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SOME INTEGRALS INVOLVING ℵ-FUNCTIONS AND LAGUERRE POLYNOMIALS

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Our aim is to establish some new integral formulas involving ℵ-functions associated with Laguerre-type polynomials. We also show that the main results presented in the paper are general by demonstrating 18 integral formulas that involve simpler known functions, e.g., the generalized hypergeometric function ${}_pF_q$ in a fairly systematic way.

1. Introduction and Preliminaries

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively, and let

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}, \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The Aleph (ℵ)-function, which is a very general higher transcendental function introduced by Südland, et al. [15, 16], is defined in terms of the Mellin–Barnes type integral in the following way (see, e.g., [8, 9]):

$$\aleph[z] = \aleph^{m,n}_{p_k,q_k,\delta_k;r}\left[z \left| \begin{matrix} (a_j, A_j)_{1,n}, [\delta_j(a_{j_k}, A_{j_k})]_{n+1,p_k;r} \\ \\ (b_j, B_j)_{1,m}, [\delta_j(b_{j_k}, B_{j_k})]_{m+1,q_k;r} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int\limits_L \Omega^{m,n}_{p_k,q_k,\delta_k;r}(s) z^{-s}\, ds, \tag{1.1}$$

where $z \in \mathbb{C} \setminus \{0\},\ i = \sqrt{-1}$, and

$$\Omega^{m,n}_{p_k,q_k,\delta_k;r}(s) := \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^{r} \delta_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{j_k} - B_{j_k} s) \prod_{j=n+1}^{p_k} \Gamma(a_{j_k} + A_{j_k} s)}. \tag{1.2}$$

Here, Γ is the known Gamma function (see, e.g., [13], Section 1.1); the integration path $L = L_{i\gamma\infty},\ \gamma \in \mathbb{R}$, extends from $\gamma - i\infty$ to $\gamma + i\infty$ with indentations, if necessary; the poles of the Gamma function $\Gamma(1 - a_j - A_j s)$,

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$j, n \in \mathbb{N},\ 1 \le j \le n$, do not coincide with the poles of $\Gamma(b_j + B_j s),\ j, m \in \mathbb{N},\ 1 \le j \le m$; the parameters $p_k, q_k \in \mathbb{N}$ satisfy the conditions $0 \le n \le p_k,\ 1 \le m \le q_k,\ \delta_k \in \mathbb{R}^+,\ 1 \le k \le r$; the parameters $A_j,\ B_j,\ A_{j_k},\ B_{j_k} \in \mathbb{R}^+$ and $a_j, b_j, a_{j_k}, b_{j_k} \in \mathbb{C}$, and the empty product in (1.2) (and elsewhere) is (as usual) understood as equal to 1. The existence conditions for the defining integral (1.1) are as follows:

$$\varphi_\ell \in \mathbb{R}^+ \qquad \text{and} \qquad |\arg(z)| < \frac{\pi}{2}\varphi_\ell, \qquad \ell \in \mathbb{N}, \quad 1 \le \ell \le r,$$

and

$$\varphi_\ell \ge 0, \qquad |\arg(z)| < \frac{\pi}{2}\,\varphi_\ell \qquad \text{and} \qquad \Re(\varsigma_\ell) + 1 < 0,$$

where

$$\varphi_\ell := \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \delta_\ell \left(\sum_{j=n+1}^{p_\ell} A_{j_\ell} + \sum_{j=m+1}^{q_\ell} B_{j_\ell} \right)$$

and

$$\varsigma_\ell := \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \delta_\ell \left(\sum_{j=n+1}^{p_\ell} A_{j_\ell} + \sum_{j=m+1}^{q_\ell} B_{j_\ell} \right) + \frac{1}{2}(p_\ell - q_\ell),$$

$$\ell \in \mathbb{N}, \quad 1 \le \ell \le r.$$

Remark 1. The expression for the Aleph-function in (1.1) does not completely follow the notational convention of the Fox's H-function. Indeed, in the $\aleph$-functions, the kernel $\Omega^{m,n}_{p_k,q_k,\delta_k;r}(s)$ and the couples of parameters

$$\left(a_j, A_j\right)_{1,n} \qquad \text{and} \qquad \left(b_j, B_j\right)_{1,m}$$

form the Gamma-function terms exclusively in the numerator, whereas

$$\left[\delta_j \left(a_{j_k}, A_{j_k}\right)_{n+1,p_k}\right] \qquad \text{and} \qquad \left[\delta_j \left(b_{j_k}, B_{j_k}\right)_{n+1,q_k}\right]$$

form the linear combination exclusively in the denominator. At the same time, for $H^{m,n}_{p,q}[z]$, both the upper and lower couples of parameters $\left(a_j, A_j\right)_{1,p}$ and $\left(b_j, B_j\right)_{1,q}$ play their roles in the formation of terms both in the numerator and in the denominator according to m and n.

Remark 2. Setting $\delta_j = 1,\ j \in \mathbb{N},\ 1 \le j \le r$, in (1.1), we get the I-function (see [7]) whose special case for $r = 1$ reduces to a familiar function (see [3, 4]).

Prabhaker and Suman [5] defined the following Laguerre-type polynomials $L_n^{(\alpha,\beta)}(x)$:

$$L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n + \beta + 1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, x^k}{k!\, \Gamma(\alpha k + \beta + 1)}, \tag{1.3}$$

$$n \in \mathbb{N}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > -1,$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)\dots(\lambda+n-1), & n \in \mathbb{N}, \end{cases}$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

A special case of (1.3) where $\alpha = 1$ reduces to the familiar generalized Laguerre polynomials $L_n^{(\beta)}(x)$ (see, e.g., [6], Chapter 12):

$$L_n^{(1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{k!\,\Gamma(k+\beta+1)} = L_n^{(\beta)}(x).$$

The Konhauser polynomials of the second kind (see [12]) are defined by

$$Z_n^{\beta}(x;k) = \frac{\Gamma(kn+\beta+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{cn}{j} \frac{x^{kj}}{\Gamma(kj+\beta+1)}, \tag{1.4}$$

$$\Re(\beta) > -1, \quad k \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

It is easy to see that

$$L_n^{(0,\beta)}(x^k) = Z_n^{\beta}(x;k)$$

and

$$L_n^{(\beta)}(x) = Z_n^{\beta}(x;1). \tag{1.5}$$

The polynomials $Z_n^{(\alpha,\beta)}(x;k)$ are defined as follows (see [10]):

$$Z_n^{(\alpha,\beta)}(x;k) = \sum_{j=0}^{n} \frac{\Gamma(kn+\beta+1)(-1)^j x^{kj}}{j!\Gamma(kj+\beta+1)\Gamma(\alpha n-\alpha j+1)}, \tag{1.6}$$

$$\Re(\alpha) > 0, \quad \Re(\beta) > -1, \quad n \in \mathbb{N}, \quad k \in \mathbb{Z}.$$

It follows from (1.4) and (1.6) that

$$Z_n^{(1,\beta)}(x;k) = Z_n^{\beta}(x;k).$$

For $\alpha \in \mathbb{N}$, relation (1.6) can be rewritten in the following form:

$$Z_n^{(\alpha,\beta)}(x;k) = \frac{\Gamma(kn+\beta+1)}{\Gamma(\alpha n+1)} \sum_{m=0}^{n} \frac{(-\alpha n)_{\alpha m} x^{km}}{m!\Gamma(km+\beta+1)(-1)^{(\alpha-1)m}}.$$

The polynomials $L_n^{(\alpha,\beta)}(\gamma;x)$ are defined by (see [10])

$$L_n^{(\alpha,\beta)}(\gamma;x)=\sum_{r=0}^{n}\frac{\Gamma(\alpha n+\beta+1)(-1)^r x^r}{r!\Gamma(\alpha r+\beta+1)\Gamma(\gamma n-\gamma r+1)},$$

$$\min\{\Re(\alpha),\,\Re(\gamma)\}>0,\qquad \Re(\beta)>-1,\quad n\in\mathbb{N}.$$

We also recall some properties of the Pochhammer symbol (see, e.g., [11])

$$(-x)_n=(-1)^n(x-n+1)_n, \tag{1.7}$$

$$(x+y)_n=\sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j}, \tag{1.8}$$

$$(x)_{n+m}=(x)_n(x+n)_m, \tag{1.9}$$

$$\binom{x}{n}=\frac{(-1)^n}{n!}(-x)_n, \tag{1.10}$$

where $x,y\in\mathbb{C}$ and $m,\ n\in\mathbb{N}_0$.

In the present paper, our aim is to establish some new integral formulas involving $\aleph$-function associated with the Laguerre-type polynomials. We also show that the main results presented here are general by choosing to demonstrate 18 integral formulas involving simpler known and familiar functions, e.g., the generalized hypergeometric function ${}_pF_q$, in a rather systematic way.

2. Integral Formulas

We now present integral formulas mainly involving the $\aleph$-functions.

Theorem 1. *Let $z,\ \lambda,\ \delta\in\mathbb{C}$ with $\min\{\Re(\delta),\ \Re(\lambda)\}>0$ and $|z|<1$. Also let $C\in\mathbb{R}^+$ and $\Re(\lambda)>-C\,\gamma$, where $\gamma\in\mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following integral formula is true:*

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right]du$$

$$=\Gamma(\delta)\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta,C),(a_j,A_j)_{1,n},[\delta_j(a_{j_k},A_{j_k})]_{n+1,\rho_k;r}\\ \\ (b_j,B_j)_{1,m},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{m+1,\sigma_k;r}\end{matrix}\right] \tag{2.1}$$

provided the other involved parameters are constrained to guarantee that each member can exist.

Proof. Let $\mathcal{L}_1$ be the left-hand side of (2.1). Then, by using (1.1) and changing the order in the double integrals, which can be done under the given conditions, we obtain

$$\mathcal{L}_1 = \frac{1}{2\pi i}\int\limits_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s) z^{-s}\left\{\int\limits_0^1 u^{\lambda+Cs-1}(1-u)^{\delta-1}\,du\right\} ds. \tag{2.2}$$

We recall the familiar Beta function $B(x,\, y)$ is defined by (and expressed in terms of) the Gamma function as follows (see, e.g., [13, p. 8]):

$$B(x,\, y) = \begin{cases} \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt, & \min\{\Re(x),\, \Re(y)\} > 0, \\[2ex] \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, & x,\, y \in \mathbb{C}\setminus\mathbb{Z}_0^-. \end{cases}$$

Further, we evaluate the inner integral in (2.2) and obtain

$$\mathcal{L}_1 = \Gamma(\delta)\frac{1}{2\pi i}\int\limits_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s) z^{-s}\frac{\Gamma(\lambda+Cs)}{\Gamma(\lambda+\delta+Cs)}\,ds. \tag{2.3}$$

Finally, interpreting the right-hand side of (2.3) in terms of definition (1.1), we arrive at the right-hand side of (2.1).

Theorem 2. *Let $z,\, \lambda,\, \delta \in \mathbb{C}$ with $\min\{\Re(\delta),\, \Re(\lambda)\} > 0$ and $|z| < 1$. Also let $x,\, t \in \mathbb{R}$ with $x \geq t$. Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following integral formula is true:*

$$\int\limits_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1}\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta, C), (a_j, A_j)_{1,n}, [\delta_j(a_{j_k}, A_{j_k})]_{n+1,\rho_k;r} \\[2ex] (b_j, B_j)_{1,m}, (\lambda, C), [\delta_j(b_{j_k}, B_{j_k})]_{m+1,\sigma_k;r} \end{matrix}\right] \tag{2.4}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Proof. Let $\mathcal{L}_2$ be the left-hand side of (2.4). We change the variable u into $v = \dfrac{u-t}{x-t}$. Similarly, as in the proof of Theorem 1, we obtain

$$\mathcal{L}_2 = \frac{(x-t)^{\delta+\lambda-1}}{2\pi i}\int\limits_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s) z^{-s}(x-t)^{Cs}\left\{\int\limits_0^1 (1-v)^{\delta-1}v^{\lambda+Cs-1}\,dv\right\} ds$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{1}{2\pi i}\int\limits_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s) z^{-s}(x-t)^{Cs}\frac{\Gamma(\lambda+Cs)}{\Gamma(\delta+\lambda+Cs)}\,ds.$$

It is now easy to see that, in view of definition (1.1), the last equality can be interpreted as the right-hand side of (2.4).

Theorem 3. *Let* $z,\ \nu,\ \mu \in \mathbb{C}$ *with* $\min\{\Re(\nu),\ \Re(\mu)\} > 0$ *and* $x \in \mathbb{R}^+$. *Also let* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following integral formula is true:*

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\big[z(x-t)^{-C}\big]\,dt$$
$$= x^{\nu+\mu-1}\Gamma(\nu)\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\mu+\nu,C),(a_j,A_j)_{1,n},[\delta_j(a_{j_k},A_{j_k})]_{n+1,\rho_k;r}\\ \\ (b_j,B_j)_{1,m},(\mu,C),[\delta_j(b_{j_k},B_{j_k})]_{m+1,\sigma_k;r}\end{matrix}\right] \tag{2.5}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Proof. By using the reasoning similar to that used either in the proof of Theorem 1 or in the proof of Theorem 2, we can establish equality (2.5). Hence, we omit the details of the proof.

As the sequel of the theorems proved above, we need the following formula (see [1]) presented in Lemma 1:

Lemma 1. *Let* $\min\big\{\Re(a),\ \Re(c),\ \Re(\zeta),\ \Re(\xi)\big\} > 0,\ \min\big\{\Re(b),\ \Re(d)\big\} > -1,$ *and* $h,\ m,n \in \mathbb{N}$. *Then the following formula is true:*

$$L_n^{(a,b)}(\xi;x)L_m^{(c,d)}(\zeta;x) = \sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(an+b+1)\Gamma(cm+d+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}$$
$$\times\frac{(-x)^h}{\Gamma(\xi(n-k)+1)\Gamma(ak+b+1)\Gamma(c(h-k)+d+1)}. \tag{2.6}$$

Theorem 4. *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\big\{\Re(\delta),\Re(\lambda)\big\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\big\{\Re(\sigma),\Re(\xi),\Re(\zeta)\big\} > 0 \qquad \text{and} \qquad \min\big\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\big\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$, *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following formula is true:*

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\zeta;\sigma(1-u))\,L_n^{(c',d')}(\xi;\sigma(1-u))\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\big[zu^{-C}\big]\,du$$
$$= \sum_{h=0}^{m+n}\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\Gamma(\delta+h)\,\sigma^h$$
$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ \\ (b_j,B_j)_{1,\alpha},(\lambda,C)[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right] \tag{2.7}$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ *is given by the formula*

$$\Delta^{n,m,\xi,\zeta}_{a',b',c',d'} := \sum_{k=0}^{h} \binom{h}{k} \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(-1)^h}{\Gamma(\zeta(m-h+k)+1)\Gamma(\xi(n-k)+1)} \times \frac{1}{\Gamma(h+1)\Gamma(a'k+b'+1)\Gamma\left(c'\left(h-k'\right)+d'+1\right)}. \tag{2.8}$$

Proof. Let $\mathcal{L}_3$ be the left-hand side of (2.7). Then, by using (2.6), we get

$$\mathcal{L}_3 = \sum_{h=0}^{m+n}\sum_{k=0}^{h} \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(\sigma)^h}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)} \times \frac{(-1)^h}{\Gamma(\xi(n-k)+1)\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \times \int_0^1 u^{\lambda-1}(1-u)^{\delta+h-1}\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right]du. \tag{2.9}$$

Applying (2.1) to the integral in (2.9), we obtain

$$\mathcal{L}_3 = \sum_{h=0}^{m+n}\sum_{k=0}^{h} \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(\sigma)^h}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)} \times \frac{(-1)^h}{\Gamma(\xi(n-k)+1)\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\Gamma(\delta+h) \times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\left|\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ (b_j,B_j)_{1,\alpha},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right.\right]. \tag{2.10}$$

Finally, it is easy to see that the expression in (2.10) corresponds to the right-hand side of (2.7).

In what follows, five integral formulas are presented in Theorems 5–9 without proofs because each of these proofs is similar to the proof of Theorem 4.

Theorem 5. *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\left\{\Re(\delta),\Re(\lambda)\right\} > 0$ *and* $|z| < 1$*. Also let*

$$\min\left\{\Re(\sigma),\Re(\xi),\Re(\zeta)\right\} > 0 \qquad \text{and} \qquad \min\left\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\right\} > -1.$$

Further, let $x, t \in \mathbb{R}$ *with* $x \geq t$*,* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$*, where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following formula is true:*

$$
\begin{aligned}
&\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} L_m^{(a',b')}(\zeta;\sigma(u-t)) \\
&\qquad \times L_n^{(c',d')}(\xi;\sigma(u-t)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du \\
&\qquad = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \sum_{h=0}^{m+n} \Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\, \sigma^h \\
&\qquad \times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]
\end{aligned}
$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ is defined as in (2.8).

Theorem 6. *Let $z,\, \mu,\, \nu \in \mathbb{C}$ with $\min\left\{\Re(\mu),\, \Re(\nu)\right\} > 0$ and $|z| < 1$. Also let*

$$
\min\left\{\Re(\sigma),\, \Re(\xi),\, \Re(\zeta)\right\} > 0 \quad \text{and} \quad \min\left\{\Re(a'),\, \Re(b'),\, \Re(c'),\, \Re(d')\right\} > -1.
$$

Further, let $x \in \mathbb{R}^+$, $C \in \mathbb{R}^+$, and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following formula is true:

$$
\begin{aligned}
&\int_0^x t^{\nu-1}(x-t)^{\mu-1} L_m^{(a',b')}(\zeta;\sigma(x-t)) \\
&\qquad \times L_n^{(c',d')}(\xi;\sigma(x-t)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right] dt \\
&\qquad = x^{\mu+\nu-1} \sum_{h=0}^{m+n} \Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\, \sigma^h\, x^h \\
&\qquad \times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \left| \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\mu, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]
\end{aligned}
$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ is given as in (2.8).

Theorem 7. *Let $z,\, \delta,\, \lambda \in \mathbb{C}$ with $\min\left\{\Re(\delta), \Re(\lambda)\right\} > 0$ and $|z| < 1$. Also let*

$$
\min\left\{\Re(\sigma), \Re(\xi), \Re(\zeta)\right\} > 0 \qquad \text{and} \qquad \min\left\{\Re(a'),\, \Re(b'),\, \Re(c'),\, \Re(d')\right\} > -1.
$$

Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following formula is true:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\zeta;\sigma(1-u))$$

$$\times L_n^{(c',d')}(\xi;\sigma(1-u)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z u^{-C}\right] du$$

$$= \sum_{h=0}^{m+n} \nabla^{n,m,\xi,\zeta}_{a',b',c',d'} \Gamma(\delta+h)\,\sigma^h$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right] \tag{2.11}$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\nabla^{n,m,\xi,\zeta}_{a',b',c',d'}$ *is given by the formula*

$$\nabla^{n,m,\xi,\zeta}_{a',b',c',d'} := \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{\Gamma(\zeta m+1)\Gamma(\xi n+1)}$$

$$\times \sum_{k=0}^{h} \left[\binom{h}{k} \frac{(-1)^{h-\zeta(h-k)-\xi k}(-\zeta m)_{\zeta(h-k)}(-\xi n)_{\xi k}}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right]. \tag{2.12}$$

Theorem 8. *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(\sigma), \Re(\xi), \Re(\zeta)\} > 0 \qquad \textit{and} \qquad \min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Further, let $x,\ t \in \mathbb{R}$ *with* $x \geq t$, $C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\gamma$, *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following formula is true:*

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} L_m^{(a',b')}(\zeta;\sigma(u-t))$$

$$\times L_n^{(c',d')}(\xi;\sigma(u-t)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1} \sum_{h=0}^{m+n} \nabla^{n,m,\xi,\zeta}_{a',b',c',d'}\,\sigma^h$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right] \tag{2.13}$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\nabla^{n,m,\xi,\zeta}_{a',b',c',d'}$ *is given as in (2.12).*

Theorem 9. *Let* $z,\, \mu,\, \nu \in \mathbb{C}$ *with* $\min\left\{\Re(\mu), \Re(\nu)\right\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\left\{\Re(\sigma), \Re(\xi), \Re(\zeta)\right\} > 0 \qquad \textit{and} \qquad \min\left\{\Re(a'),\, \Re(b'),\, \Re(c'),\, \Re(d')\right\} > -1.$$

Further, let $x \in \mathbb{R}^+$, $C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\,\gamma$, *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following relation is true:*

$$\begin{aligned}
&\int_0^x (t)^{\nu-1}(x-t)^{\mu-1} L_m^{(a',b')}(\zeta;\sigma(x-t)) \\
&\qquad \times L_n^{(c',d')}(\xi;\sigma(x-t)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right] dt \\
&\quad = x^{\mu+\nu-1} \sum_{h=0}^{m+n} \nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}\, \sigma^h\, x^h \\
&\qquad \times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\mu, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right]
\end{aligned} \tag{2.14}$$

provided that the other involved parameters satisfy the constraints such that each member may exist. Here, $\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}$ *is given as in (2.12).*

3. Special Cases

Note that the results obtained in Section 2 are sufficiently general and can be specialized to yield various simpler integral formulas. We now present some of these formulas.

Corollary 1. *Let* $z,\, \delta,\, \lambda,\, \sigma \in \mathbb{C}$ *with* $\min\left\{\Re(\delta),\, \Re(\lambda),\, \Re(\sigma)\right\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\left\{\Re(a'),\, \Re(b'),\, \Re(c'),\, \Re(d')\right\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$, *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following formula is true:*

$$\begin{aligned}
&\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u)) \\
&\qquad \times L_n^{(c',d')}(\sigma(1-u)) \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right] du \\
&\quad = \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!} \sum_{h=0}^{m+n} \sigma^h\, \Gamma(\delta+h) \\
&\qquad \times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right]
\end{aligned}$$

$$\times\, \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ (b_j,B_j)_{1,\alpha},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Proof. Setting $\zeta = \xi = 1$ in (2.11), after minor simplifications, we arrive at the desired result.

Corollary 2. *Let z, δ, $\lambda \in \mathbb{C}$ with $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ and $|z| < 1$. Also let x, $t \in \mathbb{R}$ with $x \geq t$. Further, let*

$$\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1,$$

$C \in \mathbb{R}^+$, and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1)$$

$$\times\, Z_n^{(1,d')}(\sigma(u-t);1)\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}$$

$$\times \sum_{h=0}^{m+n}\sigma^h\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right]$$

$$\times\, \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ (b_j,B_j)_{1,\alpha},(\lambda+h,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Proof. Setting $a' = c' = \xi = \zeta = 1$ in (2.13) and using (1.5) to consider $L_n^{1,b}(1;x) = Z_n^{(1,b)}(x;1)$, after minor simplifications, we get the desired result.

Corollary 3. *Let z, μ, $\nu \in \mathbb{C}$ with $\min\{\Re(\mu), \Re(\nu), \Re(\sigma)\} > 0$ and $|z| < 1$. Also let $x \in \mathbb{R}^+$ and*

$$\min\{\Re(b'), \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\left[1-\sigma(x-t)\right]^n \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right] dt$$

$$= x^{\mu+\nu-1} \sum_{h=0}^{n} (-n)_h \, \sigma^h \, x^h$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r} \left[z \left| \begin{matrix} (\mu+\nu,p), (a_j,A_j)_{1,\beta}, [\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r} \\ \\ (b_j,B_j)_{1,\alpha}, (\mu,p), [\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Proof. Setting $a' = c' = 0$ and $\xi = \zeta = 1$ in (2.14) and using some suitable identities from Section 1, including (1.7)–(1.10), after minor simplifications, we arrive at the desired result.

For $\delta_1 = \ldots = \delta_r = 1$ in (1.1), we get the definition of the I-function (see [7]):

$$I[z] = I^{m,n}_{p_k,q_k;r}[z]$$

$$= \aleph^{m,n}_{p_k,q_k,1;r} \left[z \left| \begin{matrix} (a_j,A_j)_{1,n}, [1(a_{j_k},A_{j_k})]_{n+1,p_k;r} \\ \\ (b_j,B_j)_{1,m}, [1(b_{j_k},B_{j_k})]_{m+1,q_k;r} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \Omega^{m,n}_{p_k,q_k,1;r}(s) z^{-s} \, ds, \tag{3.1}$$

where $z \in \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$, $\Omega^{m,n}_{p_k,q_k,1;r}(s)$ is defined in (1.2), and the integration path L can be used as in (1.1). Otherwise, for this (3.1), we can choose a new integration path. The existence conditions for integral (3.1) can be easily deduced from the conditions for the $\aleph$-function (1.1) with $\delta_1 = \ldots = \delta_r = 1$.

Then the integral relations in Corollaries 1–3 can be reduced to yield the following integral formulas involving the I-function and presented in Corollaries 4–6, respectively:

Corollary 4. *Let* $z, \delta, \lambda, \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$, *where* $\gamma \in \mathbb{R}$ *is a chosen number from the integration path* $L_{i\gamma\infty}$ *in (1.1). Then the following relation is true:*

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u))$$

$$\times L_n^{(c',d')}(\sigma(1-u)) I^{\alpha,\beta}_{\rho_k,\sigma_k;r}\left[zu^{-C}\right] du$$

$$= \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!} \sum_{h=0}^{m+n} \sigma^h \, \Gamma(\delta+h)$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right]$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 5. *Let $z,\ \delta,\ \lambda \in \mathbb{C}$ with $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ and $|z| < 1$. Also let $x,\ t \in \mathbb{R}$ with $x \geq t$. Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1,$$

$C \in \mathbb{R}^+$, and $\Re(\lambda) > -C\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following formula is true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1)$$

$$\times Z_n^{(1,d')}(\sigma(u-t);1) I^{\alpha,\beta}_{\rho_k,\sigma_k;r}\left[z(u-t)^{-C}\right] du$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)} \right]$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 6. *Let $z,\ \mu,\ \nu \in \mathbb{C}$ with $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ and $|z| < 1$. Also let $x \in \mathbb{R}^+$ and*

$$\min\{\Re(b'),\ \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\left[1-\sigma(x-t)\right]^n I^{\alpha,\beta}_{\rho_k,\sigma_k;r}\left[z(x-t)^{-C}\right] dt$$

$$= x^{\mu+\nu-1} \sum_{h=0}^{n} (-n)_h \, \sigma^h \, x^h$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \left| \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ \\ (b_j, B_j)_{1,\alpha}, (\mu, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right. \right]$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Further, a special case $r = 1$ of the I-function (3.1) reduces to become the H-function (see [3, 4]). Then the formulas in Corollaries 4–6 are reduced to yield the integral formulas involving the H-function presented in Corollaries 7–9, respectively.

Corollary 7. *Let $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ with $\min\left\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\right\} > 0$ and $|z| < 1$. Also let*

$$\min\left\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\right\} > -1.$$

Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u))$$

$$\times L_n^{(c',d')}(\sigma(1-u)) H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[zu^{-C}\right] du$$

$$= \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!} \sum_{h=0}^{m+n} \sigma^h \, \Gamma(\delta+h)$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right]$$

$$\times H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1} \left[z \left| \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\rho_1} \\ \\ (b_j, B_j)_{1,\sigma_1}, (\lambda, C) \end{matrix} \right. \right] \tag{3.2}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 8. *Let $z,\ \delta,\ \lambda \in \mathbb{C}$ with $\min\left\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\right\} > 0$ and $|z| < 1$. Also let $x,\ t \in \mathbb{R}$ with $x \geq t$. Further, let*

$$\min\left\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\right\} > -1,$$

$C \in \mathbb{R}^+$, and $\Re(\lambda) > -C\,\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1)$$

$$\times Z_n^{(1,d')}(\sigma(u-t);1) H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[z(u-t)^{-C}\right] du$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right]$$

$$\times H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z \,\middle|\, \begin{matrix}(\lambda+\delta+h, C), (a_j, A_j)_{1,\rho_1} \\ (b_j, B_j)_{1,\sigma_1}, (\lambda+h, C)\end{matrix}\right] \tag{3.3}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 9. *Let $z,\ \mu,\ \nu \in \mathbb{C}$ with $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ and $|z| < 1$. Also let $x \in \mathbb{R}^+$ and*

$$\min\{\Re(b'),\ \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\gamma$, where $\gamma \in \mathbb{R}$ is a chosen number from the integration path $L_{i\gamma\infty}$ in (1.1). Then the following relation is true:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\left[1-\sigma(x-t)\right]^n H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[z(x-t)^{-C}\right] dt$$

$$= x^{\mu+\nu-1} \sum_{h=0}^{n} (-n)_h\, \sigma^h\, x^h H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z \,\middle|\, \begin{matrix}(\mu+\nu, C), (a_j, A_j)_{1,\rho_1} \\ (b_j, B_j)_{1,\sigma_1}, (\mu, C)\end{matrix}\right] \tag{3.4}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Note that a special case of the H-function with $A_j = 1,\ j = 1,\dots,p$, and $B_j = 1,\ j = 1,\dots,q$, reduces to the following Meijer's G-function (see, e.g., [2], Section 8.2):

$$H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[x \,\middle|\, \begin{matrix}(a_j, 1)_{1,\rho_1} \\ (b_j, 1)_{1,\sigma_1}\end{matrix}\right] = G^{\alpha,\beta}_{\rho_1,\sigma_1}\left[x \,\middle|\, \begin{matrix}(a_{\rho_1}) \\ (b_{\sigma_1})\end{matrix}\right].$$

Then the formulas in Corollaries 7–9 are reduced to the corresponding integral formulas involving Meijer's G-function (3.5)–(3.7).

Corollary 10. *Let $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ with $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ and $|z| < 1$. Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Then the following relation is true:

$$\begin{aligned}
&\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\sigma(1-u)) \\
&\qquad \times L_n^{(c',d')}(\sigma(1-u))G^{\alpha,\beta}_{\rho_1,\sigma_1}[zu^{-1}]\,du \\
&= \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h) \\
&\qquad \times \sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right] \\
&\qquad \times \frac{\Gamma(\lambda+\delta+h)}{\Gamma(\lambda)}G^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h),(a_{\rho_1})\\(b_{\sigma_1}),(\lambda)\end{matrix}\right]
\end{aligned} \tag{3.5}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 11. *Let $z,\,\delta,\,\lambda,\,\sigma \in \mathbb{C}$ with $\min\{\Re(\delta),\,\Re(\lambda),\,\Re(\sigma)\} > 0$ and $|z| < 1$. Also let $x,\,t \in \mathbb{R}$ with $x \geq t$. Further, let*

$$\min\{\Re(a'),\,\Re(b'),\,\Re(c'),\,\Re(d')\} > -1.$$

Then the following relation is true:

$$\begin{aligned}
&\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1}Z_m^{(1,b')}(\sigma(u-t);1) \\
&\qquad \times Z_n^{(1,d')}(\sigma(u-t);1)G^{\alpha,\beta}_{\rho_1,\sigma_1}[z(u-t)^{-1}]\,du \\
&= \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!} \\
&\qquad \times \sum_{h=0}^{m+n}\sigma^h\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right] \\
&\qquad \times \frac{\Gamma(\lambda+\delta+h)}{\Gamma(\lambda+h)}G^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h),(a_{\rho_1})\\(b_{\sigma_1}),(\lambda+h)\end{matrix}\right]
\end{aligned} \tag{3.6}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 12. *Let* $z,\ \mu,\ \nu,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and*

$$\min\{\Re(b'),\ \Re(d')\} > -1.$$

Then the following relation is true:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\,[1-\sigma(x-t)]^n\, G^{\alpha,\beta}_{\rho_1,\sigma_1}[z(x-t)^{-1}]\,dt$$

$$= x^{\mu+\nu-1}\sum_{h=0}^{n}(-n)_h\,\sigma^h\,x^h G^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z\,\middle|\,\begin{matrix}(\mu+\nu),(a_{\rho_1})\\(b_{\sigma_1}),(\mu)\end{matrix}\right], \tag{3.7}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

In this case, if we replace $\sigma_1,\ a_j,\ b_j$ by $\sigma_1+1,\ 1-a_j,\ 1-b_j$ with $b_1 = 0$, respectively, and set $\alpha = 1$ in the H-function, then we get Wright's generalized hypergeometric function ${}_p\Psi_q$ (see, e.g., [14, p. 50]):

$$H^{1,\rho_1}_{\rho_1,\sigma_1+1}\left[-x\,\middle|\,\begin{matrix}(1-a_j,A_j)_{1,p}\\(0,1),(1-b_j,B_j)_{1,q}\end{matrix}\right] = {}_{\rho_1}\Psi_{\sigma_1}\left[\begin{matrix}(a_j,A_j)_{1,p}\,;\\(b_j,B_j)_{1,q}\,;\end{matrix}\;x\right]. \tag{3.8}$$

Further, by applying relation (3.8) to relations (3.2), (3.3), and (3.4), we arrive at the following respective integral relations containing the Wright's generalized hypergeometric function ${}_p\Psi_q$ (3.9)–(3.11).

Corollary 13. *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Then the following relation is true:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\sigma(1-u))$$

$$\times\, L_n^{(c',d')}(\sigma(1-u))\,{}_{\rho_1}\Psi_{\sigma_1}[zu^{-p}]\,du$$

$$= \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h)$$

$$\times\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right]$$

$$\times\,{}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\lambda+\delta+h,p),(a_j,A_j)_{1,\rho_1}\\(b_j,B_j)_{1,\sigma_1},(\lambda,p)\end{matrix}\;;z\right] \tag{3.9}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 14. *Let* $z, \delta, \lambda, \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x, t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1.$$

Then the following relation is true:

$$\begin{aligned}
&\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) \\
&\qquad \times Z_n^{(1,d')}(\sigma(u-t);1)\, {}_{\rho_1}\Psi_{\sigma_1}[z(u-t)^{-p}]\,du \\
&\quad = \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!} \\
&\qquad \times \sum_{h=0}^{m+n}\sigma^h \sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right] \\
&\qquad \times {}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\lambda+\delta+h,p),(a_j,A_j)_{1,\rho_1} \\ (b_j,B_j)_{1,\sigma_1},(\lambda+h,p)\end{matrix};z\right]
\end{aligned} \tag{3.10}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 15. *Let* $z, \mu, \nu, \sigma \in \mathbb{C}$ *with* $\min\{\Re(\mu), \Re(\nu), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and*

$$\min\{\Re(b'), \Re(d')\} > -1.$$

Then the following formula is true:

$$\begin{aligned}
&\int_0^x t^{\nu-1}(x-t)^{\mu-1}\,[1-\sigma(x-t)]^n\, {}_{\rho_1}\Psi_{\sigma_1}[z(x-t)^{-p}]\,dt \\
&\quad = x^{\mu+\nu-1}\sum_{h=0}^{n}(-n)_h\,\sigma^h\,x^h\, {}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\mu+\nu,p),(a_j,A_j)_{1,\rho_1} \\ (b_j,B_j)_{1,\sigma_1},(\mu,p)\end{matrix};z\right]
\end{aligned} \tag{3.11}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Furthermore, if we choose $p = 1;\ \alpha = 1,\ \beta = 2,\ \rho_1 = \sigma_1 = 2;\ A_j = B_j = 1;\ b_1 = 0$ and replace a_1, a_2, b_2 with $1-a_1,\ 1-a_2,\ 1-b_2$, respectively, then we reduce the H-function to the Gaussian hypergeometric function ${}_2F_1$ as follows:

$$H_{2,2}^{1,2}\left[x\,\middle|\,\begin{matrix}(1-a_1,1),(1-a_2,1) \\ (0,1),(1-b_2,1)\end{matrix}\right] = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_2)}\,{}_2F_1[a_1,a_2;b_2;-x]. \tag{3.12}$$

Finally, applying relation (3.12) to relations (3.2), (3.3) and (3.4), we get relations (3.13)–(3.15) presented in what follows whose integrands and the resulting formulas contain ${}_2F_1$ and the generalized hypergeometric function ${}_3F_2$, respectively.

Corollary 16. *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Then the following relation is true:

$$\begin{aligned}&\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\sigma(1-u))\\ &\qquad\times L_n^{(c',d')}(\sigma(1-u))\,{}_2F_1[a_1,a_2;b_2;zu^{-1}]\,du\\ &\quad=\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h)\\ &\qquad\times(\delta+h)_\lambda\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right]\\ &\qquad\times{}_3F_2[\lambda+\delta+h,a_1,a_2;b_2,\lambda;z]\end{aligned}\tag{3.13}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 17. *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\Re(\lambda),\Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x,\ t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Then the following relation is true:

$$\begin{aligned}&\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1}Z_m^{(1,b')}(\sigma(u-t);1)\\ &\qquad\times Z_n^{(1,d')}(\sigma(u-t);1)\,{}_2F_1[a_1,a_2;b_2;z(u-t)^{-1}]\,du\\ &\quad=\Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}\\ &\qquad\times\sum_{h=0}^{m+n}\sigma^h\,(\lambda+h)_\delta\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right]\\ &\qquad\times{}_3F_2[\lambda+\delta+h,a_1,a_2;b_2,\lambda+h;z]\end{aligned}\tag{3.14}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

Corollary 18. *Let* $z,\ \mu,\ \nu,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\mu), \Re(\nu), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and*

$$\min\{\Re(b'), \Re(d')\} > -1.$$

Then the following relation is true:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\,[1-\sigma(x-t)]^n\,{}_2F_1[a_1, a_2; b_2; z(x-t)^{-1}]\,dt$$

$$= x^{\mu+\nu-1}(\mu)_\nu \sum_{h=0}^{n} (-n)_h\,\sigma^h\,x^h\,{}_3F_2[\mu+\nu, a_1, a_2; b_2, \mu; z] \tag{3.15}$$

provided that the other involved parameters satisfy the constraints such that each member may exist.

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