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EXISTENCE RESULTS FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH TWO LOWER-ORDER TERMS AND L^1-DATA

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We study the existence of a renormalized solution for a class of nonlinear parabolic equations with two lower-order terms and L^1-data.

1. Introduction

We consider the following nonlinear parabolic problem:

$$\begin{aligned}
&\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f \quad \text{in} \quad Q_T,\\
&b(x,u)(t=0) = b(x,u_0) \quad \text{in} \quad \Omega, \qquad\qquad (1.1)\\
&u = 0 \quad \text{on} \quad \partial\Omega \times (0,T),
\end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$, $p > 1$, and Q_T is a cylinder $\Omega \times (0,T)$. The operator $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator, which is coercive and grows as $|\nabla u|^{p-1}$ with respect to ∇u. The function $b(x,u)$ is unbounded on u and $b(x,u_0) \in L^1(\Omega)$. The functions g and H are two Carathéodory functions satisfying certain assumptions imposed in what follows. Finally, the function $f \in L^1(Q_T)$.

Problem (1.1) is encountered in a variety of physical phenomena and applications. Thus, if

$$b(x,u) = u, \qquad a(x,t,u,\nabla u) = |\nabla u|^{p-2}\nabla u, \qquad g = f = 0,$$

and

$$H(x,t,\nabla u) = \lambda|\nabla u|^q,$$

where q and λ are positive parameters, then the equation in problem (1.1) can be regarded as the viscosity approximation of the Hamilton–Jacobi-type equation from the stochastic control theory [18]. In particular, for

$$b(x,u) = u, \qquad a(x,t,u,\nabla u) = \nabla u, \qquad g = f = 0,$$

and

$$H(x,t,\nabla u) = \lambda|\nabla u|^2,$$

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where λ is a positive parameter, the equation of problem (1.1) appears in the physical theory of growth and roughening of the surfaces in which it is known as the Kardar–Parisi–Zhang equation [14]. We introduce the definition of renormalized solutions for problem (1.1) as follows: This notion was introduced by Lions and DiPerna [12] for the investigation of the Boltzmann equation (see also [17] for the applications to models of the fluid mechanics). This notion was later adapted to the elliptic version of (1.1) by Boccardo, et al. [9] in the case where the right-hand side lies in $W^{-1,p'}(\Omega)$, by Rakotoson [24] in the case where the right-hand side is in $L^1(\Omega)$, and by Dal Maso, Murat, Orsina, and Prignet [10] in the case where the right-hand side is a general measure data; see also [19, 20].

For

$$b(x,u) = u \qquad \text{and} \qquad H = 0,$$

the existence of a weak solution to problem (1.1) that belongs to $L^m(0,T;W_0^{1,m}(\Omega))$ with

$$p > 2 - \frac{1}{N+1} \qquad \text{and} \qquad m < \frac{p(N+1)-N}{N+1}$$

was proved in [8] (see also [7]) in the case where $g = 0$. Moreover, this problem was also studied in [23] for $g = 0$ and in [11, 21, 22]. In the case where the function $g(x,t,u,\nabla u) \equiv g(u)$ is independent of $(x,t,\nabla u)$ and g is continuous, the existence of renormalized solution to problem (1.1) was proved in [5]. The existence of the renormalized solution to problem (1.1) in the variational case has been recently proved in [1].

The aim of the present paper is to prove an existence result for renormalized solutions of a class of problems (1.1) with two lower-order terms and L^1-data. The difficulties encountered in our problem (1.1) are caused by the presence of the terms g and H responsible for the lack of coercivity, uncontrolled growth of the function $b(x,s)$ with respect to s, and the facts that the functions $a(x,t,u,\nabla u)$ do not, in general, belong to $\left(L^1_{\text{loc}}(Q_T)\right)^N$ and the data $b(x,u_0), f$ are only integrable.

The remaining part of the present article is organized as follows: In Section 2, we give precise formulations of all assumptions about $b,\ a,\ g,\ H,$ and u_0. In addition, we introduce the concept of renormalized solution for problem (1.1). In Section 3, we establish the existence of our main results.

2. Essential Assumptions and Various Notions of Solutions

We now introduce several assumptions used throughout the paper.

Let Ω be a bounded open set in $\mathbb{R}^N,\ N \geq 1,$ let $T > 0$ be a given number, let $Q_T = \Omega \times (0,T),$ and let

$$b\colon\ \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{be a Carathéodory function}$$

such that, for every $x \in \Omega,\ b(x,.)$ is a strictly increasing C^1-function with $b(x,0) = 0$. Further, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \qquad \text{and} \qquad \left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_k(x), \tag{2.1}$$

for almost all $x \in \Omega$. For any s such that $|s| \leq k$, by $\nabla_x\left(\dfrac{\partial b(x,s)}{\partial s}\right)$ we denote the gradient of $\dfrac{\partial b(x,s)}{\partial s}$ defined in a sense of distributions.

Let

$$a\colon\ Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

be a Carathéodory function such that

$$\big|a(x,t,s,\xi)\big| \le \beta\big[k(x,t) + |s|^{p-1} + |\xi|^{p-1}\big], \tag{2.2}$$

for $(x,t) \in Q_T$ a.e., all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, some positive function $k(x,t) \in L^{p'}(Q_T)$, and $\beta > 0$,

$$\big[a(x,t,s,\xi) - a(x,t,s,\eta)\big](\xi - \eta) > 0 \quad \text{for all} \quad (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N \quad \text{with} \quad \xi \neq \eta, \tag{2.3}$$

and

$$a(x,t,s,\xi)\xi \ge \alpha|\xi|^p, \quad \text{where} \quad \alpha \quad \text{is a strictly positive constant.} \tag{2.4}$$

Furthermore, let

$$g(x,t,s,\xi)\colon Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \qquad \text{and} \qquad H(x,t,\xi)\colon Q_T \times \mathbb{R}^N \to \mathbb{R}$$

be two Carathéodory functions satisfying the following conditions for almost all $(x,t) \in Q_T$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$:

$$\big|g(x,t,s,\xi)\big| \le L_1(|s|)\big(L_2(x,t) + |\xi|^p\big), \tag{2.5}$$

$$g(x,t,s,\xi)s \ge 0, \tag{2.6}$$

where $L_1\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $L_2(x,t)$ is positive and belongs to $L^1(Q_T)$,

$$\exists\delta > 0, \quad \nu > 0 \quad \forall|s| \ge \delta\colon \big|g(x,t,s,\xi)\big| \ge \nu|\xi|^p, \tag{2.7}$$

$$\big|H(x,t,\xi)\big| \le h(x,t)|\xi|^{p-1}, \quad \text{where} \quad h(x,t) \ \text{is positive and belongs to} \ L^p(Q_T). \tag{2.8}$$

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as follows:

$$T_k(s) = \max(-k, \min(k,s)).$$

We use the following definition of renormalized solution to problem (1.1):

Definition 1. *Let $f \in L^1(Q_T)$ and let $b\big(\cdot,u_0(\cdot)\big) \in L^1(\Omega)$. A renormalized solution of problem (1.1) is a function u defined on Q_T and satisfying the following conditions:*

$$T_k(u) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \quad \textit{for all} \quad k \ge 0 \quad \textit{and} \quad b(x,u) \in L^\infty\big(0,T;L^1(\Omega)\big), \tag{2.9}$$

$$\int\limits_{\{m \le |u| \le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt \to 0 \quad \textit{as} \quad m \to +\infty, \tag{2.10}$$

$$\begin{aligned}\frac{\partial B_S(x,u)}{\partial t} &- \operatorname{div}\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &+ g(x,t,u,\nabla u)S'(u) + H(x,t,\nabla u)S'(u) = fS'(u) \quad \textit{in} \quad \mathcal{D}'(Q_T),\end{aligned} \tag{2.11}$$

for all functions $S \in W^{2,\infty}(\mathbb{R})$ *that are piecewise* $\mathcal{C}^1(\mathbb{R})$ *and such that* S' *has a compact support in* $\mathbb{R}$ *and, in addition,*

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega, \quad where \quad B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)\,dr. \tag{2.12}$$

Remark 1. Equation (2.11) is formally obtained by the pointwise multiplication of (1.1) by $S'(u)$. However, since $a(x,t,u,\nabla u)$, $g(x,t,u,\nabla u)$, and $H(x,t,\nabla u)$ are, generally speaking, meaningless in $\mathcal{D}'(Q_T)$, all terms in (2.11) have a meaning in $\mathcal{D}'(Q_T)$.

Indeed, if M is such that $\operatorname{supp} S' \subset [-M,M]$, then the following identifications can be made in (2.11):

- $|B_S(x,u)| = |B_S(x,T_M(u))| \le M\|S'\|_{L^\infty(\mathbb{R})} A_M(x)$ belongs to $L^\infty(\Omega)$ because A_M is a bounded function;
- $S'(u)a(x,t,u,\nabla u)$ is identified with

  $$S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$$

  a.e. in Q_T; since $|T_M(u)| \le M$ a.e. in Q_T and $S'(u) \in L^\infty(Q_T)$, it follows from (2.2) and (2.9) that

  $$S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big) \in \big(L^{p'}(Q_T)\big)^N;$$

- $S''(u)a(x,t,u,\nabla u)\nabla u$ is identified with

  $$S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u)$$

  and

  $$S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u) \in L^1(Q_T);$$

- $S'(u)\Big(g\big(x,t,u,\nabla u\big) + H(x,t,\nabla u)\Big)$ is identified with

  $$S'(u)\Big(g\big(x,t,T_M(u),\nabla T_M(u)\big) + H\big(x,t,\nabla T_M(u)\big)\Big)$$

  a.e. in Q_T; since $|T_M(u)| \le M$ a.e. in Q_T and $S'(u) \in L^\infty(Q_T)$, it follows from (2.2), (2.5), and (2.8) that

  $$S'(u)\Big(g\big(x,t,T_M(u),\nabla T_M(u)\big) + H(x,t,\nabla T_M(u))\Big) \in L^1(Q_T);$$

- $S'(u)f$ belongs to $L^1(Q_T)$.

The analysis presented above shows that (2.11) holds in $\mathcal{D}'(Q_T)$ and

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}\big(0,T;W^{-1,\,p'}(\Omega)\big) + L^1(Q_T). \tag{2.13}$$

The properties of S and assumptions (2.1) and (2.10) imply that

$$\big|\nabla B_S(x,u)\big| \le \|A_M\|_{L^\infty(\Omega)}\big|\nabla T_M(u)\big|\|S'\|_{L^\infty(\mathbb{R})} + M\|S'\|_{L^\infty(\mathbb{R})}B_M(x) \tag{2.14}$$

and

$$B_S(x,u) \quad \text{belongs to} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big). \tag{2.15}$$

Thus, (2.13) and (2.15) imply that $B_S(x,u)$ belongs to $C^0\big([0,T];L^1(\Omega)\big)$ (for the proof of this trace result, see [21]). Hence, the initial condition (2.12) is meaningful.

We also note that, for every $S \in W^{1,\infty}(\mathbb{R})$ defined as a nondecreasing function such that $\operatorname{supp} S' \subset [-M,M]$, in view of (2.1), we have

$$\begin{aligned}\lambda_M|S(r)-S(r')| &\le \Big|B_S(x,r)-B_S(x,r')\Big| \\ &\le \|A_M\|_{L^\infty(\Omega)}|S(r)-S(r')| \quad \text{a.e.} \quad x\in\Omega, \quad \forall r,r'\in\mathbb{R}.\end{aligned}$$

3. Statements of the Results

We now formulate the main results of the present paper.

Theorem 1. *Let $f \in L^1(Q_T)$ and let u_0 be a measurable function such that $b(\cdot,u_0)\in L^1(\Omega)$. Assume that (2.1)–(2.8) is true. Then there exists a renormalized solution u of problem (1.1) in the sense of Definition 1.*

Proof. The proof of Theorem 1 is done in five steps.

Step 1: Approximate problem and *a priori* estimates. For $n>0$, we define the following approximations of b, f, and u_0 :

First, let

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n} r b_n$$

be a Carathéodory function satisfying (2.1). Then there exist $\lambda_n>0$ and functions $A_n\in L^\infty(\Omega)$ and $B_n\in L^p(\Omega)$ such that

$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \qquad \text{and} \qquad \left|\nabla_x\left(\frac{\partial b_n(x,s)}{\partial s}\right)\right| \le B_n(x)$$

a.e. in Ω, $s\in\mathbb{R}$.

Further, we set

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1+\dfrac{1}{n}|g(x,t,s,\xi)|} \qquad \text{and} \qquad H_n(x,t,\xi) = \frac{H(x,t,\xi)}{1+\dfrac{1}{n}|H(x,t,\xi)|}.$$

Note that

$$\big|g_n(x,t,s,\xi)\big| \le \max\big\{|g(x,t,s,\xi)|;n\big\} \qquad \text{and} \qquad \big|H_n(x,t,\xi)\big| \le \max\big\{|H(x,t,\xi)|;n\big\}.$$

Moreover, since $f_n \in L^{p'}(Q_T)$ and $f_n \to f$ a.e. in Q_T and strongly in $L^1(Q_T)$ as $n \to \infty$, we get

$$u_{0n} \in \mathcal{D}(\Omega),$$
$$b_n(x, u_{0n}) \to b(x, u_0) \quad \text{a.e. in} \quad \Omega \quad \text{and strongly in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.1}$$

We now consider an approximate problem

$$\begin{aligned}
&\frac{\partial b_n(x, u_n)}{\partial t} - \operatorname{div}(a(x, t, u_n, \nabla u_n)) + g_n(x, t, u_n, \nabla u_n) + H_n(x, t, \nabla u_n) = f_n \quad \text{in} \quad Q_T, \\
&b_n(x, u_n)(t = 0) = b_n(x, u_{0n}) \quad \text{in} \quad \Omega, \\
&u_n = 0 \quad \text{in} \quad \partial\Omega \times (0, T).
\end{aligned} \tag{3.2}$$

Since $f_n \in L^{p'}\big(0, T; W^{-1, p'}(\Omega)\big)$, we can easily prove the existence of a weak solution

$$u_n \in L^p\big(0, T; W_0^{1, p}(\Omega)\big)$$

of problem (3.2) (see, e.g., [16, p. 271]), namely,

$$\int_0^T \left\langle \frac{\partial b_n(x, u_n)}{\partial t}, v \right\rangle dt + \int_{Q_T} a(x, t, u_n, \nabla u_n) \nabla v \, dx \, dt$$
$$+ \int_{Q_T} g_n(x, t, u_n, \nabla u_n) v \, dx \, dt + \int_{Q_T} H_n(x, t, \nabla u_n) v \, dx \, dt = \int_{Q_T} f_n v \, dx \, dt$$

for all $v \in L^p\big(0, T; W^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$.

We now prove that the solution u_n of problem (3.2) is bounded in $L^p\big(0, T; W_0^{1, p}(\Omega)\big)$.

Lemma 1. *Let $u_n \in L^p\big(0, T; W_0^{1, p}(\Omega)\big)$ be a weak solution of (3.2). Then the following estimates hold:*

$$\|u_n\|_{L^p\left(0, T; W_0^{1, p}(\Omega)\right)} \le D, \tag{3.3}$$

where D depends only on Ω, T, N, p, p', f, and $\|h\|_{L^p(Q_T)}$.

Proof. To get (3.3), we split the integral $\int_{Q_T} |\nabla u_n|^p \, dx \, dt$ in two parts and prove the following estimates: for all $k \ge 0$,

$$\int_{\{|u_n| \le k\}} \big|\nabla u_n\big|^p \, dx \, dt \le M_1 k, \tag{3.4}$$

and

$$\int\limits_{\{|u_n|>k\}} \left|\nabla u_n\right|^p \,dx\,dt \le M_2, \tag{3.5}$$

where M_1 and M_2 are positive constants. In what follows, by $M_i,\ i = 3, 4, \ldots,$ we denote some generic positive constants. Suppose that $p < N$ (the case $p \ge N$ is similar). For $\varepsilon > 0$ and $s \ge 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \operatorname{sign}(r) & \text{for} \quad |r| > s+\varepsilon, \\ \dfrac{\operatorname{sign}(r)(|r|-s)}{\varepsilon} & \text{for} \quad s < |r| \le s+\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

We choose $v = \varphi_\varepsilon(u_n)$ as the test function in (3.2) and obtain

$$\left[\int\limits_\Omega B^n_{\varphi_\varepsilon}(x,u_n)\,dx\right]_0^T + \int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla(\varphi_\varepsilon(u_n))\,dx\,dt$$

$$+ \int\limits_{Q_T} g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt + \int\limits_{Q_T} H_n(x,t,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt$$

$$= \int\limits_{Q_T} f_n\varphi_\varepsilon(u_n)\,dx\,dt,$$

where

$$B^n_{\varphi_\varepsilon}(x,r) = \int\limits_0^r \frac{\partial b_n(x,s)}{\partial s}\varphi_\varepsilon(s)\,ds.$$

By using the inequalities $B^n_{\varphi_\varepsilon}(x,r) \ge 0$ and $g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n) \ge 0$, (2.4), (2.8), and the Hölder inequality and letting ε go to zero, we obtain

$$-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt \le \int\limits_{\{s<|u_n|\}} |f_n|\,dx\,dt$$

$$+ \int\limits_s^{+\infty} \left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}} \left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma,$$

where $\{s<|u_n|\}$ denotes the set

$$\big\{(x,t)\in Q_T, s<|u_n(x,t)|\big\}$$

and $\mu(s)$ stands for the distribution function of u_n, i.e.,

$$\mu(s) = \big|\big\{(x,t) \in Q_T,\, |u_n(x,t)| > s\big\}\big|$$

for all $s \geq 0$.

On the other hand, from the Fleming–Rishel coarea formula and the isoperimetric inequality, we get, for almost all $s > 0$,

$$NC_N^{\frac{1}{N}}\big(\mu(s)\big)^{\frac{N-1}{N}} \leq -\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p dx\, dt, \tag{3.6}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. By using the Hölder's inequality, we conclude that, for almost every $s > 0$,

$$-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\, dx\, dt \leq \big(-\mu'(s)\big)^{\frac{1}{p'}}\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\, dx\, dt\right)^{\frac{1}{p}}. \tag{3.7}$$

Thus, combining (3.6) and (3.7), for almost all $s > 0$, we obtain

$$1 \leq \Big(NC_N^{\frac{1}{N}}\Big)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p dx\, dt\right)^{\frac{1}{p}}. \tag{3.8}$$

By using (3.8), we get

$$\begin{aligned}
&\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\, dx\, dt\right)^{\frac{1}{p'}} \\
&\qquad \leq \Big(NC_N^{\frac{1}{N}}\Big)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\left(\int\limits_{\{s<|u_n|\}} |f_n|\, dx\, dt\right) \\
&\qquad\quad + \Big(NC_N^{\frac{1}{N}}\Big)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}} \\
&\qquad\quad \times \int\limits_s^{+\infty}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} h^p\, dx\, dt\right)^{\frac{1}{p}}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} |\nabla u_n|^p\, dx\, dt\right)^{\frac{1}{p'}} d\sigma.
\end{aligned} \tag{3.9}$$

We now consider two functions B and ψ (see Lemma 2.2 of [2]) defined as

$$\int\limits_{\{s<|u_n|\}} h^p(x,t)\, dx\, dt = \int\limits_0^{\mu(s)} B^p(\sigma)\, d\sigma \tag{3.10}$$

and

$$\psi(s) = \int\limits_{\{s<|u_n|\}} |f_n|\,dx\,dt. \tag{3.11}$$

We have

$$\|B\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} \le \|h\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} \qquad \text{and} \qquad |\psi(s)| \le \|f_n\|_{L^1(Q_T)}.$$

It follows from (3.9), (3.10), and (3.11) that

$$\begin{aligned}
&\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} \\
&\qquad \le \left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\psi(s) + \left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}} \\
&\qquad\quad \times \int_s^{+\infty} B\big(\mu(\nu)\big)\big(-\mu'(\nu)\big)^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int\limits_{\{\nu<|u_n|\}} |\nabla u_n|^p dxdt\right)^{\frac{1}{p'}} d\nu.
\end{aligned}$$

By the Gronwall lemma (see [3]), we get

$$\begin{aligned}
&\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} \\
&\qquad \le \left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\psi(s) + (NC_N^{\frac{1}{N}})^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}} \\
&\qquad\quad \times \int_s^{+\infty} \Big[\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(\sigma)\big)^{\frac{1}{N}-1}\psi(\sigma)\Big] B\big(\mu(\sigma)\big)\big(-\mu'(\sigma)\big) \\
&\qquad\quad\quad \times \exp\left(\int_s^{\sigma}\left(NC_N^{\frac{1}{N}}\right)^{-1} B\big(\mu(r)\big)\big(\mu(r)\big)^{\frac{1}{N}-1}\big(-\mu'(r)\big)dr\right) d\sigma.
\end{aligned} \tag{3.12}$$

Further, by the change of variables and the Hölder inequality, we estimate the argument of the exponential function on the right-hand side of (3.12) as follows:

$$\int_s^{\sigma} B\big(\mu(r)\big)\big(\mu(r)\big)^{\frac{1}{N}-1}\big(-\mu'(r)\big)dr = \int_s^{\sigma} B(z) z^{\frac{1}{N}-1} dz$$

$$\leq \int_0^{|\Omega|} B(z) z^{\frac{1}{N}-1} dz \ \leq \ \|B\|_{L^p} \left(\int_0^{|\Omega|} z^{(\frac{1}{N}-1)p'} \right)^{\frac{1}{p'}} .$$

Raising to the (p') th power in (3.12), we can write

$$-\frac{d}{ds} \int_{\{s<|u_n|\}} |\nabla u_n|^p \, dx \, dt \leq M_1,$$

where M_1 depends only on $\Omega,\ N,\ p,\ p',\ f,\ \alpha,$ and $\|h\|_{L^p(Q_T)}$. Thus, integrating from 0 to $k,$ we prove (3.4).

We now present the proof of (3.5) by using $T_k(u_n)$ as the test function in (3.2). Thus, we get

$$\left[\int_\Omega B_k^n(x, u_n) \, dx \right]_0^T + \int_\Omega a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt$$

$$+ \int_\Omega (g_n(x, t, u_n, \nabla u_n) + H_n(x, t, \nabla u_n)) T_k(u_n) \, dx \, dt$$

$$= \int_\Omega f_n T_k(u_n) \, dx \, dt,$$

where

$$B_k^n(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s} T_k(s) \, ds.$$

By using (2.8), we conclude that

$$\left[\int_\Omega B_k^n(x, u_n) \, dx \right]_0^T + \int_{\{|u_n| \leq k\}} a(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt$$

$$+ \int_{\{|u_n| \leq k\}} g_n(x, t, u_n, \nabla u_n) u_n \, dx$$

$$+ \int_{\{|u_n| > k\}} g_n(x, t, u_n, \nabla u_n) T_k(u_n) \, dx \, dt$$

$$\leq \int_\Omega f_n T_k(u_n) \, dx \, dt + \int_\Omega h(x, t) |\nabla u_n|^{p-1} |T_k(u_n)| \, dx \, dt.$$

Further, in view of the facts that $B_k^n(x,r) \geq 0$ and $g_n(x,t,u_n,\nabla u_n)u_n \geq 0$ and (2.4), we find

$$\alpha \int\limits_{\{|u_n|\leq k\}} |\nabla u_n|^p \, dx \, dt \; + \int\limits_{\{|u_n|>k\}} g(x,u_n,\nabla u_n) T_k(u_n) \, dx \, dt$$

$$\leq k\|f\|_{L^1} + k \int\limits_{\{|u_n|\leq k\}} h(x,t)|\nabla u_n|^{p-1} \, dx \, dt$$

$$+ \; k \int\limits_{\{|u_n|\geq k\}} h(x,t)|\nabla u_n|^{p-1} \, dx \, dt.$$

By virtue of the Hölder inequality, (3.4), (2.7), and Young's inequality, for all $k > \delta,$ we get

$$\nu k \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx \, dt \leq k\|f\|_{L^1(Q_T)} + k^{1+\frac{1}{p'}} M_1 \|h\|_{L^p(Q_T)}$$

$$+ \, k \int\limits_{\{|u_n|>k\}} h(x,t)|\nabla u_n|^{p-1} \, dx \, dt$$

$$\leq k\|f\|_{L^1(Q_T)} + k^{1+\frac{1}{p'}} M_1 \|h\|_{L^p(Q_T)}$$

$$+ \, M_6 k \|h\|_{L^p}^p + \frac{1}{p'} \nu k \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx \, dt.$$

Hence,

$$\left(1 - \frac{1}{p'}\right) \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx \, dt \leq M_3 \|f\|_{L^1(Q_T)} + k^{\frac{1}{p'}} M_5 \|h\|_{L^p(Q_T)} + M_7 \|h\|_{L^p}^p. \tag{3.13}$$

Lemma 1 is proved.

Then there exists $u \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ such that, for some subsequence

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big), \tag{3.14}$$

we conclude that

$$\big\|T_k(u_n)\big\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p \leq c_2 k. \tag{3.15}$$

It follows from inequalities (2.1) and (3.15) that

$$\int\limits_{\Omega} B_k^n(x,u_n) \, dx \leq Ck, \tag{3.16}$$

where

$$B_k^n(x,z) = \int_0^z \frac{\partial b_n(x,s)}{\partial s} T_k(s)\,ds.$$

We now prove that u_n and $b_n(x,u_n)$ converge almost everywhere. Consider a nondecreasing function $\xi_k \in C^2(\mathbb{R})$ such that

$$\xi_k(s) = s \quad \text{for} \quad |s| \le \frac{k}{2} \qquad \text{and} \qquad \xi_k(s) = k \quad \text{for} \quad |s| \ge k.$$

Multiplying the approximate equation by $\xi_k'(u_n)$, we obtain

$$\begin{aligned}\frac{\partial B_\xi^n(x,u_n)}{\partial t} &- \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\xi_k'(u_n)\Big)\\ &+ a(x,t,u_n,\nabla u_n)\xi_k''(u_n)\nabla u_n\\ &+ \Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big)\xi_k'(u_n)\\ &= f_n\xi_k'(u_n),\end{aligned} \tag{3.17}$$

in the sense of distributions, where

$$B_\xi^n(x,z) = \int_0^z \frac{\partial b_n(x,s)}{\partial s} \xi_k'(s)\,ds.$$

As a consequence of (3.15), we conclude that

$$\xi_k(u_n) \quad \text{is bounded in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big)$$

and

$$\frac{\partial B_\xi^n(x,u_n)}{\partial t} \quad \text{is bounded in} \quad L^1(Q_T) + L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big).$$

In view of the properties of ξ_k and (2.1), the derivative

$$\frac{\partial \xi_k(u_n)}{\partial t} \quad \text{is bounded in} \quad L^1(Q_T) + L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big),$$

which implies that $\xi_k(u_n)$ strongly converges in $L^1(Q_T)$ (see [21]).

In view of the choice of ξ_k, we conclude that, for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q_T, which implies that u_n converges almost everywhere to some measurable function u in Q_T. Thus, by using the same arguments as in [4, 5, 25], we can show that

$$\begin{gathered} u_n \to u \quad \text{a.e. in} \quad Q_T,\\ b_n(x,u_n) \to b(x,u) \quad \text{a.e. in} \quad Q_T. \end{gathered} \tag{3.18}$$

It follows from (3.15) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big).$$

In view of (2.2), for all $k>0$, this implies that there exists a function $\overline{a} \in (L^{p'}(Q_T))^N$ such that

$$a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup \overline{a} \quad \text{weakly in} \quad \big(L^{p'}(Q_T)\big)^N. \tag{3.19}$$

We now show that $b(.,u)$ belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$. By using (3.18) and passing to the limit-inf in (3.16) as n tends to $+\infty$, we obtain

$$\frac{1}{k}\int_\Omega B_k(x,u)(\tau)\,dx \le C$$

for almost all τ in $(0,T)$. In view of the definition of $B_k(x,s)$ and the fact that $\frac{1}{k}B_k(x,u)$ converges (pointwise) to $b(x,u)$ as k tends to $+\infty$, we show that $b(x,u)$ belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$.

Lemma 2. *Let u_n be a solution of the approximate problem (3.2). Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.20}$$

Proof. We use

$$T_1\big(u_n - T_m(u_n)\big)^+ = \alpha_m(u_n) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)\cap L^\infty(Q_T)$$

as the test function in (3.2). Thus, we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t};\,\alpha_m(u_n)\right\rangle dt\\
&\qquad + \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n \alpha_m'(u_n)\,dx\,dt\\
&\qquad + \int_{Q_T}\big(g_n(x,t,u_n,\nabla u_n)+H_n(x,t,\nabla u_n)\big)\alpha_m(u_n)\,dx\,dt\\
&\qquad \le \int_{Q_T}\big|f_n\alpha_m(u_n)\big|\,dx\,dt.
\end{aligned}$$

Hence, by setting

$$B_m^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\alpha_m(s)\,ds$$

and using (2.6) and (2.8), we find

$$\int_\Omega B_m^n(x,u_n)(T)\,dx \;+ \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt$$

$$\le \int_{\{m\le u_n\}} |f_n|\,dx\,dt \;+ \int_{Q_T} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt.$$

We now use the Hölder inequality and (3.3) in order to deduce the inequality

$$\int_\Omega B_m^n(x,u_n)(T)\,dx \;+ \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt$$

$$\le \int_{\{m\le u_n\}} |f_n|\,dx\,dt + c_1\left(\int_{\{m\le u_n\}} |h(x,t)|^p\,dx\,dt\right)^{\frac{1}{p'}}.$$

Since $B_m^n(x,u_n)(T)\ge 0$, in view of the strong convergence of f_n in $L^1(Q_T)$, by the Lebesgue theorem, we find

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le u_n\}} |f_n|\,dx\,dt = 0.$$

Similarly, since $h\in L^p(Q_T)$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\left(\int_{\{m\le u_n\}} |h(x,t)|^p\,dx\,dt\right)^{\frac{1}{p'}} = 0.$$

We conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.21}$$

On the other hand, using $T_1\big(u_n - T_m(u_n)\big)^-$ as the test function in (3.2) and the same reasoning as in the proof of (3.21), we show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.22}$$

Therefore, (3.20) follows from (3.21) and (3.22).

Step 2: Convergence of gradients almost everywhere. In this step, for fixed $k \geq 0$, we introduce the time of regularization of the function $T_k(u)$ with an aim to apply the monotonicity method (the proof of this step is similar to the proof of Step 4 in [5]). This kind of regularization was first introduced by Landes (see Lemma 6 and Proposition 3 in [15, p. 230] and Proposition 4 in [15, p. 231]). For fixed $k > 0$, we set

$$\varphi(t) = te^{\gamma t^2}, \quad \gamma > 0.$$

It is well known that, for $\gamma > \left(\dfrac{L_1(k)}{2\alpha}\right)^2$, we have

$$\varphi'(s) - \left(\frac{L_1(k)}{\alpha}\right)|\varphi(s)| \geq \frac{1}{2} \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.23}$$

Let $\{\psi_i\} \subset \mathcal{D}(\Omega)$ be a sequence strongly convergent to u_0 in $L^1(\Omega)$. We set

$$w_\mu^i = (T_k(u))_\mu + e^{-\mu t}\, T_k(\psi_i),$$

where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that w_μ^i is a smooth function with the following properties:

$$\frac{\partial w_\mu^i}{\partial t} = \mu(T_k(u) - w_\mu^i), \qquad w_\mu^i(0) = T_k(\psi_i), \qquad \left|w_\mu^i\right| \leq k, \tag{3.24}$$

$$w_\mu^i \to T_k(u) \quad \text{strongly in} \ \ L^p\big(0,T;W_0^{1,p}(\Omega)\big) \quad \text{as} \quad \mu \to \infty. \tag{3.25}$$

Further, we introduce the following function of one real variable:

$$h_m(s) = \begin{cases} 1 & \text{for} \quad |s| \leq m, \\ 0 & \text{for} \quad |s| \geq m+1, \\ m+1-|s| & \text{for} \quad m \leq |s| \leq m+1, \end{cases}$$

where $m > k$. Let

$$\theta_n^{\mu,i} = T_k(u_n) - w_\mu^i \qquad \text{and} \qquad z_{n,m}^{\mu,i} = \varphi\big(\theta_n^{\mu,i}\big)h_m(u_n).$$

By using the test function $z_{n,m}^{\mu,i}$ in system (3.2), in view of the fact that

$$g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n) - w_\mu^i\big)h_m(u_n) \geq 0 \quad \text{on} \quad \{|u_n| > k\},$$

we obtain

$$\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t};\, \varphi(T_k(u_n) - w_\mu^i)h_m(u_n) \right\rangle dt$$

$$+\int_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n)-\nabla w_\mu^i\big)\varphi'\big(\theta_n^{\mu,i}\big)h_m(u_n)\,dx\,dt$$

$$+\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n\varphi\big(\theta_n^{\mu,i}\big)h_m'(u_n)\,dx\,dt$$

$$+\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt$$

$$\le \int_{Q_T}\big|f_n z_{n,m}^{\mu,i}\big|\,dx\,dt+\int_{Q_T}\big|H_n(x,t,\nabla u_n)z_{n,m}^{\mu,i}\big|\,dx\,dt. \tag{3.26}$$

In the remaining part of the present paper, for the sake of simplicity, we omit the notation $\varepsilon(n,\mu,i,m)$ in all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}\varepsilon(n,\mu,i,m)=0.$$

Moreover, this is the order in which the analyzed parameters tend to infinity, i.e., first n, then μ and i and finally, m. Similarly, we write $\varepsilon(n)$ or $\varepsilon(n,\mu),\ldots$ to denote that we pass to the limits only with respect to the indicated parameters.

We now consider each term in (3.26). First, we note that

$$\int_{Q_T}\big|f_n z_{n,m}^{\mu,i}\big|\,dx\,dt+\int_{Q_T}\big|H_n(x,t,\nabla u_n)z_{n,m}^{\mu,i}\big|\,dx\,dt=\varepsilon(n,\mu),$$

because

$$\varphi\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\quad\text{converges to}\quad \varphi\big(T_k(u)-(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i)\big)h_m(u)$$

strongly in $L^p(Q_T)$ and weakly $-*$ in $L^\infty(Q_T)$ as $n\to\infty$ and, in addition,

$$\varphi\big(T_k(u)-(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i)\big)h_m(u)\quad\text{converges to}\quad 0$$

strongly in $L^p(Q_T)$ and weakly $-*$ in $L^\infty(Q_T)$ as $\mu\to\infty$. In view of (3.20), the third and fourth integrals on the right-hand side of (3.26) tend to zero as n and m tend to infinity and, by the Lebesgue theorem and $F\in(L^{p'}(Q_T))^N$, we deduce that the right-hand side of (3.26) converges to zero as $n,\ m,$ and μ tend to infinity. Since

$$(T_k(u_n)-w_\mu^i)h_m(u_n)\rightharpoonup(T_k(u)-w_\mu^i)h_m(u)$$

weakly* in $L^1(Q_T)$ and strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$, we have

$$(T_k(u)-w_\mu^i)h_m(u)\rightharpoonup 0$$

weakly* in $L^1(Q_T)$ and strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ as $\mu\to+\infty$.

On the one hand, the definition of the sequence w_μ^i makes it possible to establish the following lemma:

Lemma 3. *For $k \geq 0$, the following inequality is true:*

$$\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}; \varphi(T_k(u_n) - w_\mu^i) h_m(u_n) \right\rangle dt \;\geq\; \varepsilon(n,m,\mu,i). \tag{3.27}$$

Proof (see Blanchard and Redwane [6]).

On the other hand, the second term on the left-hand side of (3.26) can be rewritten as follows:

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt \\
&\quad = \int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt \\
&\qquad + \int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt \\
&\quad = \int_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)\,dx\,dt \\
&\qquad + \int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt.
\end{aligned}$$

Since $m > k$ and $h_m(u_n) = 1$ on $\{|u_n| \leq k\}$, we conclude that

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt \\
&\quad = \int_{Q_T} \Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big) \\
&\qquad\quad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)\,dx\,dt \\
&\qquad + \int_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big) \\
&\qquad\quad \times \varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt
\end{aligned}$$

$$+\int_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u)\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt$$

$$-\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt$$

$$=K_1+K_2+K_3+K_4. \tag{3.28}$$

By using (2.2), (3.19), and the Lebesgue theorem, we show that

$$a\big(x,t,T_k(u_n),\nabla T_k(u)\big) \text{ converges to } a\big(x,t,T_k(u),\nabla T_k(u)\big) \text{ strongly in } \big(L^{p'}(Q_T)\big)^N$$

and

$$\nabla T_k(u_n) \text{ converges to } \nabla T_k(u) \text{ weakly in } \big(L^p(Q_T)\big)^N.$$

Then

$$K_2=\varepsilon(n). \tag{3.29}$$

By using (3.19) and (3.25), we find

$$K_3=\int_{Q_T}\overline{a}\,\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{3.30}$$

Since $h_m(u_n)=0$ on $\{|u_n|>m+1\}$, for K_4, we can write:

$$K_4=-\int_{Q_T} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt$$

$$=-\int_{\{|u_n|\le k\}} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt$$

$$-\int_{\{k<|u_n|\le m+1\}} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla w_\mu^i$$

$$\times\,\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt.$$

As above, we get

$$K_4=-\int_{\{|u|\le k\}}\overline{a}\,\nabla w_\mu^i\varphi'\big(T_k(u)-w_\mu^i\big)\,dx\,dt$$

$$-\int_{\{k<|u|\le m+1\}}\overline{a}\,\nabla w_\mu^i\varphi'\big(T_k(u)-w_\mu^i\big)h_m(u)\,dx\,dt+\varepsilon(n)\quad\text{as}\quad n\to\infty.$$

Hence,

$$K_4 = -\int_{Q_T} \overline{a}\, \nabla T_k(u)\, dx\, dt + \varepsilon(n,\mu) \tag{3.31}$$

as $\mu \to \infty$. In view of (3.28), (3.29), (3.30), and (3.31), we conclude that

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\, dx\, dt \\
&\quad = \int_{Q_T} \Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big) \\
&\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)\, dx\, dt + \varepsilon(n,\mu).
\end{aligned} \tag{3.32}$$

As for the third term on the left-hand side of (3.26), we observe that

$$\left| \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \varphi\big(\theta_n^{\mu,i}\big)h'_m(u_n)\, dx\, dt \right| \le \varphi(2k) \int_{\{m \le |u_n| \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\, dx\, dt.$$

In view of (3.20), we obtain

$$\left| \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \varphi\big(\theta_n^{\mu,i}\big)h'_m(u_n)\, dx\, dt \right| \le \varepsilon(n,m). \tag{3.33}$$

We now consider the fourth term on the left-hand side of (3.26). Thus, we get

$$\begin{aligned}
&\left| \int_{\{|u_n| \le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\, dx\, dt \right| \\
&\quad \le \int_{\{|u_n| \le k\}} L_1(k)L_2(x,t) + \big|\nabla T_k(u_n)\big|^p \big|\varphi\big(T_k(u_n) - w_\mu^i\big)\big| h_m(u_n)\, dx\, dt \\
&\quad \le L_1(k) \int_{Q_T} L_2(x,t)\big|\varphi\big(T_k(u_n) - w_\mu^i\big)\big|\, dx\, dt \\
&\qquad + \frac{L_1(k)}{\alpha} \int_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\varphi\big(T_k(u_n) - w_\mu^i\big)\big|\, dx\, dt
\end{aligned} \tag{3.34}$$

because $L_2(x,t)$ belongs to $L^1(Q_T)$.

It is easy to see that

$$L_1(k)\int\limits_{Q_T} L_2(x,t)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt=\varepsilon(n,\mu).$$

On the other hand, the second term on the right-hand side of (3.34) can be represented in the form

$$\frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt$$

$$=\frac{L_1(k)}{\alpha}\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt$$

$$+\ \frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt$$

$$+\ \frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt.$$

Thus, as above, we first let n tend to infinity, and then pass to the limit as μ goes to infinity. As a result, we readily conclude that each of the last two integrals has the form $\varepsilon(n,\mu)$. This yields

$$\left|\int\limits_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt\right|$$

$$\le\frac{L_1(k)}{\alpha}\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt+\varepsilon(n,\mu). \tag{3.35}$$

Combining (3.26), (3.27), (3.32), (3.33), and (3.35), we get

$$\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\bigg(\varphi'(T_k(u)-w_\mu^i)-\frac{L_1(k)}{\alpha}\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\bigg)\,dx\,dt$$

$$\le\varepsilon(n,\mu,i,m)$$

and, therefore, in view of (3.23), we obtain

$$\int_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\,dx\,dt\ \le\ \varepsilon(n).$$

Hence, passing to the lim-sup with respect to n, we find

$$\limsup_{n\to\infty}\int_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\ =\ 0.$$

This implies that

$$T_k(u_n)\to T_k(u)\quad\text{strongly in}\ \ L^p\big(0,T;W_0^{1,p}(\Omega)\big)\quad\text{for all}\quad k. \tag{3.36}$$

Note that, for every $\sigma>0$,

$$\begin{aligned}\text{meas}\Big\{(x,t)\in Q_T:\ |\nabla u_n-\nabla u|>\sigma\Big\}\\ \le \text{meas}\Big\{(x,t)\in Q_T:\ |\nabla u_n|>k\Big\}+\text{meas}\Big\{(x,t)\in Q_T:\ |u|>k\Big\}\\ +\text{meas}\Big\{(x,t)\in Q_T:\ \big|\nabla T_k(u_n)-\nabla T_k(u)\big|>\sigma\Big\}.\end{aligned}$$

Thus, in view of (3.36), we see that ∇u_n converges to ∇u in measure and, hence, always reasoning for a subsequence, we get

$$\nabla u_n\to\nabla u\quad\text{a.e. in}\quad Q_T.$$

This yields

$$a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\rightharpoonup a\big(x,t,T_k(u),\nabla T_k(u)\big)\quad\text{weakly in}\ \ \big(L^{p'}(Q_T)\big)^N. \tag{3.37}$$

Step 3: Equiintegrability of $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$. We now prove that $H_n(x,t,\nabla u_n)$ converges to $H(x,t,\nabla u)$ and $g_n(x,t,u_n,\nabla u_n)$ converges to $g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ by using the Vitali theorem. Since $H_n(x,t,\nabla u_n)\to H(x,t,\nabla u)$ a.e. on Q_T and $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ a.e. on Q_T, by virtue of (2.5) and (2.8), it suffices to prove that $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$ are uniformly equiintegrable in Q_T. We now prove that $H(x,\nabla u_n)$ is uniformly equiintegrable. To do this, we use the Hölder inequality and (3.3). As a result, for any measurable subset $E\subset Q_T$, we get

$$\begin{aligned}\int_E\big|H(x,\nabla u_n)\big|\,dx\,dt &\le\left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac1p}\left(\int_{Q_T}|\nabla u_n|^p\,dx\,dt\right)^{\frac1{p'}}\\ &\le c_1\left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac1p},\end{aligned}$$

which is small uniformly in n when the measure of E is small.

We now prove the uniform equiintegrability of $g_n(x,t,u_n,\nabla u_n)$. For any measurable subset $E \subset Q_T$ and $m \geq 0$, we obtain

$$\begin{aligned}\int_E \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt &= \int_{E\cap\{|u_n|\leq m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \\ &\quad + \int_{E\cap\{|u_n|>m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \\ &\leq L_1(m) \int_{E\cap\{|u_n|\leq m\}} \big[L_2(x,t) + |\nabla u_n|^p\big]\,dx\,dt \\ &\quad + \int_{E\cap\{|u_n|>m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \\ &= K_1 + K_2. \end{aligned} \tag{3.38}$$

For fixed $m,$ we get

$$K_1 \leq L_1(m) \int_E \Big[L_2(x,t) + \big|\nabla T_m(u_n)\big|^p\Big]\,dx\,dt,$$

which is thus small uniformly in n for fixed m when the measure of E is small $\big($recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)\big)$. We now discuss the behavior of the second integral on the right-hand side of (3.38). Assume that ψ_m is a function such that

$$\psi_m(s) = \begin{cases} 0 & \text{for} \quad |s| \leq m-1, \\ \operatorname{sign}(s) & \text{for} \quad |s| \geq m, \end{cases}$$

$$\psi_m'(s) = 1 \quad \text{for} \quad m-1 < |s| < m.$$

For $m > 1,$ we choose $\psi_m(u_n)$ as the test function in (3.2) and obtain

$$\begin{aligned}\bigg[\int_\Omega B_m^n(x,u_n)dx\bigg]_0^T &+ \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \psi_m'(u_n)\,dx\,dt \\ &+ \int_{Q_T} g_n(x,t,u_n,\nabla u_n)\psi_m(u_n)\,dx\,dt \\ &+ \int_{Q_T} H_n(x,t,\nabla u_n)\psi_m(u_n)\,dx\,dt \\ = &\int_{Q_T} f_n \psi_m(u_n)\,dx\,dt,\end{aligned}$$

where

$$B_m^n(x,r)=\int\limits_0^r \frac{\partial b_n(x,s)}{\partial s}\psi_m(s)\,ds.$$

Since $B_m^n(x,r)\geq 0,$ in view of (2.4), the Hölder inequality

$$\int\limits_{\{m-1\leq|u_n|\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \;\leq\; \int\limits_E |H_n(x,t,\nabla u_n)|\,dx\,dt \;+\; \int\limits_{\{m-1\leq|u_n|\}} |f|\,dx\,dt,$$

and (3.3), this yields

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} \int\limits_{\{|u_n|>m-1\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt=0.$$

Thus, we have proved that the second term on the right-hand side of (3.38) is also small uniformly in n and in E if m is sufficiently large. This means that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equiintegrable in $Q_T,$ as required. Thus, we conclude that

$$\begin{aligned} H_n(x,t,\nabla u_n)\to H(x,t,\nabla u) \quad &\text{strongly in} \quad L^1(Q_T),\\ g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) \quad &\text{strongly in} \quad L^1(Q_T). \end{aligned} \tag{3.39}$$

Step 4: We now prove that u satisfies (2.10).

Lemma 4. *The limit u of the approximate solution u_n of system (3.2) satisfies the relation*

$$\lim_{m\to+\infty} \int\limits_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt=0.$$

Proof. Note that, for any fixed $m\geq 0,$ we can write

$$\begin{aligned} &\int\limits_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &\qquad=\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_{m+1}(u_n)-\nabla T_m(u_n)\big)\,dx\,dt\\ &\qquad=\int\limits_{Q_T} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla T_{m+1}(u_n)\,dx\,dt\\ &\qquad\quad-\int\limits_{Q_T} a\big(x,t,T_m(u_n),\nabla T_m(u_n)\big)\nabla T_m(u_n)\,dx\,dt. \end{aligned}$$

According to (3.37) and (3.36), we can pass to the limit as $n \to +\infty$ for fixed $m \geq 0$ and obtain

$$\lim_{n\to+\infty} \int\limits_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt$$

$$= \int\limits_{Q_T} a\big(x,t,T_{m+1}(u),\nabla T_{m+1}(u)\big)\nabla T_{m+1}(u)\,dx\,dt$$

$$- \int\limits_{Q_T} a\big(x,t,T_m(u),\nabla T_m(u)\big)\nabla T_m(u_n)\,dx\,dt$$

$$= \int\limits_{\{m\leq|u_n|\leq m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt. \tag{3.40}$$

Passing to the limit as $m \to +\infty$ in (3.40) and using estimate (3.20), we show that u satisfies (2.10). The proof is completed.

Step 5: We now prove that u satisfies (2.11) and (2.12).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that the support of S' is a subset of $[-M,M]$. As a result of the pointwise multiplication of the approximate equation (3.2) by $S'(u_n)$, we get

$$\frac{\partial B_S^n(x,u_n)}{\partial t} - \operatorname{div}\Big(S'(u_n)a(x,t,u_n,\nabla u_n)\Big)$$

$$+ S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$$

$$+ \; S'(u_n)\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big) = fS'(u_n) \quad \text{in} \quad \mathcal{D}'(Q_T), \tag{3.41}$$

where

$$B_S^n(x,z) = \int\limits_0^z \frac{\partial b_n(x,r)}{\partial r} S'(r)\,dr.$$

We now pass to the limit in (3.41) as n tends to $+\infty$:

We find the limit of $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$. Since S is bounded and continuous, the fact that $u_n \to u$ a.e. in Q_T implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in Q_T and $L^\infty(Q_T)$-weakly*. Thus, $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\dfrac{\partial B_S(x,u)}{\partial t}$ in $\mathcal{D}'(Q_T)$ as n tends to $+\infty$.

We find the limit of $-\operatorname{div}\big(S'(u_n)a(x,t,u_n,\nabla u_n)\big)$. Since $\operatorname{supp}(S') \subset [-M,M]$, for $n \geq M$, we obtain

$$S'(u_n)a_n(x,t,u_n,\nabla u_n) = S'(u_n)a\big(x,t,T_M(u_n),\nabla T_M(u_n)\big) \quad \text{a.e. in} \quad Q_T.$$

The pointwise convergence of u_n to u, relation (3.37), and the bounded character of S' imply that, as n tends to $+\infty$, the quantity $S'(u_n)a_n\big(x,t,u_n,\nabla u_n\big)$ converges to $S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$ in $\big(L^{p'}(Q_T)\big)^N$. Note that, in equation (2.11), $S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$ is denoted by $S'(u)a(x,t,u,\nabla u)$.

We find the limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$. Consider the "energy" term

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a\big(x,t,T_M(u_n),\nabla T_M(u_n)\big)\nabla T_M(u_n) \quad \text{a.e. in} \quad Q_T.$$

The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (3.37) as n tends to $+\infty$ and the bounded character of S'' enable us to conclude that $S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n$ converges to $S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u)$ weakly in $L^1(Q_T)$. Recall that

$$S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u) = S''(u)a(x,t,u,\nabla u)\nabla u \quad \text{a.e. in} \quad Q_T.$$

We find the limit of $S'(u_n)\big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\big)$. From $\operatorname{supp}(S') \subset [-M,M]$, in view of (3.39), we conclude that $S'(u_n)g_n(x,t,u_n,\nabla u_n)$ converges to $S'(u)g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ and $S'(u_n)H_n(x,t,\nabla u_n)$ converges to $S'(u)H(x,t,\nabla u)$ strongly in $L^1(Q_T)$ as n tends to $+\infty$.

We find the limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q_T, we see that $S'(u_n)f_n$ converges to $S'(u)f$ strongly in $L^1(Q_T)$ as n tends to $+\infty$.

As a consequence of this convergence result, we now pass to the limit as n tends to $+\infty$ in equation (3.41) and conclude that u satisfies (2.11).

It remains to show that $B_S(x,u)$ satisfies the initial condition (2.12). To this end, we first note that S is bounded. Thus, in view of (2.14), (3.15), we conclude that $B_S^n(x,u_n)$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$. Second, in view of (3.41) and the analysis of behavior of the terms of this equation presented above, we can show that

$$\frac{\partial B_S^n(x,u_n)}{\partial t}$$

is bounded in $L^1(Q_T) + L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$. As a consequence (see [21]), we conclude that

$$B_S^n(x,u_n)(t=0) = B_S^n(x,u_{0n})$$

converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, in view of the smoothness of S and (3.1), we conclude that

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad \text{in} \quad \Omega.$$

Finally, we note that Steps 1–5 complete the proof of Theorem 1.

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