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SHARP KOLMOGOROV–REMEZ-TYPE INEQUALITIES FOR PERIODIC FUNCTIONS OF LOW SMOOTHNESS

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In the case where either $r = 2,\ k = 1$ or $r = 3,\ k = 1, 2,$ for any $q, p \geq 1,\ \beta \in [0, 2\pi),$ and a Lebesgue-measurable set $B \subset I_{2\pi} := [-\pi/2, 3\pi/2],\ \mu B \leq \beta,$ we prove a sharp Kolmogorov–Remez-type inequality

$$\left\|f^{(k)}\right\|_q \leq \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)^{\alpha}_{L_p(I_{2\pi}\setminus B_{2m})}} \|f\|^{\alpha}_{L_p(I_{2\pi}\setminus B)} \left\|f^{(r)}\right\|^{1-\alpha}_{\infty}, \quad f \in L^r_{\infty},$$

with $\alpha = \min\left\{1 - k/r, (r - k + 1/q)/(r + 1/p)\right\},$ where φ_r is the perfect Euler spline of order $r,$ $E_0(f)_{L_p(G)}$ is the best approximation of f by constants in $L_p(G),\ B_{2m} = \left[\dfrac{\pi - 2m}{2}, \dfrac{\pi + 2m}{2}\right],$ and $m = m(\beta) \in [0, \pi)$ is uniquely defined by $\beta.$ We also establish a sharp Kolmogorov–Remez-type inequality, which takes into account the number of sign changes of the derivatives.

1. Introduction

Let G be a measurable subset of the real axis and let $L_p(G)$ be a space of measurable functions $x\colon G \to \mathbf{R}$ with finite norm (quasinorm)

$$\|x\|_{L_p(G)} := \begin{cases} \left(\displaystyle\int_G \left|x(t)\right|^p dt\right)^{1/p} & \text{for} \quad 0 < p < \infty, \\ \operatorname{vraisup}_{t\in G} |x(t)| & \text{for} \quad p = \infty. \end{cases}$$

By I_d we denote a circle realized in the form of the segment of length d with identified ends. For the sake of brevity, instead of $\|x\|_{L_p(I_{2\pi})}$ and $\|x\|_{L_\infty(\mathbf{R})}$, we write $\|x\|_p$ and $\|x\|_\infty$, respectively.

For $r \in \mathbf{N}$ and $G = \mathbf{R}$ (or $G = I_d$), by $L^r_\infty(G)$ we denote the space of all functions $x \in L_\infty(G)$ with locally absolutely continuous derivatives up to the $(r-1)$th order such that $x^{(r)} \in L_\infty(G)$.

By $\varphi_r(t),\ r \in \mathbf{N},$ we denote a shift of the rth 2π-periodic integral with zero mean value over a period of the function $\varphi_0(t) = \operatorname{sgn}\sin t$ satisfying the condition $\varphi_r(0) = 0.$

The following theorem was proved in [1]:

Theorem 1. *Suppose that $r \in \mathbf{N}$ and that the function $f \in L^r_\infty(I_{2\pi})$ is such that, for any segment $I = [a, b]$ satisfying the conditions*

$$f'(a) = f'(b) = 0, \qquad f'(t) \neq 0, \quad t \in (a, b),$$

there exists a function $f_I \in L^r_\infty(\mathbf{R})$ such that

$$f_I(t) = f(t), \quad t \in (a, b),$$

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$$E_0(f_I)_\infty \le E_0(f)_{L_\infty[a,b]},$$

and

$$\|f_I^{(r)}\|_\infty \le \|f^{(r)}\|_\infty.$$

Then, for any $k \in \mathbf{N},\ k < r,\ q \ge 1,$ *and* $p \in (0, \infty]$ *for* $k = 1$ *or* $p \in [1 - k/r, \infty]$ *for* $k > 1,$ *a sharp inequality*

$$\|f^{(k)}\|_q \le \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)_p^\alpha} \|f\|_p^\alpha \|f^{(r)}\|_\infty^{1-\alpha} \tag{1.1}$$

is true in the class $L_\infty^r(I_{2\pi})$ *with maximum possible exponent*

$$\alpha = \min\left\{1 - \frac{k}{r},\ \frac{r - k + 1/q}{r + 1/p}\right\},$$

where

$$E_0(f)_p := \inf_{c \in \mathbf{R}} \|f - c\|_p.$$

In particular, inequality (1.1) was proved in [1] for functions of low smoothness (i.e., for $r = 2,\ k = 1,$ and for $r = 3,\ k = 1, 2$).

Note that the problem of coincidence of sharp constants in inequalities of the form (1.1) for periodic and nonperiodic functions on the axis was investigated in [2].

In the present paper, we obtain a generalization of inequality (1.1) containing the so-called "Remez effect." We now present necessary definitions.

We say that $f \in L_\infty^1(\mathbf{R})$ is a comparison function for $x \in L_\infty^1(\mathbf{R})$ if there exists $c \in \mathbf{R}$ such that

$$\min_{t \in \mathbf{R}} f(t) + c \le x(t) \le \max_{t \in \mathbf{R}} f(t) + c, \quad t \in \mathbf{R},$$

and the equality $x(\xi) = f(\eta) + c,$ where $\xi, \eta \in \mathbf{R},$ yields the inequality $|x'(\xi)| \le |f'(\eta)|$ provided that these derivatives exist.

An odd 2ω-periodic function $\varphi \in L_\infty^1(I_{2\omega})$ is called an S-function if it has the following properties: φ is even with respect to $\omega/2$ and $|\varphi|$ is convex downward on $[0, \omega]$ and strictly monotone on $[0, \omega/2]$.

For a 2ω-periodic S-function φ, by $S_\varphi(\omega)$ we denote a class of functions $x \in L_\infty^1(I_d)$ for which φ is a comparison function. Note that the classes $S_\varphi(\omega)$ were considered in [3, 4]. As examples of the classes $S_\varphi(\omega)$, we can mention the Sobolev classes $\left\{f \in L_\infty^r(I_d) \colon \|f^{(r)}\|_\infty \le 1\right\}$ and bounded subsets of the space T_n (of trigonometric polynomials of degree of at most n) and the space $S_{n,r}$ (of 2π-periodic splines of order r and defect 1 with nodes at the points $k\pi/n,\ k \in \mathbf{Z}$).

In the approximation theory, an important role is played by the Remez-type inequalities

$$\|T\|_{L_\infty(I_{2\pi})} \le C(n, \beta) \|T\|_{L_\infty(I_{2\pi} \setminus B)} \tag{1.2}$$

on the class $T_n,$ where B is an arbitrary Lebesgue-measurable set $B \subset I_{2\pi},\ \mu B \le \beta.$

The investigations in this direction were originated by Remez in [5] who found the sharp constant $C(n, \beta)$ in an inequality of the form (1.2) for algebraic polynomials. Two-sided estimates for the sharp constant $C(n, \beta)$ in inequality (1.2) for trigonometric polynomials were established in a series of works. Moreover, the asymptotic behaviors of the constants $C(n, \beta)$ as $\beta \to 2\pi$ [6] and $\beta \to 0$ [7] are known. For the bibliography in this field, see [6–9]. In [7], the inequality

$$\|T\|_{L_\infty(I_{2\pi})} \le \left(1 + 2\tan^2 \frac{n\beta}{4m}\right) \|T\|_{L_\infty(I_{2\pi} \setminus B)} \tag{1.3}$$

was proved for an arbitrary polynomial $T \in T_n$ with the minimum period $2\pi/m$ and any Lebesgue measurable set $B \subset I_{2\pi},\ \mu B \le \beta$, where $\beta \in (0, 2\pi m/n)$. Inequality (1.3) turns into the equality for the polynomial

$$T(t) = \cos nx + \frac{1}{2}\left(1 - \cos \frac{\beta}{2}\right).$$

Recently [10], the sharp constant in inequality (1.2) was found for the case of trigonometric polynomials.

The result obtained in [7] was generalized to the classes $S_\varphi(\omega)$ in [11]. As a consequence, an analog of inequality (1.3) was obtained for polynomial splines and functions from the classes $L^r_\infty(I_{2\pi})$.

In [12, 13], sharp Remez-type inequalities were proved for various metrics in the classes $S_\varphi(\omega)$; in particular, for differentiable periodic functions, trigonometric polynomials, and splines.

In the present paper, we establish a sharp Kolmogorov–Remez-type inequality (Theorem 2) for functions satisfying the conditions of Theorem 1 with arbitrary $q, p \ge 1$. As a consequence, inequalities of this type are proved for functions of low smoothness for any $q, p \ge 1$ (Corollary 1). In addition, we prove the sharp Kolmogorov–Remez-type inequality (see Theorem 3 and its Corollary 2) taking into account the number of sign changes of the derivatives.

2. Necessary Information

Let $\alpha,\ y > 0$. For a 2ω-periodic S-function φ, we set

$$E^\alpha_y(\varphi) := \left\{t \in I_{2\omega} \colon\ |\varphi(t) + \alpha| > y\right\}. \tag{2.1}$$

It is clear that, for any $\beta \in (0, 2\omega)$, there exists a unique number $y = y(\beta)$ for which

$$\mu E^\alpha_{y(\beta)}(\varphi) = \beta, \tag{2.2}$$

where μ is the Lebesgue measure.

Lemma 1 [13]. *Suppose that $p \in [1, \infty]$. For any 2ω-periodic S-function φ and any $\beta \in (0, 2\omega)$, the following relation is true:*

$$\min_{\alpha \in \mathbf{R}} \left\{ \int\limits_{I_{2\omega} \setminus E^\alpha_{y(\beta)}(\varphi)} |\varphi(t) + \alpha|^p \, dt \right\}^{1/p} = E_0(\varphi)_{L_p(I_{2\omega} \setminus B_\beta)},$$

where

$$B_\beta := \left[\frac{\omega - \beta}{2}, \frac{\omega + \beta}{2}\right].$$

For a function $f \in L_1[a,b]$, by $m(f,y),\ y>0,$ we denote its distribution function given by the equality

$$m(f,y) := \mu\big\{t \in [a,b]\colon\ |f(t)| > y\big\}. \tag{2.3}$$

Moreover, by $r(f,t)$ we denote a decreasing permutation of the function $|f|$ (see, e.g., [14], Sec. 1.3). We set

$$r(f,t) = 0 \quad \text{for} \quad t > b-a.$$

For $f \in L_\infty(G),$ we introduce the definition

$$E_0(f)_{L_\infty(G)} := \inf_{c\in\mathbf{R}} \|f-c\|_{L_\infty(G)}$$

and, for $\lambda > 0$ and $r \in \mathbf{N},$ we set

$$\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t), \quad t \in \mathbf{R}.$$

Let $K_r := \|\varphi_r\|_\infty$ be the Favard constant.

Lemma 2. *Suppose that $r \in \mathbf{N}$, a function $f \in L^r_\infty(I_{2\pi})$ and a segment $I=[a,b]$ satisfy the conditions*

$$\big\|f^{(r)}\big\|_\infty = 1, \tag{2.4}$$

$$f'(a) = f'(b) = 0, \qquad f'(t) \neq 0, \quad t \in (a,b). \tag{2.5}$$

If there exists a function $f_I \in L^r_\infty(\mathbf{R})$ such that

$$f_I(t) = f(t), \qquad t \in (a,b), \tag{2.6}$$

$$E_0(f_I)_\infty \le E_0(f)_{L_\infty(I)} \tag{2.7}$$

and

$$\big\|f_I^{(r)}\big\|_\infty \le \big\|f^{(r)}\big\|_\infty = 1, \tag{2.8}$$

and λ is chosen from the condition

$$E_0(f)_{L_\infty(I)} = \|\varphi_{\lambda,r}\|_\infty, \tag{2.9}$$

then, for any $q,p \ge 1$ and $\beta \in [0, b-a)$, and an arbitrary Lebesgue measurable set $B \subset [a,b],\ \mu B \le \beta,$ the following inequality is true:

$$\|f\|_{L_p(I\setminus B)} \ge 2^{-1/p}\lambda^{-(r+1/p)}E_0(\varphi_r)_{L_p(I_{2\pi}\setminus B_{2\gamma})}, \tag{2.10}$$

where

$$B_{2\gamma} = \left[\frac{\pi-2\gamma}{2}, \frac{\pi+2\gamma}{2}\right]$$

and the number $\gamma = \gamma(\beta)$ *is uniquely defined by the condition*

$$r(\overline{\varphi}, \gamma/\lambda) = r(\overline{f}, \beta), \tag{2.11}$$

where, in turn, $\gamma < \min\{\pi, \beta\lambda\}$. *The inequality [15]*

$$\|f'\|_q \le 2^{-1/q} \lambda^{-(r-1+1/q)} \|\varphi_{r-1}\|_q \tag{2.12}$$

also holds.

Proof. We prove inequality (2.10). By virtue of condition (2.9), there exists $\alpha \in \mathbf{R}$ such that

$$\max_{t\in I} f(t) = \max_{t\in\mathbf{R}} \left[\varphi_{\lambda,r}(t) + \lambda^{-r}\alpha\right] = \lambda^{-r}[K_r + \alpha]$$

and

$$\min_{t\in I} f(t) = \min_{t\in\mathbf{R}} \left[\varphi_{\lambda,r}(t) + \lambda^{-r}\alpha\right] = \lambda^{-r}[\alpha - K_r].$$

According to (2.7)–(2.9), the function f_I satisfies the conditions of the Kolmogorov comparison theorem [16]. By this theorem, a spline $\varphi_{\lambda,r}(t) + \lambda^{-r}\alpha$ is a comparison function for the function f_I. For the sake of definiteness, we assume that f increases on $I = [a, b]$. Passing, if necessary, to a shift of the function

$$\varphi(t) := \varphi_{\lambda,r}(t) + \lambda^{-r}\alpha,$$

we can assume that $\varphi(t)$ also increases on $[-\omega/2,\ \omega/2]$, where $\omega := \pi/\lambda$. For $\tau \in \mathbf{R}$, we set

$$f_\tau(t) := f(t+\tau)$$

and choose τ_1 and τ_2 such that

$$f_{\tau_1}\left(\frac{\omega}{2}\right) = \max_{t\in I} f(t) \qquad \text{and} \qquad f_{\tau_2}\left(-\frac{\omega}{2}\right) = \min_{t\in I} f(t).$$

By the Kolmogorov comparison theorem, the following inequalities are true:

$$\big(f_{\tau_1}(t)\big)_+ \ge \varphi_+(t), \quad t \in \left[-\frac{\omega}{2}, \frac{\omega}{2}\right], \tag{2.13}$$

and

$$\big(f_{\tau_2}(t)\big)_- \ge \varphi_-(t), \quad t \in \left[-\frac{\omega}{2}, \frac{\omega}{2}\right], \tag{2.14}$$

where $u_\pm := \max\{\pm\, u, 0\}$.

By $\overline{f}$ we denote the restriction of f to $[a, b]$. By the symbol $\overline{\varphi}$ we denote the restriction of φ to $[-\omega/2,\ \omega/2]$, where $\omega := \pi/\lambda$. It follows from inequalities (2.13) and (2.14) that

$$b - a \ge \pi/\lambda \qquad \text{and} \qquad m\big(\overline{f}_\pm, y\big) \ge m(\overline{\varphi}_\pm, y), \quad y \ge 0,$$

where the function $m(f, y)$ is given by relation (2.3). Therefore,

$$m(\overline{f}, y) \ge m(\overline{\varphi}, y), \quad y \ge 0.$$

This directly implies that

$$r(\overline{f}, t) \ge r(\overline{\varphi}, t), \quad t \ge 0. \tag{2.15}$$

Further, we note that, for any measurable set $B \subset I,\ \mu B \le \beta,$ the inequality

$$\int_B |f(t)|^p\, dt \le \int_0^\beta r^p(\overline{f}, t) dt$$

is true. Since the permutation preserves the L_p-norm, we get

$$\begin{aligned} \|f\|^p_{L_p(I \setminus B)} &= \int_I |f(t)|^p\, dt - \int_B |f(t)|^p\, dt \\ &\ge \int_0^{b-a} r^p(\overline{f}, t) dt - \int_0^\beta r^p(\overline{f}, t) dt = \int_\beta^{b-a} r^p(\overline{f}, t) dt. \end{aligned} \tag{2.16}$$

According to the Kolmogorov comparison theorem, in view of relations (2.11), (2.15), and (2.16), we obtain

$$\begin{aligned} \|f\|^p_{L_p(I \setminus B)} &\ge \int_{\gamma/\lambda}^{\pi/\lambda} r^p(\overline{\varphi}, t)\, dt \\ &= 2^{-1}\lambda^{-(rp+1)} \int_{2\gamma}^{2\pi} r^p(\varphi_r + \alpha, t)\, dt \\ &= 2^{-1}\lambda^{-(rp+1)} \int_{I_{2\pi} \setminus E^\alpha_{y(2\gamma)}(\varphi_r)} |\varphi_r(t) + \alpha|^p\, dt, \end{aligned}$$

where $r(\varphi_r + \alpha, t)$ is a permutation of the restriction of $\varphi_r + \alpha$ to $I_{2\pi}$ and the set $E^\alpha_{y(\beta)}(\varphi)$ is given by relations (2.1) and (2.2). By virtue of Lemma 1, this inequality implies that

$$\|f\|^p_{L_p(I \setminus B)} \ge 2^{-1}\lambda^{-(rp+1)} E^p_0(\varphi_r)_{L_p(I_{2\pi} \setminus B_{2\gamma})},$$

which is equivalent to (2.10). Inequality (2.12) was proved in [15].

Lemma 2 is proved.

Remark 1. Inequality (2.10) with $\beta = 0$ was proved in [15]. Both inequalities [(2.10) with $\beta = 0$ and (2.12)] were formulated in [1] (in the same form as in Lemma 2).

Lemma 3 [1]. *Suppose that $r, k \in \mathbf{N},\ k < r,\ q \geq 1,$ and $p \in (0, \infty]$ for $k = 1$ or $p \in [1 - k/r, \infty]$ for $k > 1$. Further, assume that the numbers $\lambda_i > 0,\ i = 1, 2, \ldots, 2\nu,$ satisfy the conditions*

$$\sum_{i=1}^{2\nu} \frac{1}{\lambda_i} \leq 2, \tag{2.17}$$

$$\sum_{i=1}^{\nu} \frac{1}{\lambda_{2i}^r} = \sum_{i=0}^{\nu-1} \frac{1}{\lambda_{2i+1}^r}. \tag{2.18}$$

If

$$\alpha = \min\left\{1 - \frac{k}{r},\ \frac{r - k + 1/q}{r + 1/p}\right\},$$

then the inequality

$$C := \frac{\left(\frac{1}{2}\sum_{i=1}^{2\nu} \lambda_i^{-(q(r-1)+1)}\right)^{1/q}}{\left(\frac{1}{2}\sum_{i=1}^{2\nu} \lambda_i^{-(rp+1)}\right)^{\alpha/p}} \leq 1 \tag{2.19}$$

is true. If

$$\alpha = \frac{r - k + 1/q}{r + 1/p}, \qquad q < \frac{rp}{r - k},$$

then the inequality

$$C \leq \nu^{1/q - \alpha/p} \tag{2.20}$$

holds.

3. Main Results

Theorem 2. *Suppose that $r \in \mathbf{N}$ and a function $f \in L_\infty^r(I_{2\pi})$ is such that, for any segment $I = [a, b]$ satisfying the conditions*

$$f'(a) = f'(b) = 0 \quad \text{and} \quad f'(t) \neq 0, \quad t \in (a, b), \tag{3.1}$$

there exists a function $f_I \in L_\infty^r(\mathbf{R})$ such that

$$f_I(t) = f(t), \quad t \in (a, b), \tag{3.2}$$

$$E_0(f_I)_\infty \leq E_0(f)_{L_\infty[a,b]}, \tag{3.3}$$

and

$$\|f_I^{(r)}\|_\infty \le \|f^{(r)}\|_\infty. \tag{3.4}$$

Then, for any $q, p \ge 1,\ \beta \in [0, 2\pi),$ *and an arbitrary Lebesgue measurable set* $B \subset I_{2\pi},\ \mu B \le \beta,$ *the inequality*

$$\|f^{(k)}\|_q \le \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)^\alpha_{L_p(I_{2\pi}\setminus B_{2m})}} \|f\|^\alpha_{L_p(I_{2\pi}\setminus B)} \|f^{(r)}\|^{1-\alpha}_\infty \tag{3.5}$$

is true with the following exponent:

$$\alpha = \min\left\{1 - \frac{k}{r},\ \frac{r-k+1/q}{r+1/p}\right\}.$$

Here,

$$B_{2m} = \left[\frac{\pi - 2m}{2}, \frac{\pi + 2m}{2}\right],$$

$$m := \max_I\{\gamma_I\}$$

[the maximum is taken over all segments I *satisfying conditions (3.1)] and the numbers* γ_I *are uniquely determined by the relation*

$$r(\overline{\varphi_i}, \gamma_i/\lambda_i) = r(\overline{f_i}, \beta_i), \tag{3.6}$$

where $\overline{\varphi_i}$ *is the restriction of the spline* $\varphi_{\lambda_i, r}(t) + \lambda_i^{-r}\alpha_i$ *to the segment* $\left[-\frac{\pi}{2\lambda_i}, \frac{\pi}{2\lambda_i}\right]$ *and* $\overline{f_i}$ *is the restriction of the function* f *to the segment* I_i*.*

Inequality (3.5) is sharp in the class $L^r_\infty(I_{2\pi})$ *and turns into the equality for the function* $f(t) := \varphi_r - c_p$ *and the set* $B = B_{2m}$ *(for* $m = \beta/2$*), where* c_p *is the constant of best approximation of the spline* φ_r *in the metric of the space* $L_p(I_{2\pi} \setminus B_{2m})$*.*

Proof. We now prove inequality (3.5) for $k = 1$. We fix a function $f \in L^r_\infty(I_{2\pi})$ satisfying the conditions of the theorem. Without loss of generality, we can assume that

$$\|f^{(r)}\|_\infty = 1 \tag{3.7}$$

and $f'(0) = 0$. We set

$$E(f') = \{t \in [0, 2\pi] :\ f'(t) \ne 0\}.$$

It is clear that $E(f')$ is open and, hence, can be represented in the form

$$E(f') = \bigcup_{i=0}^{\nu(f')} (a_i, b_i),$$

where $\nu(f')$ is the number of sign changes of f' in a period.

Further, we fix $\beta \in [0, 2\pi)$ and an arbitrary measurable set $B \subset I_{2\pi},\ \mu B \le \beta$. Let

$$I_i := [a_i, b_i], \quad B^i = B \bigcap I_i, \quad \beta_i = \mu B^i, \quad \text{and} \quad i = 1, \dots, \nu(f').$$

We choose $\lambda_i > 0$ from the conditions

$$E_0(f)_{L_\infty(I_i)} = \|\varphi_{\lambda_i, r}\|_\infty, \quad i = 1, \dots, \nu(f'). \tag{3.8}$$

The numbers α_i are chosen to guarantee that

$$\max_{t \in I_i} f(t) = \max_{t \in \mathbf{R}} \left[\varphi_{\lambda_i, r}(t) + \lambda_i^{-r}\alpha_i\right] \qquad \text{and} \qquad \min_{t \in I_i} f(t) = \min_{t \in \mathbf{R}} \left[\varphi_{\lambda_i, r}(t) + \lambda_i^{-r}\alpha_i\right].$$

Thus, applying Lemma 2 to each segment $I_i,\ i = 1, \dots, \nu(f')$, we arrive at the inequalities

$$\|f\|_{L_p(I_i \setminus B^i)} \ge 2^{-1/p}\lambda_i^{-(r+1/p)} E_0(\varphi_r)_{L_p\left(I_{2\pi} \setminus B_{2\gamma_i}\right)} \tag{3.9}$$

and

$$\|f'\|_q \le 2^{-1/q}\lambda_i^{-(r-1+1/q)}\|\varphi_{r-1}\|_q, \tag{3.10}$$

where

$$B_{2\gamma_i} = \left[\frac{\pi - 2\gamma_i}{2}, \frac{\pi + 2\gamma_i}{2}\right],$$

and the numbers $\gamma_i = \gamma_i(\beta_i)$ are uniquely determined by conditions (2.11).

We now set

$$m := \max\left\{\gamma_i,\ i = 1, \dots, \nu(f')\right\}.$$

It is clear that

$$E_0(\varphi_r)_{L_p(I_{2\pi} \setminus B_{2\gamma_i})} \ge E_0(\varphi_r)_{L_p(I_{2\pi} \setminus B_{2m})}.$$

Finding the sum of estimates (3.9) and (3.10), we obtain

$$\|f'\|_q^q = \sum_{i=1}^{\nu(f')} \|f'\|_{L_q(I_i)}^q \le \frac{1}{2}\|\varphi_{r-1}\|_q^q \sum_{i=1}^{\nu(f')} \lambda_i^{-((r-1)q+1)}$$

and

$$\|f\|_{L_p(I_{2\pi} \setminus B)}^p = \sum_{i=1}^{\nu(f')} \|f\|_{L_p(I_i \setminus B^i)}^p \ge \frac{1}{2} E_0^p(\varphi_r)_{L_p(I_{2\pi} \setminus B_{2m})} \sum_{i=1}^{\nu(f')} \lambda_i^{-(rp+1)}.$$

Therefore,

$$\frac{\|f'\|_q}{\|f\|_{L_p(I_{2\pi} \setminus B)}^\alpha} \le \frac{\|\varphi_{r-1}\|_q}{E_0^\alpha(\varphi_r)_{L_p(I_{2\pi} \setminus B_{2m})}} C, \tag{3.11}$$

where

$$C = \frac{\left(\frac{1}{2}\sum_{i=1}^{\nu(f')} \lambda_i^{-(q(r-1)+1)}\right)^{1/q}}{\left(\frac{1}{2}\sum_{i=1}^{\nu(f')} \lambda_i^{-(rp+1)}\right)^{\alpha/p}}. \tag{3.12}$$

By virtue of relations (3.3), (3.4), (3.7), and (3.8), the conditions of the Kolmogorov comparison theorem are satisfied [16]. According to this theorem, the inequalities $\pi/\lambda_i \le b_i - a_i,\ i = 1, \ldots, \nu(f')$, are true. In view of an obvious estimate

$$\sum_{i=1}^{\nu(f')} (b_i - a_i) \le 2\pi,$$

these inequalities imply that conditions (2.17) are satisfied. In addition, in view of the periodicity of the function f, the sum of variations of this function in all segments of increasing is equal to the sum of variations of the function f in all segments of decreasing. Thus, by virtue of equality (3.8) used to find λ_i, the same conclusion is also true for the comparison function $\varphi_{\lambda_i, r}$. Since

$$\|\varphi_{\lambda_i, r}\|_\infty = \lambda_i^{-r}\|\varphi_r\|_\infty,$$

condition (2.18) is also satisfied. Hence, all conditions of Lemma 3 are satisfied. By virtue of this lemma, estimate (2.19) is true. Applying this estimate to relation (3.11), we complete the proof of inequality (3.5) for $k = 1$ by virtue of condition (3.7).

For $k > 1$, inequality (3.5) is proved by induction in exactly the same way as in [1] where it was proved for $\beta = 0$.

Theorem 2 is proved.

Remark 2. For $\beta = 0$, Theorem 2 was proved in [1].

In [15], it was shown that, for $r = 2$ and $r = 3$, any function $f \in L_\infty^r(I_{2\pi})$, and any segment $I = [a, b]$ satisfying conditions (3.1), there exists a function $f_I \in L_\infty^r(\mathbf{R})$ satisfying requirements (3.2)–(3.4). For its construction, it is sufficient to extend the restriction of the function f to $I = [a, b]$ onto the segment $[b, 2b - a]$ as an even function with respect to the point b and then $2(b - a)$-periodically onto the entire axis. This fact and Theorem 2 imply the following assertion:

Corollary 1. *Suppose that $q, p \in [1, \infty],\ r = 2,\ k = 1$ or $r = 3,\ k = 1, 2,\ \beta \in [0, 2\pi)$. Then, for any function $f \in L_\infty^r(I_{2\pi})$ and an arbitrary Lebesgue measurable set $B \subset I_{2\pi} := [-\pi/2, 3\pi/2],\ \mu B \le \beta$, the following inequality, which is sharp in the class $L_\infty^r(I_{2\pi})$, is true with the same exponent α as in Theorem 2:*

$$\left\|f^{(k)}\right\|_q \le \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)^\alpha_{L_p(I_{2\pi}\setminus B_{2m})}} \|f\|^\alpha_{L_p(I_{2\pi}\setminus B)} \left\|f^{(r)}\right\|_\infty^{1-\alpha}.$$

Remark 3. For $\beta = 0$, Corollary 1 was proved in [1].

In [17], it was shown that, for sufficiently large $r \in \mathbf{N}$, there exist a function $f \in L_\infty^r(I_{2\pi})$ and a segment I for which conditions (3.1) are satisfied. However, for the indicated function and segment, the class $L_\infty^r(\mathbf{R})$ does not contain a function f_I satisfying requirements (3.2)–(3.4). In other words, there are functions $f \in L_\infty^r(I_{2\pi})$ that cannot be extended from an arbitrary segment of monotonicity I to the entire axis with preservation of smoothness of the extended function, the L_∞-norm of its higher derivative, and the best uniform approximation by the same constant as in the segment I.

Thus, as follows from [18], the exponent α in inequality (3.5) is maximally possible. In particular, if

$$q < \frac{rp}{r-k},$$

then

$$\alpha = \min\left\{1 - \frac{k}{r}, \frac{r-k+1/q}{r+1/p}\right\} = 1 - \frac{k}{r}.$$

The next theorem shows that this exponent can be increased if we take into account the number $\nu(f^{(k)})$ of sign changes of the derivative $f^{(k)}$ in inequality (3.5).

Theorem 3. *Under the conditions of Theorem 2, for* $q < \dfrac{rp}{r-k}$ *and* $\alpha = \dfrac{r-k+1/q}{r+1/p}$, *the following inequality is true:*

$$\left\|f^{(k)}\right\|_q \le \left(\frac{\nu(f^{(k)})}{2}\right)^{1/q-\alpha/p} \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)^\alpha_{L_p(I_{2\pi}\setminus B_{2m})}} \|f\|^\alpha_{L_p(I_{2\pi}\setminus B)} \left\|f^{(r)}\right\|^{1-\alpha}_\infty, \tag{3.13}$$

where

$$B_{2m} = \left[\frac{\pi - 2m}{2}, \frac{\pi + 2m}{2}\right]$$

and the number $m = m(\beta)$ *is uniquely determined by the number* β *as in Theorem 2.*

Inequality (3.13) is sharp in the class $L^r_\infty(I_{2\pi})$ *and turns into the equality for the same function* f *and set* B *as in Theorem 2.*

Proof. We prove inequality (3.13) for $k = 1$. We fix a function $f \in L^r_\infty(I_{2\pi})$ satisfying the conditions of the theorem. Without loss of generality, we can assume that

$$\left\|f^{(r)}\right\|_\infty = 1. \tag{3.14}$$

Repeating the reasoning used in the proof of Theorem 2 with

$$\alpha = \frac{r-k+1/q}{r+1/p},$$

we arrive at inequality (3.11) with the same exponent, where C is a constant given by equality (3.12). Moreover, as in Theorem 2, we can show that the numbers λ_i satisfy the conditions of Lemma 3. By applying estimate (2.20)

$$C \le \left(\frac{\nu(f^{(k)})}{2}\right)^{1/q-\alpha/p}$$

to inequality (3.11), by virtue of condition (3.14), we arrive at inequality (3.13) with $k = 1$.

For $k > 1$, inequality (3.13) is proved by induction in exactly the same way as in [1], where it was proved for $\beta = 0$.

Theorem 3 is proved.

Remark 4. For $\beta = 0$, Theorem 3 was proved in [1].

For the first time, Kolmogorov-type inequalities taking into account the number of sign changes of the derivatives were proved by Ligun [19]. He applied these inequalities to the solution of extreme problems of the approximation theory. Some other inequalities of this type can be found in [20–22].

By using the property of extension of the functions of low smoothness from the segment of monotonicity established in [15] and used earlier to obtain Corollary 1 of Theorem 2, we arrive at the following corollary of Theorem 3:

Corollary 2. *Suppose that* $q, p \in [1, \infty]$, $q < \dfrac{rp}{r-k}$, $r = 2$, $k = 1$ *or* $r = 3$, $k = 1, 2$, $\beta \in [0, 2\pi)$. *Then, for any function* $f \in L^r_\infty(I_{2\pi})$ *and an arbitrary Lebesgue measurable set* $B \subset I_{2\pi} := [-\pi/2, 3\pi/2]$, $\mu B \le \beta$, *the following inequality, which is sharp in the class* $L^r_\infty(I_{2\pi})$, *is true:*

$$\left\| f^{(k)} \right\|_q \le \left(\frac{\nu(f^{(k)})}{2} \right)^{1/q - \alpha/p} \frac{\|\varphi_{r-k}\|_q}{E_0(\varphi_r)^{\alpha}_{L_p(I_{2\pi} \setminus B_{2m})}} \|f\|^{\alpha}_{L_p(I_{2\pi} \setminus B)} \left\| f^{(r)} \right\|^{1-\alpha}_\infty,$$

where

$$\alpha = \frac{r - k + 1/q}{r + 1/p}.$$

Remark 5. For $\beta = 0$, Corollary 2 was proved in [1].

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