

## CLASSIFICATIONS OF TRANSLATION SURFACES IN ISOTROPIC GEOMETRY WITH CONSTANT CURVATURE

M. E. Aydin

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We classify translation surfaces in isotropic geometry with arbitrary constant isotropic Gaussian and mean curvatures under the condition that at least one translating curve lies in the plane.

### 1. Introduction

A *translation surface* in the Euclidean space  $\mathbb{R}^3$  expressed as the sum of two curves can be locally parametrized by [5]

$$r(x, y) = \alpha(x) + \beta(y), \quad (1.1)$$

where  $\alpha$  and  $\beta$  are referred to as *translating curves*. Recent results and progress in the field of translation surfaces in  $\mathbb{R}^3$  with constant Gaussian and mean curvatures were well structured in [14–16, 25].

If  $\alpha$  and  $\beta$  lie in orthogonal planes, then, up to a change of coordinates, the surface can be locally described in the explicit form

$$z(x, y) = f(x) + g(y),$$

where  $f, g$  are smooth real-valued functions of one variable. In this case, parallel with the planes, the only minimal translation surface (i.e., its mean curvature identically vanishes) is the *Scherk surface*, namely, the graph of [34]

$$z(x, y) = \frac{1}{c} \log \left| \frac{\cos(cy)}{\cos(cx)} \right|, \quad c \in \mathbb{R} - \{0\}.$$

Numerous generalizations of this result in (semi-) Euclidean and homogeneous spaces were done so far, see, e.g., [8–10, 12, 17, 19, 20, 23, 24, 26, 28, 29, 35, 37, 38].

Recently, Liu and Yu [22] introduced a new class of translation surfaces in  $\mathbb{R}^3$ , namely, so-called *affine translation surfaces*, as the graphs of

$$z(x, y) = f(x) + g(y + ax), \quad a \in \mathbb{R} - \{0\}. \quad (1.2)$$

By the change of coordinates  $x = u$ ,  $y = v - au$  in (1.2), we get the following local parametrization:

$$r(u, v) = (u, v - au, f(u) + g(v)),$$

where the translating curves lie in the planes  $x = 0$  and  $ax + y = 0$ . Since  $a \neq 0$ , these planes are not orthogonal

Firat University, Elazig, Turkey; e-mail: meaydin@firat.edu.tr.

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to each other and the obtained surface is a natural generalization of the classical translation surface. In the same paper, the authors conjectured that, parallel with the planes, solely the minimal graph surface of the form (1.2), usually called the *affine Scherk surface*, is given in explicit form

$$z(x, y) = \frac{1}{c} \log \left| \frac{\cos(c\sqrt{1+a^2}x)}{\cos(c[y+ax])} \right|, \quad c \in \mathbb{R} - \{0\}.$$

We also refer the reader to [18, 21, 39, 40] for more recent results on this kind of surfaces.

Following Liu and Yu [22] we introduce and classify a new type of translation surfaces in the isotropic geometry with constant isotropic Gaussian curvature (CIGC) and constant isotropic mean curvature (CIMC). In addition, we obtain the surfaces of CIGC and CIMC for which one of translating curves is planar and the other curve is a space curve.

## 2. Preliminaries

For the fundamental notions of curves and surfaces in isotropic geometry, i.e., in one of the Cayley–Klein geometries, we refer the reader to [3, 4, 6, 7, 11, 30–33]. These notions can be briefed by the arguments from projective geometry as in the next paragraphs.

Let  $\mathbb{P}^3$  denote the projective space and let  $\Gamma$  be a plane in  $\mathbb{P}^3$ . Then an affine space can be obtained from  $\mathbb{P}^3$  by subtracting  $\Gamma$ , which is called the *absolute plane*. If  $\Gamma$  involves a pair of complex-conjugate straight lines  $l_1$  and  $l_2$ , the so-called *absolute lines*, then the obtained affine space becomes an *isotropic space*  $\mathbb{I}^3$ , where the triple  $(\Gamma, l_1, l_2)$  is referred to as the *absolute figure* of  $\mathbb{I}^3$ .

Let a quadruple  $(\tilde{t} : \tilde{x} : \tilde{y} : \tilde{z})$  be the projective coordinates, i.e.,  $(\tilde{t} : \tilde{x} : \tilde{y} : \tilde{z}) \neq (0 : 0 : 0 : 0)$ . Then  $\Gamma$  and  $l_1, l_2$  are parametrized by  $\tilde{t} = 0$  and  $\tilde{t} = \tilde{x} \pm i\tilde{y} = 0$ , respectively. The intersection point of  $l_1$  and  $l_2$  is said to be *absolute*, i.e.,  $(0 : 0 : 0 : 1)$ .

We are interested in an affine model of  $\mathbb{I}^3$ . Thus, by means of the affine coordinates

$$x = \frac{\tilde{x}}{\tilde{t}}, \quad y = \frac{\tilde{y}}{\tilde{t}}, \quad z = \frac{\tilde{z}}{\tilde{t}}, \quad \tilde{t} \neq 0,$$

the group of motions of  $\mathbb{I}^3$  is a six-parameter group given by

$$(x, y, z) \mapsto (x', y', z') : \begin{cases} x' = a + x \cos \theta - y \sin \theta, \\ y' = b + x \sin \theta + y \cos \theta, \\ z' = c + dx + ey + z, \end{cases} \quad (2.1)$$

where  $a, b, c, d, e, \theta \in \mathbb{R}$ . The metric invariants of  $\mathbb{I}^3$  under (2.1), such as *isotropic distance* and *angle*, are Euclidean invariants in the Cartesian plane.

A line in  $\mathbb{I}^3$  is said to be *isotropic* provided that its point at infinity agrees with the absolute point. In the affine model of  $\mathbb{I}^3$ , it corresponds to a line parallel to the  $z$ -axis. Otherwise, it is called a *nonisotropic line*.

A plane in  $\mathbb{I}^3$  containing an isotropic line is called *isotropic* and, in this case, its line includes the absolute point at infinity. Otherwise, it is called a *nonisotropic plane*. Thus, the equation  $ax + by + cz = d$ ,  $a, b, c, d \in \mathbb{R}$ , determines a nonisotropic (resp., isotropic) plane for  $c \neq 0$  (resp., for  $c = 0$ ).

A unit speed curve has the form

$$\alpha : I \subseteq \mathbb{R} \longrightarrow \mathbb{I}^3, \quad s \longmapsto (f(s), g(s), h(s)), \quad (f')^2 + (g')^2 = 1,$$

where primes denote the derivatives with respect to  $s$ . Therefore, the *curvature*  $\kappa$  and *torsion*  $\tau$  are given by the formulas

$$\kappa = \sqrt{(f'')^2 + (g'')^2} \quad \text{or} \quad \kappa = f'g'' - f''g',$$

and

$$\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}, \quad \kappa \neq 0. \tag{2.2}$$

A curve that lies in an isotropic (resp., nonisotropic) plane is called *isotropic* (resp., *nonisotropic*) *planar*. Otherwise, we call it *space curve* and, in this case,  $\tau \neq 0$ .

Let  $M^2$  be an admissible surface in  $\mathbb{I}^3$ , i.e., a surface for which the tangent plane is nonisotropic at each point. Then the tangent plane  $T_p(M^2)$  has a Euclidean metric at some point  $p \in M^2$ . For this surface, the components  $E, F$ , and  $G$  of the first fundamental form are obtained by the metric on  $M^2$  induced from  $\mathbb{I}^3$ .

The unit isotropic direction  $U = (0, 0, 1)$  is assumed to be a normal vector field of  $M^2$ , which is indeed orthogonal to all tangent vectors in  $T_p(M^2)$ . Hence, the components of the second fundamental form are computed with respect to  $U$ , namely,

$$l = \frac{\det(r_{xx}, r_x, r_y)}{\sqrt{EG - F^2}}, \quad m = \frac{\det(r_{xy}, r_x, r_y)}{\sqrt{EG - F^2}}, \quad n = \frac{\det(r_{yy}, r_x, r_y)}{\sqrt{EG - F^2}},$$

where  $r = r(x, y)$  refers to a local parametrization on  $M^2$ ,  $r_x = \frac{\partial r}{\partial x}$ , and  $r_{xy} = \frac{\partial^2 r}{\partial x \partial y}$ , etc. Note that the admissibility of  $M^2$  implies that  $EG - F^2 \neq 0$ .

The *isotropic Gaussian* (so-called *relative*)  $K$  and the *mean curvatures*  $H$  are defined by

$$K = \frac{ln - m^2}{EG - F^2} \quad \text{and} \quad H = \frac{En - 2Fm + Gl}{2(EG - F^2)}.$$

A surface for which  $H$  (resp.,  $K$ ) identically vanishes is said to be *isotropic minimal* (resp., *flat*). Moreover, a surface is said to have *CIMC* (resp., *CIGC*) if  $H$  (resp.,  $K$ ) is a constant function in the entire surface.

### 3. Categorization of Translation Surfaces

The translation surfaces in  $\mathbb{I}^3$  locally given by (1.1) can be categorized in terms of the translating curves and the absolute figure as follows:

Type I:  $\alpha$  and  $\beta$  are planar:

Type I.1:  $\alpha$  and  $\beta$  are isotropic planar.

Type I.2:  $\alpha$  is isotropic planar and  $\beta$  nonisotropic planar.

Type I.3:  $\alpha$  and  $\beta$  are nonisotropic planar.

Type II:  $\alpha$  is isotropic planar and  $\beta$  is a space curve.

Type III:  $\alpha$  is nonisotropic planar and  $\beta$  is a space curve.

Type IV:  $\alpha$  and  $\beta$  are space curves.

A surface that belongs to one type is not equivalent to a surface of another type up to the absolute figure.

For a surface of Type 1, we assume that the translating curves lie in orthogonal planes. Let  $f$  and  $g$  be smooth functions. By a change of coordinates, this surface can be locally represented in one of the following explicit forms:

Type I.1\*: Both translating curves are isotropic planar  $z(x, y) = f(x) + g(y)$ .

Type I.2\*: One translating curve is nonisotropic planar and the other curve isotropic planar

$$y(x, z) = f(x) + g(z).$$

Type I.3\*: Both translating curves are nonisotropic planar

$$x(y, z) = \frac{1}{2} \left[ f \left( \frac{y + z - \pi}{2} \right) + g \left( \frac{-y + z + \pi}{2} \right) \right].$$

Surfaces of this kind with CIMC and CIGC were obtained in [27, 36].

Further, for a surface of Type 1, we assume that the translating curves lie in arbitrary planes. Let  $[a_{ij}]$  be a  $2 \times 2$  matrix and let

$$|a_{ij}| = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

More generally, the surfaces of Type I.1 are locally given by

$$r(u, v) = \left( \frac{a_{22}u}{|a_{ij}|} - \frac{a_{12}v}{|a_{ij}|}, -\frac{a_{21}u}{|a_{ij}|} + \frac{a_{11}v}{|a_{ij}|}, f(u) + g(v) \right). \quad (3.1)$$

Up to a change of coordinates, surface (3.1) turns into a graph of the form

$$z = f(a_{11}x + a_{12}y) + g(a_{21}x + a_{22}y). \quad (3.2)$$

The indicated surfaces with CIMC and CIGC, which are called *translation graphs of the first kind*, were presented in [2]. In the present paper, we are interested in the surfaces of Types I.2–III.

In the case where one curve is isotropic planar and the other curve has no conditions, the translation surfaces with CIMC and CIGC can be found in [1].

#### 4. Surfaces of Types I.2 and I.3

Let  $[a_{ij}]$  denote a  $2 \times 2$  matrix and let  $\omega = |a_{ij}| \neq 0$ . We consider the following translation surface generated by planar curves:

$$r(u, v) = \left( \frac{a_{22}u}{\omega} - \frac{a_{12}v}{\omega}, f(u) + g(v), -\frac{a_{21}u}{\omega} + \frac{a_{11}v}{\omega} \right), \quad (4.1)$$

where the translating curves and the planes including these curves are given by

$$\alpha(u) = \left( \frac{a_{22}u}{\omega}, f(u), -\frac{a_{21}u}{\omega} \right), \quad \Gamma_\alpha : a_{21}x + a_{22}z = 0$$

and

$$\beta(v) = \left( -\frac{a_{12}v}{\omega}, g(v), \frac{a_{11}v}{\omega} \right), \quad \Gamma_\beta : a_{11}x + a_{12}z = 0.$$

**Remark 4.1.** Throughout the section, in view of the fact that the roles played by  $f$  and  $g$  are symmetric, we only discuss the cases specified for a function  $f$ .

For the surface given by (4.1), we have:

The planes  $\Gamma_\alpha$  and  $\Gamma_\beta$  are orthogonal to each other provided that  $[a_{ij}]$  is an orthogonal matrix.

If  $a_{12} = 0$  then, in view of  $\omega \neq 0$ ,  $\Gamma_\alpha$  becomes a nonisotropic plane and  $\Gamma_\beta$  is an isotropic plane. Hence, the obtained surface belongs to Type I.2.

If  $a_{12} \neq 0$ , then (by symmetry)  $a_{22} \neq 0$ , and the planes  $\Gamma_\alpha, \Gamma_\beta$  are nonisotropic. Therefore, the obtained surface belongs to Type I.3.

By the change of coordinates (the so-called *affine parameter coordinates*):

$$u = a_{11}x + a_{12}z, \quad v = a_{21}x + a_{22}z,$$

the local surface given by (4.1) turns into a graph of the form

$$y = f(u) + g(v). \tag{4.2}$$

A surface of the form (4.2) is called a *translation graph of the second kind*. Note that it is not equivalent to a graph of the form (3.2) to within the absolute figure. The positive side of this notion is the possibility of expression of the surfaces of Types I.2 and I.3 in the same format.

We now want to present translation graphs of the second kind in  $\mathbb{I}^3$  with CIGC. Thus, in view of the admissibility, we obtain

$$a_{12}f' + a_{22}g' \neq 0, \quad f' = \frac{df}{du}, \quad g' = \frac{dg}{dv}.$$

After necessary calculations, the Gaussian curvature  $K$  turns into

$$K = \frac{\omega^2 f'' g''}{(a_{12}f' + a_{22}g')^4}. \tag{4.3}$$

It follows from (4.3) that  $K$  identically vanishes provided that  $f'' = 0$ , namely, the surface is a *generalized cylinder* (see [13, p. 439]) with nonisotropic rulings. Hence, next result can be formulated in order to guarantee that  $K$  is a nonvanishing constant.

**Remark 4.2.** Throughout the paper, for the sake of convenience of calculations, we denote nonzero constants by  $c_1, c_2, \dots$  and some constants by  $d_1, d_2, \dots$ , unless otherwise stated.

**Theorem 4.1.** *The following relations hold for a translation graph of the second kind in  $\mathbb{I}^3$  with nonzero CIGC:*

$$f(u) = \frac{c_1}{2}u^2 + d_1u + d_2 \quad \text{and} \quad g(v) = \frac{-c_1a_{11}^2}{2K_0a_{22}^2} \left( -\frac{3K_0a_{22}^2}{c_1a_{11}^2}v + d_3 \right)^{\frac{2}{3}} + d_4. \quad (4.4)$$

**Proof.** Since  $K = K_0 \neq 0$ ,  $K_0 \in \mathbb{R}$ , in (4.3), we get  $f''g'' \neq 0$ . The partial derivative of (4.3) with respect to  $u$  gives

$$\frac{4K_0}{\omega^2} (a_{12}f' + a_{22}g')^3 (a_{12}f'') = f'''g''. \quad (4.5)$$

In analyzing relation (4.5), we distinguish the following two cases:

*Case 1:*  $a_{12} = 0$ . Then  $a_{11}a_{22} \neq 0$  because  $\omega \neq 0$ . By (4.5), we get  $f'' = c_1$  and, therefore, it follows from (4.3) that

$$\frac{K_0a_{22}^2}{c_1a_{11}^2} = \frac{g''}{(g')^4}. \quad (4.6)$$

Solving the equations  $f'' = c_1$  and (4.6), we arrive at (4.4).

*Case 2:*  $a_{12} \neq 0$ . By virtue of symmetry, we conclude that  $a_{22} \neq 0$ . Then (4.5) can be arranged as follows:

$$\frac{(a_{12}f' + a_{22}g')^3}{g''} = \frac{\omega^2}{4K_0a_{12}} \left( \frac{f'''}{f''} \right). \quad (4.7)$$

The partial derivative of (4.7) with respect to  $v$  yields

$$3a_{22}(g'')^2 - (a_{12}f' + a_{22}g')g''' = 0, \quad (4.8)$$

where

$$g''' \neq 0 \quad \text{because} \quad a_{22}g'' \neq 0.$$

By taking the partial derivative of (4.8) with respect to  $u$ , we immediately arrive at a contradiction.

Theorem 4.1 is proved.

After necessary calculations, for the isotropic mean curvature  $H$  of (4.2), we obtain

$$H = -\frac{[a_{12}^2 + (\omega g')^2] f'' + [a_{22}^2 + (\omega f')^2] g''}{2(a_{12}f' + a_{22}g')^3}. \quad (4.9)$$

First, we consider the minimality case and prove the following result:

**Theorem 4.2.** *For a minimal translation graph of the second kind in  $\mathbb{I}^3$  one of the following assertions is true:*

(1) *it is a nonisotropic plane;*

$$(2) \quad f(u) = \frac{1}{c_1 a_{11}^2 a_{22}^2} \log |\cos (c_1 a_{11} a_{22}^2 u + d_1)| + d_2,$$

$$g(v) = \frac{-1}{c_1 a_{11}^2 a_{22}^2} \log |c_1 a_{11}^2 a_{22}^2 v + d_3| + d_4,$$

$$(3) \quad f(u) = \frac{1}{c_1 \omega^2} \log |\cos (\omega c_1 a_{22} u + d_1)| + d_2,$$

$$g(v) = \frac{-1}{c_1 \omega^2} \log |\cos (\omega c_1 a_{12} v + d_3)| + d_4.$$

**Proof.** Since  $H$  identically vanishes, (4.9) reduces to

$$[a_{12}^2 + (\omega g')^2] f'' + [a_{22}^2 + (\omega f')^2] g'' = 0. \tag{4.10}$$

Note that  $f'' = g'' = 0$  is a solution for (4.10) and, in this case, the surface is a nonisotropic plane. Suppose that  $f'' g'' \neq 0$ . Hence, (4.10) implies that

$$-\frac{f''}{a_{22}^2 + (\omega f')^2} = c_1 = -\frac{g''}{a_{12}^2 + (\omega g')^2}. \tag{4.11}$$

We have two cases:

*Case 1:*  $a_{12} = 0$ . Then  $a_{11} a_{22} \neq 0$  because  $\omega \neq 0$ . We solve (4.11) and obtain assertion (2) of the theorem.

*Case 2:*  $a_{12} \neq 0$ . By virtue of symmetry, we get  $a_{22} \neq 0$ . Thus, solving (4.11), we get the last assertion of the theorem.

**Theorem 4.3.** *For a translation graph of the second kind in  $\mathbb{I}^3$  with nonzero CIMC, one of the following assertions is true:*

$$(a) \quad f(u) = c_1 u + d_1, \quad g(v) = -H_0 \frac{a_{12} c_1}{a_{21}^2} v^2 + d_2 v + d_3,$$

or

$$(b) \quad f(u) = c_1 u + d_1, \quad g(v) = \frac{a_{22}^2 + (\omega c_1)^2}{2H_0 a_{22}^2} \left( \frac{4H_0 a_{22}}{a_{22}^2 + (\omega c_1)^2} v + d_2 \right)^{\frac{1}{2}} - \frac{a_{12} c_1}{a_{22}} v + d_3.$$

**Proof.** Assume that  $H = H_0 \neq 0$ ,  $H_0 \in \mathbb{R}$ , in (4.9). The partial derivatives of (4.9) with respect to  $u$  and  $v$  yield

$$-6H_0 \omega^{-2} a_{12} a_{22} (a_{12} f' + a_{22} g') (f'' g'') = g' g'' f''' + f' f'' g'''. \tag{4.12}$$

The situation in which both  $f''$  and  $g''$  vanish is a solution of (4.12). However, we omit this solution because  $H_0 \neq 0$ .

We distinguish the following remaining cases:

*Case 1:*  $f = c_1 u + d_1$  and  $g'' \neq 0$ . This assumption is a solution for (4.12). Thus, from (4.9), we derive

$$\frac{g''}{(a_{12}c_1 + a_{22}g')^3} = \frac{-2H_0}{a_{22}^2 + (\omega c_1)^2}. \quad (4.13)$$

We have two cases:

- (1.1)  $a_{22} = 0$ . In this case,  $a_{12}a_{21} \neq 0$  due to  $\omega \neq 0$ . Solving (4.13), we get the first assertion of the theorem.  
 (1.2)  $a_{22} \neq 0$ . By virtue of symmetry, we get  $a_{12} \neq 0$ . Solving (4.13), we arrive at the second assertion of the theorem.

*Case 2:*  $f''g'' \neq 0$ . Dividing (4.12) by  $f''g''$ , we find

$$-6H_0\omega^{-2}a_{12}a_{22}(a_{12}f' + a_{22}g') = g' \frac{f'''}{f''} + f' \frac{g'''}{g''}. \quad (4.14)$$

We have the following cases:

- (2.1)  $a_{12} = 0$ . Then  $\omega \neq 0$  implies that  $a_{11}a_{22} \neq 0$ . Hence, relation (4.14) turns into

$$\frac{f'''}{f'f''} = d_1 = -\frac{g'''}{g'g''}.$$

This gives  $f'' = c_1 e^{d_1 f}$  and  $g'' = c_2 e^{-d_1 g}$ . Substituting these formulas in (4.9), we obtain

$$-2H_0a_{22}(g')^3 = c_1a_{11}^2(g')^2e^{d_1 f} + [c_2 + c_2a_{11}^2(f')^2]e^{-d_1 g}. \quad (4.15)$$

We set  $f' = p$  and  $g' = q$  in (4.15). Thus, taking the partial derivative of (4.15) with respect to  $f$ , we get

$$0 = d_1c_1q^2e^{d_1 f} + 2c_2p\dot{p}e^{-d_1 g}, \quad (4.16)$$

where

$$\dot{p} = \frac{dp}{df} = \frac{f''}{f'}.$$

If  $d_1 = 0$  in (4.16), then we arrive at a contradiction,  $\dot{p} = 0$ . Otherwise,

$$\frac{d_1c_1e^{d_1 f}}{2c_2p\dot{p}} = c_3 = -\frac{e^{-d_1 g}}{q^2}. \quad (4.17)$$

Substituting the second equality in (4.17) in (4.15), we conclude that

$$-2H_0a_{22}q(g) = c_1a_{11}^2e^{d_1 f} - c_2c_3 \left[ 1 + a_{11}^2p(f)^2 \right]. \quad (4.18)$$

The left-hand side in (4.18) is a function of  $g$ . At the same time, the right-hand side is a function of  $f$ . This is impossible.

(2.2)  $a_{12} \neq 0$  in (4.14). The conditions of symmetry imply that  $a_{22} \neq 0$ . Dividing (4.14) by  $f'g'$ , we can write

$$D\left(\frac{a_{12}}{g'} + \frac{a_{22}}{f'}\right) = \frac{f'''}{f'f''} + \frac{g'''}{g'g''}, \tag{4.19}$$

where  $D = -6H_0\omega^{-2}a_{12}a_{22}$ .

It follows from (4.19) that

$$f''' = (-d_1f' + Da_{22})f'' \quad \text{and} \quad g''' = (d_1g' + Da_{12})g''. \tag{4.20}$$

On the other hand, if we take the partial derivative of (4.9) with respect to  $v$  and consider the second equality in (4.20), then we find

$$-6Ha_{22}(a_{12}f' + a_{22}g')^2 = 2\omega^2g'f'' + [a_{22}^2 + (\omega f')^2](d_1g' + Da_{12}),$$

which is a polynomial equation on  $g'$ . The leading coefficient coming from the term  $(g')^2$  is  $-6Ha_{22}^3$  that cannot vanish. This gives a contradiction.

Theorem 4.3 is proved.

### 5. Surfaces of Type II

Let  $\alpha$  and  $\beta$  be the isotropic planar and space curves given, respectively, by

$$\alpha(x) = (x, ax, f(x)) \quad \text{and} \quad \beta(y) = (y, g(y), h(y)),$$

where  $a \in \mathbb{R}$ . Since the torsion of  $\beta$  does not vanish, it follows from (2.2) that

$$g''h''' - g'''h'' \neq 0, \tag{5.1}$$

where

$$g' = \frac{dg}{dy}, \quad h' = \frac{dh}{dy},$$

and so on. Thus, the obtained translation surface belongs to Type II and has the form

$$r(x, y) = (x + y, ax + g(y), f(x) + h(y)). \tag{5.2}$$

Assumption (5.1) ensures the admissibility of (5.2), i.e.,  $g' - a \neq 0$ . Hence, as a result of calculations, for the Gaussian curvature  $K$ , we get

$$K = \frac{f''[h''(g' - a) - g''(h' - f')]}{(g' - a)^3}, \tag{5.3}$$

where  $f' = \frac{df}{dx}$ , etc.

**Theorem 5.1.** *A translation surface in  $\mathbb{I}^3$  of the form (5.2) with CIGC  $(K_0)$  is a generalized cylinder with nonisotropic rulings, i.e.,  $K_0 = 0$ .*

**Proof.** If  $K_0 \neq 0$ , then (5.3) can be rewritten as

$$\frac{K_0}{f''} = \frac{h''}{(g' - a)^2} - \frac{g''}{(g' - a)^3}(h' - f'). \quad (5.4)$$

Taking the partial derivative of (5.4) with respect to  $x$ , we get

$$-K_0 \frac{f'''}{(f'')^3} = \frac{g''}{(g' - a)^3}.$$

Further, solving this equation, we find

$$f(x) = \frac{1}{3c_1^2} (-2c_1x + d_1)^{\frac{3}{2}} + d_2x + d_3 \quad (5.5)$$

and

$$g(y) = \frac{1}{K_0c_1} (2K_0c_1y + d_4)^{\frac{1}{2}} + ay + d_5. \quad (5.6)$$

Substituting (5.5) and (5.6) in (5.4), we can write

$$0 = \frac{h''}{h' - d_2} + \frac{K_0c_1}{2K_0c_1y + d_4}. \quad (5.7)$$

Solving (5.7), we obtain

$$h(y) = \frac{c_2}{K_0c_1} (2K_0c_1y + d_4)^{\frac{1}{2}} + d_2y + d_6. \quad (5.8)$$

Comparing (5.6) with (5.8), we arrive at a contradiction due to (5.1). Further, we assume that  $K_0 = 0$ . If  $f'' \neq 0$ , then we get

$$h''(g' - a) = g''(h' - f').$$

Taking partial derivative of this expression with respect to  $x$ , we arrive at a contradiction  $g'' = 0$  due to (5.1). Hence, the only possibility is that  $f'' = 0$ , namely,  $\alpha$  is a nonisotropic line.

Theorem 5.1 is proved.

By direct calculations, the mean curvature  $H$  is

$$2H = \frac{[1 + (g')^2](g' - a)f'' + (1 + a^2)[h''(g' - a) - g''(h' - f')]}{(g' - a)^3}. \quad (5.9)$$

**Theorem 5.2.** A translation surface in  $\mathbb{I}^3$  of the form (5.2) cannot be isotropic minimal.

**Proof.** We proceed by contradiction. If  $H = 0$ , then (5.9) reduces to

$$[1 + (g')^2](g' - a)f'' + (1 + a^2)[h''(g' - a) - g''(h' - f')] = 0. \quad (5.10)$$

The partial derivative of (5.10) with respect to  $x$  yields

$$[1 + (g')^2](g' - a)f''' + (1 + a^2)g''f'' = 0. \tag{5.11}$$

We distinguish two cases:

*Case 1:*  $f'' = 0$ , i.e.,  $f(x) = d_1x + d_2$ . By (5.10), we deduce

$$\frac{h''}{h' - d_1} = \frac{g''}{g' - a}.$$

This implies that

$$h = c_1g + (d_1 - ac_1)y - d_3,$$

which is not possible due to (5.1).

*Case 2:*  $f'' \neq 0$ . Relation (5.11) implies that

$$\frac{f'''}{f''} = -\frac{(1 + a^2)g''}{[1 + (g')^2](g' - a)}. \tag{5.12}$$

Hence, it follows from (5.12) that

$$f'' = c_1f' + d_1, \quad [1 + (g')^2](g' - a)c_1 = -(1 + a^2)g''. \tag{5.13}$$

Substituting (5.13) in (5.10), we conclude that

$$0 = \frac{g''}{g' - a} - \frac{h''}{h' + \frac{d_1}{c_1}}. \tag{5.14}$$

Solving (5.14), we get

$$g = c_2h + \left(a + \frac{c_2d_1}{c_1}\right)y + d_2,$$

which is not possible due to (5.1).

Theorem 5.2 is proved.

**Theorem 5.3.** *For a translation surface in  $\mathbb{I}^3$  of the form (5.2) with nonzero CIMC ( $H_0$ ), one of the following relations is satisfied:*

(1)  $\alpha(x) = (x, ax, d_1x + d_2)$  and

$$\beta(y) = \left(y, g, \frac{H_0}{1 + a^2}(g - ay)^2 + d_1y + d_3(g - ay) + d_4\right);$$

(2)  $\alpha(x) = (x, ax, c_1 \exp(c_2x) + d_1x + d_2)$  and

$$\beta(y) = \left( y, g, \frac{H_0}{(1+a^2)}(g-ay)^2 - \frac{d_1}{c_2}y + d_3(g-ay) + d_4 \right),$$

where  $g = g(y)$  is a nonlinear function and

$$g - ay \neq \frac{-1}{c_3} \sqrt{-2c_3y + d_5} + d_6. \quad (5.15)$$

**Proof.** We separate the proof into two parts:

*Case 1:*  $f'' = 0$  and  $f(x) = d_1x + d_2$ . Substituting these relations in (5.9), we find

$$\frac{2H_0}{1+a^2}(g' - a) = \left( \frac{h' - d_1}{g' - a} \right)'. \quad (5.16)$$

As a result of double integration of relation (5.16), we obtain

$$h = \frac{H_0}{1+a^2}(g-ay)^2 + d_1y + d_3(g-ay) + d_4. \quad (5.17)$$

On the other hand, by using (5.1) and (5.17), we deduce (5.15). This completes the proof of assertion (1) of the theorem.

*Case 2:*  $f'' \neq 0$ . By taking the partial derivative of (5.9) with respect to  $x$ , we again obtain (5.11). This means that the next steps are similar to the corresponding steps in Theorem 5.2. Thus, we get (5.13), namely,

$$f(x) = c_1 \exp(c_2x) + d_1x + d_2 \quad (5.18)$$

and

$$[1 + (g')^2](g' - a)c_2 = -(1 + a^2)g''. \quad (5.19)$$

Substituting (5.18) and (5.19) in (5.9), we conclude that

$$\frac{2H_0}{(1+a^2)}(g' - a) = \frac{-d_1g''}{c_2(g' - a)^2} + \left( \frac{h'}{g' - a} \right)'. \quad (5.20)$$

Integrating relation (5.20) two times, we find

$$h = \frac{H_0}{(1+a^2)}(g-ay)^2 - \frac{d_1}{c_2}y + d_3(g-ay) + d_4. \quad (5.21)$$

By (5.1) and (5.21), we again arrive at (5.15).

Theorem 5.3 is proved.

### 6. Surfaces of Type III

Let  $\alpha$  and  $\beta$  be, respectively, nonisotropic planar and space curves given by

$$\alpha(x) = (x, f(x), ax) \quad \text{and} \quad \beta(y) = (y, g(y), h(y)),$$

where  $a \in \mathbb{R}$ . Then the torsion of  $\beta$  is nonvanishing, namely, (2.2) implies that

$$g''h''' - g'''h'' \neq 0, \tag{6.1}$$

where

$$g' = \frac{dg}{dy}, \quad h' = \frac{dh}{dy},$$

and so on. Hence, the surface specified by the sum of  $\alpha$  and  $\beta$  belongs to Type III and has the form

$$r(x, y) = (x + y, f(x) + g(y), ax + h(y)). \tag{6.2}$$

It follows from (6.1) that the surface is admissible, i.e.,  $g' - f' \neq 0$  and  $f' = \frac{df}{dx}$ . By calculations, for the isotropic Gaussian curvature  $K$ , we get

$$K = -\frac{f''(h' - a)[h''(g' - f') - g''(h' - a)]}{(g' - f')^4}. \tag{6.3}$$

**Theorem 6.1.** *A translation surface in  $\mathbb{I}^3$  of the form (6.2) with CIGC ( $K_0$ ) is a generalized cylinder with nonisotropic rulings, namely,  $K_0 = 0$ .*

**Proof.** Assume that  $K$  is a nonzero constant  $K_0$ . Then we have  $f'' \neq 0$  and the partial derivative of (6.3) with respect to  $x$  gives

$$\frac{4(g' - f')^3}{f''} + \frac{(g' - f')^4 f'''}{(f'')^3} = \frac{(h' - a)h''}{K_0}. \tag{6.4}$$

Thus, we have two cases:

*Case 1:*  $f''' = 0$  and  $f'' = c_1$ . Hence, it follows from (6.4) that

$$4K_0(g' - f')^3 = c_1(h' - a)h''. \tag{6.5}$$

The partial derivative of (6.5) with respect to  $x$  gives  $f'' = 0$ , which is impossible.

*Case 2:*  $f''' \neq 0$ . Taking the partial derivative of (6.4) with respect to  $x$  and dividing by  $(g' - f')^2$ , we find

$$-12 - 8(g' - f')\frac{f'''}{(f'')^2} + (g' - f')^2 \left(\frac{f'''}{(f'')^3}\right)' = 0. \tag{6.6}$$

This is a polynomial equation for  $g'$  and the leading coefficient  $\left(\frac{f'''}{(f'')^3}\right)'$  that comes from  $(g')^2$  must vanish. Therefore, relation (6.6) reduces to

$$-12 - 8(g' - f') \frac{f'''}{(f'')^2} = 0. \quad (6.7)$$

Taking the partial derivative of (6.7) with respect to  $y$ , we get  $f''' = 0$ , which is not our case.

The discussion presented above yields  $K_0 = 0$ . In this case, since  $\beta$  is a space curve, the only possibility in (6.3) is  $f'' = 0$ , i.e.,  $\alpha$  is a nonisotropic line.

Theorem 6.1 is proved.

By direct calculations, for the isotropic mean curvature, we obtain

$$2H = \frac{[1 + (f')^2] [h''(g' - f') - g''(h' - a)] - [1 + (g')^2] (h' - a)f''}{(g' - f')^3}. \quad (6.8)$$

**Theorem 6.2.** *A translation surface in  $\mathbb{I}^3$  of the form (6.2) cannot be isotropic minimal.*

**Proof.** We proceed by contradiction. If the surface is isotropic minimal, then (6.8) reduces to

$$[1 + (f')^2] [h''(g' - f') - g''(h' - a)] - [1 + (g')^2] (h' - a)f'' = 0. \quad (6.9)$$

Thus, we have two cases:

*Case 1:*  $f'' = 0$  and  $f = d_1x + d_2$ . In this case, (6.9) reduces to

$$\frac{h''}{h' - a} = \frac{g''}{g' - d_1}.$$

Solving the last equation, we arrive at a contradiction due to (6.1).

*Case 2:*  $f'' \neq 0$ . Dividing (6.9) by  $[1 + (g')^2][1 + (f')^2](h' - a)$ , we obtain

$$\frac{h''(g' - f')}{(h' - a)[1 + (g')^2]} + \frac{f''}{1 + (f')^2} - \frac{g''}{1 + (g')^2} = 0. \quad (6.10)$$

The partial derivatives of (6.10) with respect to  $x$  and  $y$  imply that

$$\frac{h''}{(h' - a)[1 + (g')^2]} = c_1. \quad (6.11)$$

Substituting (6.11) in (6.10) gives

$$\frac{f''}{1 + (f')^2} - c_1 f' = d_1 \quad \text{and} \quad \frac{g''}{1 + (g')^2} - c_1 g' = d_1. \quad (6.12)$$

By using the second equality from (6.12) in (6.11), we find

$$\frac{h''}{h' - a} = \frac{c_1 g''}{c_1 g' + d_1}. \tag{6.13}$$

Solving (6.13), we arrive at a contradiction due to (6.1).

Theorem 6.2 is proved.

**Theorem 6.3.** *For a translation surface in  $\mathbb{I}^3$  of the form (6.2) with nonzero CIMC ( $H_0$ ), the following equalities are true:*

$$\alpha(x) = (x, ax, d_1x + d_2)$$

and

$$\beta(y) = \left( y, g(y), \frac{H_0}{1 + d_1^2} (g - d_1y)^2 + ay + d_3(g - d_1y) + d_4 \right),$$

where  $g = g(y)$  is a nonlinear function and

$$g - d_1y \neq \frac{-1}{c_1} \sqrt{-2c_1y + d_5} + d_6. \tag{6.14}$$

**Proof.** We split the proof into two parts:

*Case 1:*  $f'' = 0$ . Then  $f(x) = d_1x + d_2$  and (6.8) reduces to

$$\frac{2H}{1 + d_1^2} (g' - d_1) = \left( \frac{h' - a}{g' - d_1} \right)'. \tag{6.15}$$

As a result of the double integration of (6.15), we obtain

$$h = \frac{H_0}{1 + d_1^2} (g - d_1y)^2 + ay + d_3 (g - d_1y) + d_4. \tag{6.16}$$

By (6.1) and (6.16), we get (6.14) and, thus, the hypothesis of the theorem is true.

*Case 2:*  $f'' \neq 0$ . By multiplying (6.8) by  $\frac{(g' - f')^3}{1 + (f')^2}$  and taking partial derivatives with respect to  $x$  and  $y$ , we obtain

$$\begin{aligned} 12H_0 \left[ \frac{(g' - f')f''g''}{1 + (f')^2} + \frac{(g' - f')^2 f' f'' g''}{[1 + (f')^2]^2} \right] \\ = h''' f'' + \{ 2g' g'' (h' - a) + [1 + (g')^2] h'' \} \left( \frac{f''}{1 + (f')^2} \right)'. \end{aligned} \tag{6.17}$$

Dividing (6.17) by  $12H_0 f'' g''$  and setting

$$A(y) = 2g'(h' - a) + \frac{1 + (g')^2}{g''} h'', \quad B(x) = \frac{\{f'' / [1 + (f')^2]\}'}{f''},$$

we conclude that

$$\frac{g' - f'}{1 + (f')^2} + \frac{(g' - f')^2 f'}{[1 + (f')^2]^2} = \frac{1}{12H_0} \left( \frac{h'''}{g''} + AB \right). \quad (6.18)$$

Taking the partial derivative of (6.18) with respect to  $y$ , we find

$$\frac{1}{1 + (f')^2} + \frac{2(g' - f')f'}{[1 + (f')^2]^2} = \frac{1}{12H_0} \left[ \frac{(h'''/g'')'}{g''} + \frac{A'}{g''} B \right]. \quad (6.19)$$

Taking the partial derivative of (6.19) with respect to  $y$  once again and setting

$$C(y) = \left[ \frac{(h'''/g'')'}{g''} \right]', \quad (6.20)$$

we deduce

$$\frac{f'}{[1 + (f')^2]^2} = \frac{1}{24H_0} \left[ \frac{C}{g''} + \frac{(A'/g'')'}{g''} B \right]. \quad (6.21)$$

The partial derivatives of (6.21) with respect to  $x$  and  $y$  imply that

$$0 = \left[ \frac{(A'/g'')'}{g''} \right]' B'. \quad (6.22)$$

In view of (6.22), we have the following two possibilities:

(1)  $B = \text{const.}$  Taking the partial derivative of (6.19) with respect to  $x$  and multiplying by  $\frac{[1 + (f')^2]^3}{f''}$  we get the following polynomial equation for  $f'$ :

$$(f')^3 - 3g'(f')^2 - 3f' + g' = 0,$$

which yields a contradiction.

(2)  $B \neq \text{const.}$  Then, from (6.22), we derive

$$A = d_1(g')^2 + d_2g' + d_3 \quad (6.23)$$

and it follows from (6.21) that  $C = d_4g''$ . Hence, by (6.20), we obtain

$$\frac{h'''}{g''} = d_4(g')^2 + d_5g' + d_6. \quad (6.24)$$

Substituting (6.23) and (6.24) in (6.18), we arrive at the following polynomial equation for  $g'$ :

$$\left\{ \frac{f'}{[1 + (f')^2]^2} - \frac{d_1B + d_4}{12H_0} \right\} (g')^2 + \left\{ \frac{1 - (f')^2}{[1 + (f')^2]^2} - \frac{d_2B + d_5}{12H_0} \right\} g' + \frac{f'}{[1 + (f')^2]^2} - \frac{d_3B + d_6}{12H_0} = 0.$$

The coefficients of this equation must be equal to zero, namely,

$$\begin{aligned} \frac{f'}{[1 + (f')^2]^2} &= \frac{d_4 + d_1B}{12H_0}, \\ \frac{1 - (f')^2}{[1 + (f')^2]^2} &= \frac{d_5 + d_2B}{12H_0}, \\ \frac{-f'}{[1 + (f')^2]^2} &= \frac{d_6 + d_3B}{12H_0}. \end{aligned} \tag{6.25}$$

Since  $f'' \neq 0$  none of  $d_1$ ,  $d_2$ , and  $d_3$  can vanish. By using the first and the second equations in (6.25), we get the following polynomial equation for  $f'$ :

$$d_1 - d_1(f')^2 - d_2f' = \frac{d_1d_5 - d_2d_4}{12H_0} [1 + (f')^2]^2,$$

which yields a contradiction.

Theorem 6.3 is proved.

### 7. Several Remarks

1. By analyzing the results obtained above, we can state that the following surfaces do not exist:  
 the surfaces of Types I.3, II and III with nonzero CIGC;  
 the isotropic minimal surfaces of Types II and III.
2. The isotropic minimal translation surfaces belong to the *family of isotropic Scherk surfaces*. If the translating curves lie in orthogonal planes, then the members of this family are locally given by the formulas [36]:

$$\begin{aligned} r(x, y) &= (x, y, c [x^2 - y^2]), \\ r(x, z) &= \left( x, \frac{1}{c} \log \left| \frac{cz}{\cos(cx)} \right|, z \right), \\ r(y, z) &= \frac{1}{2} \left( \frac{1}{c} \log \left| \frac{\cos(cz)}{\cos(cy)} \right|, y - z + \pi, y + z \right), \quad c \in \mathbb{R} - 0. \end{aligned}$$

If the translating curves lie in arbitrary planes, then the isotropic Scherk surfaces can be described in the following explicit forms:

$$z(x, y) = c \left[ (a_{11}x + a_{12}y)^2 - \frac{a_{11}^2 + a_{12}^2}{a_{21}^2 + a_{22}^2} (a_{21}x + a_{22}z)^2 \right] \quad (\text{see [2]}),$$

$$y(x, z) = \frac{1}{c} \log \left| \frac{\cos \left( \frac{cx}{a_{11}} \right)}{c(a_{21}x + a_{22}z)} \right|,$$

$$y(x, z) = \frac{1}{c} \log \left| \frac{\cos \left( \frac{ca_{22}}{|a_{ij}|} [a_{11}x + a_{12}z] \right)}{\cos \left( \frac{ca_{12}}{|a_{ij}|} [a_{21}x + a_{22}z] \right)} \right|, \quad c \in \mathbb{R} - 0.$$

3. To classify the surfaces of Type IV with arbitrary CIGC and CIMC is a somewhat more complicated task. However, it can be regarded as a challenging open problem.

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