f : \mathbb{R}H^n \to S^n_p$ by

$$f(x) = f(x_1, \ldots, x_n) = (u_0, u_1, \ldots, u_n) = u,$$

where

$$u_0 = \frac{1 + \left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right)}{2x_n}, \qquad u_k = \frac{x_k}{x_n}, \quad 1 \le k < n,$$

$$u_n = \frac{1 - \left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right)}{2x_n}.$$

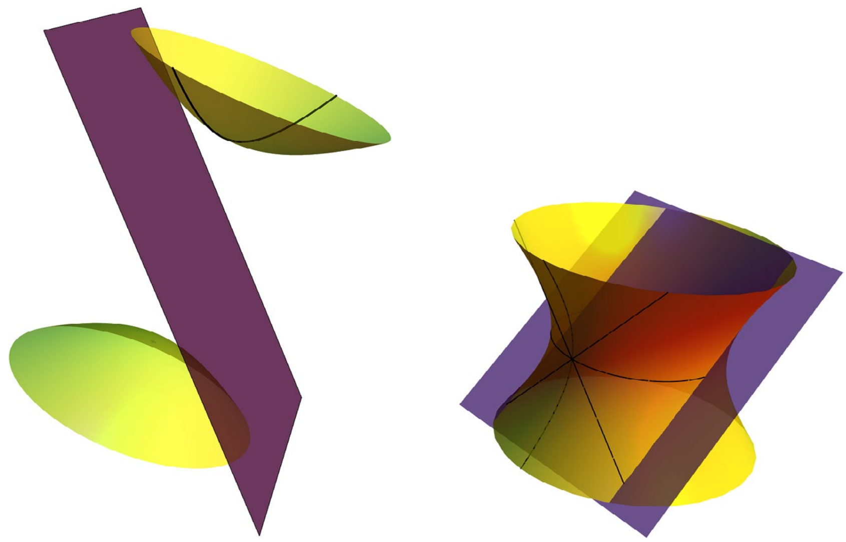

Fig. 1. Geodesics on $\mathbb{R}H^2$: Riemannian case (left), Lorentz case (right).

The image $f(\mathbb{R}H^n)$ is an open submanifold

$$\big\{u = (u_0, \dots, u_n) \in S^n_p \ \big|\ u_0 + u_n > 0\big\}.$$

The proof in the case where $\lambda > 0$ is similar, it is only necessary to replace S^n_p with H^n_p.

Theorem 4.1 is proved.

Remark 4.1. The previous theorem was proved by Nomizu [9] in the Lorentz case. Following the reasoning from the cited paper, we can show that there is an isomorphism

$$h\colon \mathbb{R}H^n \to SO^+(p, n-p)$$

such that the mapping f is equivariant, which means that the following diagram commutes for every $g \in \mathbb{R}H^n$:

$$\begin{array}{ccc} \mathbb{R}H^n & \xrightarrow{L_g} & \mathbb{R}H^n \\ \downarrow f & & \downarrow f \\ S^n_p & \xrightarrow{h(g)} & S^n_p \end{array}$$

(the same is true if we replace S^n_p with H^n_p).

Remark 4.2. Note that a geodesic curve $c(t)$ in $\mathbb{R}H^n$ is incomplete if and only if $f(c(t))$ reaches the boundary $u_0 + u_n = 0$ for a finite value of the affine parameter t. In Fig. 1, we illustrate this by an example of geodesics on $\mathbb{R}H^2$.

Theorem 4.2. *$(\mathbb{R}H^n, g_0)$ of signature $(p, n-p)$ is isometric to a part of $\mathbb{R}^n_p$ determined by the condition*

$$y_1 + y_n > 0.$$

Proof. We can define the following change of coordinates, i.e., a map from $\mathbb{R}H^n \subset \mathbb{R}^n_p$ to $\mathbb{R}^n_p$:

$$y_1 = \frac{1 + \left(2x_1x_n + \sum_{k=2}^{n-1} \epsilon_k x_k^2\right)}{2x_n}, \qquad y_k = \frac{x_k}{x_n}, \quad 1 < k < n,$$

$$y_n = \frac{1 - \left(2x_1x_n + \sum_{k=2}^{n-1} \epsilon_k x_k^2\right)}{2x_n}.$$

In these new coordinates, the metric g_0 has the form

$$g_0' = -dy_1^2 + \sum_{k=2}^{n-1} \epsilon_k dy_k^2 + dy_n^2.$$

This is a part of an open half space of the flat space form satisfying the relation $y_1 + y_n > 0$.

Theorem 4.2 is proved.

The present research was supported by the Serbian Ministry of Science (Project 174012, Geometry, Education, and Visualization with Applications).

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