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LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH ABSOLUTELY UNSTABLE SOLUTIONS

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For linear functional-differential equations of retarded and neutral types with infinitely many deviations and self-adjoint operator coefficients, we establish necessary and sufficient conditions for the absolute instability of the trivial solutions.

1. Main Object of Investigations

Let H be a Hilbert space and let $\|\cdot\|_H$ be a norm in H given by the equality $\|x\|_E = \sqrt{(x,x)}$, where (x,y) is the scalar product of x by y $(x, y \in H)$. Let $L(H,H)$ be a Banach algebra of linear continuous operators $A\colon H \to H$ with identity I and the norm

$$\|A\|_{L(H,H)} = \sup\bigl\{\|Ax\|_H\colon \|x\|_H = 1\bigr\}.$$

Let $C([-h,0],H)$ be a Banach space of functions $x = x(\theta)$ continuous on $[-h,0]$ with values in H and the norm

$$\|x\|_{C([-h,0],H)} = \max\bigl\{\|x(\theta)\|_H\colon \theta \in [-h,0]\bigr\}$$

and let $\mathfrak{D}_h$ be the set of all linear operators $D\colon C\bigl([-h,0],H\bigr) \to H$ with unit norm each of which is defined by the Riemann–Stieltjes integral

$$Dx = \int\limits_{-h}^{0} x(\theta)\, dF(\theta),$$

where $F(\theta)$ is a nondecreasing function of bounded variation defined on $[-h,0]$ with values in $\mathbb{R}$ such that $F(\theta+0) = F(\theta)$ for all $\theta \in [-h,0)$.

Note that if $-h = x_0 < x_3 < x_2 < \ldots < x_n = 0$ is a partition of the segment $[-h,0]$ and $\xi_1, \xi_2, \ldots, \xi_n$ are arbitrary points from the corresponding elements of the partition, then the symbol $\int_{-h}^{0} x(\theta)\, dF(\theta)$ denotes the limit

$$\lim_{\max(x_i - x_{i-1}) \to 0} \sum_{i=1}^{n} \bigl(F(x_i) - F(x_{i-1})\bigr) x(\xi_i).$$

This limit exists and does not depend on the partition of the segment $[-h,0]$ and the choice of the points $\xi_1, \ldots, \xi_n$.

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For any function $x(t)$ continuous on $[-h, +\infty)$ with values in H, by x_t we denote an element $x_t(\theta) = x(t+\theta)$ of the space $C\big([-h, 0], H\big)$.

Consider self-adjoint operators

$$A_n \in L(H, H), \quad n \geq 1, \qquad \text{and} \qquad B_n \in L(H, H), \quad n \geq 0,$$

satisfying the condition

$$\sum_{n=1}^{\infty} \|A_n\|_{L(H,H)} + \sum_{n=0}^{\infty} \|B_n\|_{L(H,H)} < \infty \tag{1}$$

and the operators $\mathcal{C}_n, \mathcal{D}_n \in \mathfrak{D}_h, \ n \geq 1.$

The main object of investigation in the present paper is the instability of trivial solutions of the linear functional-differential equations

$$\frac{dx(t)}{dt} = B_0 x(t) + \sum_{n=1}^{\infty} B_n \mathcal{D}_n x_t, \quad t \geq 0, \tag{2}$$

and

$$\frac{dx(t)}{dt} + \sum_{n=1}^{\infty} A_n \mathcal{C}_n \frac{dx_t}{dt} = B_0 x(t) + \sum_{n=1}^{\infty} B_n \mathcal{D}_n x_t, \quad t \geq 0, \tag{3}$$

for any $\mathcal{C}_n, \mathcal{D}_n \in \mathfrak{D}_h, \ n \geq 1$, and $h > 0$.

It is clear that the following differential-difference equations are special cases of these equations:

$$\frac{dx(t)}{dt} = B_0 x(t) + \sum_{n=1}^{\infty} B_n x(t - \Delta_n), \quad t \geq 0, \tag{4}$$

and

$$\frac{dx(t)}{dt} + \sum_{n=1}^{\infty} A_n \frac{dx(t-\tau_n)}{dt} = B_0 x(t) + \sum_{n=1}^{\infty} B_n x(t - \Delta_n), \quad t \geq 0, \tag{5}$$

where $\Delta_n, \ \tau_n, \ n \geq 1$, are nonnegative numbers such that

$$\sup_{n \geq 1} \Delta_n + \sup_{n \geq 1} \tau_n < \infty. \tag{6}$$

The trivial solutions of Eqs. (2) and (3) are called *absolutely unstable* if these solutions are unstable for all $\mathcal{D}_n \in \mathfrak{D}_h, \ n \geq 1$, and $\mathcal{C}_n, \mathcal{D}_n \in \mathfrak{D}_h, \ n \geq 1$, respectively, and $h > 0$ (for the definition of unstable solutions of differential equations with deviating argument, see, e.g., [1, 2]).

The aim of the present paper is to establish necessary and sufficient conditions for the absolute instability of the trivial solutions of Eqs. (2) and (3).

2. Statement of Main Results

By $\sigma(A)$ we denote the spectrum of the operator $A \in L(H, H)$. Moreover, by $\mathbb{C}_+$ we denote the set $\{z \in \mathbb{C} : \Re z > 0\}$. The following assertions are true:

Theorem 1. *For the absolute instability of the trivial solution of Eq. (2), it is necessary and sufficient that*

$$\sigma\left(\sum_{n=0}^{\infty} B_n\right) \bigcap \mathbb{C}_+ \neq \varnothing. \tag{7}$$

Theorem 2. *Suppose that*

$$\sum_{n=1}^{\infty} \|A_n\|_{L(H,H)} < 1 \tag{8}$$

and

$$\left(\sum_{n=1}^{\infty} p_n A_n\right)\left(B_0 + \sum_{n=1}^{\infty} q_n B_n\right) = \left(B_0 + \sum_{n=1}^{\infty} q_n B_n\right)\left(\sum_{n=1}^{\infty} p_n A_n\right) \tag{9}$$

for all $p_n \in [0,1]$ *and* $q_n \in [0,1],\ n \geq 1.$

For the absolute instability of the trivial solution of Eq. (3), it is necessary and sufficient that relation (7) be true.

These theorems are proved in Secs. 4 and 5.

3. Auxiliary Statements

We now present some results for self-adjoint continuous operators, which are used in what follows.

Recall that an operator $A \in L(H,H)$ is called self-adjoint if it coincides with its adjoint operator A^*, i.e., $(Ax,y) = (x,Ay)$ for all $x,y \in H$ [3–5]. For a self-adjoint operator $A \in L(H,H)$, its Hermitian form (Ax,x) $(x \in H)$ takes only real values. The spectrum of the self-adjoint operator $\sigma(A)$ is a nonempty bounded closed set on the real axis. By $\left[\lambda_m(A), \lambda_M(A)\right]$ we denote the least segment containing the spectrum $\sigma(A)$. It is known (see [4]) that

$$\lambda_m(A) = \inf\left\{(Ax,x)\colon\ \|x\|_H = 1\right\},$$

$$\lambda_M(A) = \sup\left\{(Ax,x)\colon\ \|x\|_H = 1\right\},$$

$$\|A\|_{L(H,H)} = \max\left\{\lambda_M(A), -\lambda_m(A)\right\}.$$

It is clear that $\lambda_m(A),\ \lambda_M(A)$, and $\|A\|_{L(H,H)}$ are continuous functions of A.

Note that the sum of self-adjoint operators is a self-adjoint operator and a linear combination of self-adjoint operators with real coefficients is also a self-adjoint operator. In view of the continuity of the scalar product, the limit of a sequence of self-adjoint operators with respect to the norm is a self-adjoint operator. The product BA of self-adjoint operators A and B is a self-adjoint operator if and only if $BA = AB$.

The following statements are important for the investigation of the instability of solutions of Eqs. (2) and (3):

Theorem 3 ([4], Chap. VII, Sec. 4)**.** *A point* λ *belongs to the spectrum of a self-adjoint operator* $A \in L(H,H)$ *if and only if there exists a sequence of normed vectors* $x_n,\ n \geq 1$, *for which*

$$\lim_{n\to\infty} \|Ax_n - \lambda x_n\|_H = 0.$$

Theorem 4. *If a self-adjoint operator $A \in L(H,H)$ satisfies the relation $\sigma(A) \bigcap \mathbb{C}_+ = \varnothing$, then*

$$\sup_{t \geq 0} \left\| e^{tA} \right\|_{L(H,H)} \leq 1.$$

Theorem 4 can be easily proved by using the Murray lemma [3, pp. 109, 110].

4. Proof of Theorem 1

Necessity. Assume that the trivial solution of Eq. (2) is absolutely unstable for

$$\mathcal{D}_n x_t = \int_{-h}^{0} x_t(\theta)\, dF_n(\theta) = x(t), \quad n \geq 1,$$

where

$$F_n(\theta) = \begin{cases} 1 & \text{for} \quad \theta = 0, \\ 0 & \text{for} \quad \theta \in [-h, 0). \end{cases}$$

In this case, Eq. (2) takes the form

$$\frac{dx(t)}{dt} = \left(\sum_{n=0}^{\infty} B_n \right) x(t), \quad t \geq 0, \tag{10}$$

and the function

$$x = e^{tA} c, \tag{11}$$

where $A = \sum_{n=0}^{\infty} B_n$ and c is an arbitrary vector of the space $H,$ is the general solution of Eq. (10) [6]. In view of (1) and the self-adjointness of the operators $B_n,\ n \geq 0,$ the operator $\sum_{n=0}^{\infty} B_n$ is also self-adjoint.

Assume that relation (7) is not true, i.e.,

$$\sigma \left(\sum_{n=0}^{\infty} B_n \right) \bigcap \mathbb{C}_+ = \varnothing. \tag{12}$$

Then, by Theorem 4, each solution (11) of Eq. (10) is bounded on $[0, +\infty),$ which contradicts the instability of the trivial solution of this equation.

Thus, the assumption of validity of relation (12) is not true.

Sufficiency. Assume that relation (7) is true.

We fix an arbitrary number $h > 0$ and operators $\mathcal{D}_n \in \mathfrak{D}_h,\ n \geq 1.$ Suppose that the operator $\mathcal{D}_n$ is given by the equality

$$\mathcal{D}_n x = \int_{-h}^{0} x(\theta)\, d\Psi_n(\theta),$$

where the function $\Psi_n(\theta)$ has the same properties as the functions specifying the elements of the set $\mathfrak{D}_h.$

Consider Eq. (2) and an operator function

$$P(z) = zI - B_0 - \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta} d\Psi_n(\theta) B_n, \quad z \geq 0. \tag{13}$$

In view of (1) and the self-adjointness of the operators $B_n,\ n \geq 0,$ the values of the function $P(z)$ for $z \in [0, +\infty)$ are self-adjoint operators and this function is continuous on $[0, +\infty)$. Hence, the function $\lambda_m\big(P(z)\big)$ is continuous on $[0, +\infty)$.

Since

$$\lambda_m\big(P(0)\big) < 0$$

in view of (7) and (13) and

$$\lim_{z \to +\infty} \lambda_m(P(z)) = +\infty$$

according to (13), by the Bolzano–Cauchy theorem [7], there exists a point $z_0 \in (0, +\infty)$ such that

$$\lambda_m\big(P(z_0)\big) = 0.$$

This equality means that

$$0 \in \sigma\big(P(z_0)\big).$$

We now show that the trivial solution of Eq. (2) is unstable.

By Theorem 3, there exists a sequence of normed vectors $a_m,\ m \geq 1,$ such that

$$\lim_{m \to \infty} \|P(z_0) a_m\|_H = 0. \tag{14}$$

We fix arbitrary numbers $\varepsilon \in (0, 1)$ and $m \in \mathbb{N}$ and consider a positive number $T(\varepsilon)$ for which

$$\left| e^{z_0 T(\varepsilon)} \right| = 2. \tag{15}$$

This number exists because $z_0 > 0$. By $x(t, \varepsilon a_m)$ we denote a continuous solution of Eq. (2) satisfying the condition

$$x(t, \varepsilon a_m) = e^{z_0 t} \varepsilon a_m$$

for all $t \in [-h, 0)$. Consider a function

$$\delta_m(t) = x(t, \varepsilon a_m) - e^{z_0 t} \varepsilon a_m. \tag{16}$$

It is clear that

$$\frac{d\delta_m(t)}{dt} \equiv B_0 \delta_m(t) + \sum_{n=1}^{\infty} B_n \int_{-h}^{0} \delta_m(t + \theta) d\Psi_n(\theta) - \varepsilon e^{z_0 t} P(z_0) a_m.$$

This yields

$$\delta_m(t) = \varepsilon \frac{1 - e^{z_0 t}}{z_0} P(z_0) a_m + \int\limits_0^t \left(B_0 \delta_m(s) + \sum_{n=1}^{\infty} B_n \int\limits_{-h}^{0} \delta_m(s + \theta) d\Psi_n(\theta) \right) ds, \quad t \geq 0.$$

Hence,

$$\max_{\tau \in [0,t]} \left\| \delta_m(\tau) \right\|_H \leq \frac{\varepsilon}{z_0} e^{z_0 t} \left\| P(z_0) a_m \right\|_H + \int\limits_0^t \sum_{n=0}^{\infty} \| B_n \|_{L(H,H)} \max_{\tau \in [0,s]} \left\| \delta_m(\tau) \right\|_H ds, \quad t \geq 0,$$

and, by the Gronwall–Bellman inequality (see, e.g., [8]),

$$\max_{\tau \in [0,T(\varepsilon)]} \left\| \delta_m(\tau) \right\|_H \leq \left(\frac{\varepsilon}{z_0} e^{z_0 T(\varepsilon)} \left\| P(z_0) a_m \right\|_H \right) e^{T(\varepsilon) \sum_{n=0}^{\infty} \| B_n \|_{L(H,H)}}.$$

By using this result and relation (14), we obtain

$$\lim_{m \to \infty} \max_{\tau \in [0,T(\varepsilon)]} \| \delta_m(\tau) \|_H = 0.$$

Thus, in view of (15) and (16),

$$\left\| x(T(\varepsilon), \varepsilon a_m) \right\|_H \geq 1$$

for sufficiently large $m \in \mathbb{N}$. Since the choice of ε is arbitrary, this means that the trivial solution of Eq. (2) is unstable. In view of the arbitrariness of the choice of the operators $\mathcal{D}_n \in \mathfrak{D}_h,\ n \geq 1,$ and the number $h > 0,$ the trivial solution of this equation is absolutely unstable.

Theorem 1 is proved.

5. Proof of Theorem 2

Sufficiency. Assume that relations (8) and (9) are true and the trivial solution of Eq. (3) is absolutely unstable. Then this solution is also unstable for

$$\mathcal{C}_n x_t = \mathcal{D}_n x_t = \int\limits_{-h}^{0} x_t(\theta)\, dF_n(\theta) = x(t), \quad n \geq 1,$$

where

$$F_n(\theta) = \begin{cases} 1 & \text{for} \quad \theta = 0, \\ 0 & \text{for} \quad \theta \in [-h, 0). \end{cases}$$

In this case, Eq. (3) takes the form

$$\left(I + \sum_{n=1}^{\infty} A_n \right) \frac{dx(t)}{dt} = \left(\sum_{n=0}^{\infty} B_n \right) x(t), \quad t \geq 0. \tag{17}$$

In view of (8), the operator

$$I + \sum_{n=1}^{\infty} A_n$$

possesses the continuous inverse operator $\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1}$ (see, e.g., [9]) and

$$\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} = I + \sum_{n=1}^{\infty} (-1)^n \left(I + \sum_{n=1}^{\infty} A_n\right)^n. \tag{18}$$

Hence, Eq. (17) is equivalent to the equation

$$\frac{dx(t)}{dt} = \left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \left(\sum_{n=0}^{\infty} B_n\right) x(t), \quad t \geq 0. \tag{19}$$

In view of (9) and (18), the operator

$$\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \left(\sum_{n=0}^{\infty} B_n\right)$$

in Eq. (19) is self-adjoint. Since the trivial solution of Eq. (19) is unstable, by using the same reasoning as in the proof of necessity in Theorem 1, we conclude that the relation

$$\sigma\left(\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \sum_{n=0}^{\infty} B_n\right) \bigcap \mathbb{C}_+ \neq \varnothing \tag{20}$$

is true. This relation implies (7). Indeed, in view of (9) and (18), the self-adjoint operators

$$\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n$$

are permutable (commuting) and, hence, by virtue of (8),

$$\sigma\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \subset (0, +\infty). \tag{21}$$

Since, for the commuting operators $\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1}$ and $\sum_{n=0}^{\infty} B_n$, we have

$$\sigma\left(\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1} \sum_{n=0}^{\infty} B_n\right) \subset \left\{\lambda\mu \colon \lambda \in \sigma\left(\left(I + \sum_{n=1}^{\infty} A_n\right)^{-1}\right), \ \mu \in \sigma\left(\sum_{n=0}^{\infty} B_n\right)\right\}$$

(see, e.g., [9, pp. 229, 230]), in view of (20) and (21), relation (7) is true.

Sufficiency. Assume that relations (7)–(9) hold.

It is necessary to show that the trivial solution of Eq. (3) is absolutely unstable.

We fix arbitrary operators $\mathcal{C}_n, \mathcal{D}_n \in \mathfrak{D}_h,\ n \geq 1,$ and a number $h > 0.$ Assume that the operators $\mathcal{C}_n$ and $\mathcal{D}_n$ are given by the equalities

$$\mathcal{C}_n x = \int_{-h}^{0} x(\theta)\, d\Phi_n(\theta) \qquad \text{and} \qquad \mathcal{D}_n x = \int_{-h}^{0} x(\theta)\, d\Psi_n(\theta),$$

where the functions $\Phi_n(\theta)$ and $\Psi_n(\theta)$ have the same properties as the functions specifying the elements of the set $\mathfrak{D}_h$.

Consider Eq. (3) and an operator function

$$Q(z) = z\left(I + \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta}\, d\Phi_n(\theta) A_n\right) - B_0 - \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta}\, d\Psi_n(\theta) B_n, \quad z \geq 0. \tag{22}$$

In view of (1) and the self-adjointness of the operators $A_n,\ n \geq 1,$ and $B_n,\ n \geq 0,$ the values of the function $Q(z)$ for $z \in [0, +\infty)$ are self-adjoint operators. Moreover, this function is continuous on $[0, +\infty)$. Hence, the function $\lambda_m(Q(z))$ is also continuous on $[0, +\infty)$.

In view of (7) and (22), it is clear that

$$\lambda_m(Q(0)) < 0,$$

and, therefore,

$$\lim_{z \to +\infty} \lambda_m(Q(z)) = +\infty. \tag{23}$$

Indeed, by virtue of the relations

$$\begin{aligned}
\lambda_m(Q(z)) &= \inf_{\|x\|_H = 1} \left(\left(z\left(I + \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta} d\Phi_n(\theta) A_n\right) - B_0 - \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta} d\Psi_n(\theta) B_n\right) x, x\right) \\
&\geq \inf_{\|x\|_H = 1} \left(z\left(I + \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta} d\Phi_n(\theta)\ A_n\right) x, x\right) \\
&\quad - \sup_{\|x\|_H = 1} \left(\left(B_0 + \sum_{n=1}^{\infty} \int_{-h}^{0} e^{z\theta}\ d\Psi_n(\theta)\ B_n\right) x, x\right) \\
&\geq z - \sup_{\|x\|_H = 1} \left(z\left(\sum_{n=1}^{\infty} e^{-z\tau_n} A_n\right) x, x\right) - \sum_{n=0}^{\infty} \|B_n\|_{L(H,H)} \\
&\geq z\left(1 - \sum_{n=1}^{\infty} \|A_n\|_{L(H,H)}\right) - \sum_{n=0}^{\infty} \|B_n\|_{L(H,H)}, \quad z \in [0, +\infty),
\end{aligned}$$

and (8), relation (23) is true.

By the Bolzano–Cauchy theorem, there exists a point $z_0 \in (0, +\infty)$ such that

$$\lambda_m(Q(z_0)) = 0.$$

This equality means that

$$0 \in \sigma(Q(z_0)). \tag{24}$$

We now show that the trivial solution of Eq. (3) is unstable.

By Theorem 3 and inclusion (24), there exists a sequence of normed vectors $a_m,\ m \geq 1,$ for which

$$\lim_{m\to\infty} \left\| Q(z_0)a_m \right\|_H = 0. \tag{25}$$

We consider vector functions $v_m = e^{z_0 t} a_m,\ m \geq 1.$ These functions are solutions of the equations

$$\frac{dv(t)}{dt} + \sum_{n=1}^{\infty} A_n \int_{-h}^{0} \frac{dv(t+\theta)}{dt}\, d\Phi_n(\theta)$$

$$- B_0 v(t) - \sum_{n=1}^{\infty} B_n \int_{-h}^{0} v(t+\theta)\, d\Psi_n(\theta)$$

$$= e^{z_0 t} Q(z_0) a_m, \quad m \geq 1. \tag{26}$$

Further, we consider functions $\varepsilon_m = \varepsilon_m(t),\ m \geq 1,$ continuously differentiable on $[-h, 0]$ and such that

$$\frac{d\varepsilon_m(0)}{dt} + \sum_{n=1}^{\infty} A_n \int_{-h}^{0} \frac{d\varepsilon_m(\theta)}{dt}\, d\Phi_n(\theta) - B_0 \varepsilon_m(0) - \sum_{n=1}^{\infty} B_n \int_{-h}^{0} \varepsilon_m(\theta)\, d\Psi_n(\theta) = Q(z_0)a_m, \quad m \geq 1,$$

and

$$\lim_{m\to\infty} \left(\sup_{t\in[-h,0]} \|\varepsilon_m(t)\|_H + \sup_{t\in[-h,0]} \left\| \frac{d\varepsilon_m(t)}{dt} \right\|_H \right) = 0, \tag{27}$$

where

$$\frac{d\varepsilon_m(0)}{dt} \quad \text{and} \quad \frac{d\varepsilon_m(-h)}{dt}$$

denote the left and right derivatives of the function $\varepsilon_m(t)$ at the points 0 and $-h,$ respectively. Functions with these properties exist in view of relations (8) and (25) and the linearity of Eq. (3).

By $\gamma_m(t)$ we denote the solution of Eq. (26) satisfying the initial condition

$$v(\theta) = \varepsilon_m(\theta), \quad \theta \in [-h, 0].$$

Then $v_m(t) - \gamma_m(t)$ is a solution of Eq. (3).

By using (25) and (27), we easily show that

$$\lim_{m\to\infty}\left(\sup_{t\in[0,T]}\|\gamma_m(t)\|_H+\sup_{t\in[0,T]}\left\|\frac{d\gamma_m(t)}{dt}\right\|_H\right)=0 \tag{28}$$

for any $T>0.$ Since the solutions $v_m(t)-\gamma_m(t),\ m\geq 1,$ of Eq. (3) satisfy the relations

$$e^{z_0 t}+\big\|\gamma_m(t)\big\|_H\geq\big\|e^{z_0 t}a_m-\gamma_m(t)\big\|_H\geq e^{z_0 t}-\big\|\gamma_m(t)\big\|_H$$

for all $m\geq 1$ and $t\geq 0,$ we conclude that, by virtue of (28) and $z_0>0,$ the trivial solution of Eq. (3) is unstable. Finally, in view of the arbitrary choice of the operators $\mathcal{C}_n,\mathcal{D}_n\in\mathfrak{D}_h,\ n\geq 1,$ and the number $h>0$ in Eq. (3), the trivial solution of this equation is absolutely unstable.

Theorem 2 is proved.

6. Remarks and Bibliographical Comments

Theorems 1 and 2 are new. They are similar to the corresponding statements on the necessary and sufficient conditions for the absolute instability of solutions of linear scalar differential-difference equations obtained in [1, 10], where one can also find sufficient conditions for the absolute instability of solutions of linear systems of differential-difference equations with delay.

Sufficient conditions for the instability of solutions of differential equations in Banach spaces with finitely many arbitrary delays, which continuously depend on time, were established in [1].

Necessary and sufficient conditions for the instability of trivial solutions of Eqs. (4) and (5) for any Δ_n and τ_n satisfying (6) were obtained in [11].

The problem of absolute instability of the solutions of functional-differential equations is similar to the problem of absolute stability of the solutions of differential-difference equations and functional-differential equations solved, e.g., in [1, 12 – 22] and [23], respectively.

The applications of differential-difference equations with absolutely stable and unstable solutions are presented in [1].

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