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RESONANT EQUATIONS WITH CLASSICAL ORTHOGONAL POLYNOMIALS. II

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We study some resonant equations related to the classical orthogonal polynomials on infinite intervals, i.e., the Hermite and the Laguerre orthogonal polynomials, and propose an algorithm for finding their particular and general solutions in the closed form. This algorithm is especially suitable for the computer-algebra tools, such as Maple. The resonant equations form an essential part of various applications, e.g., of the efficient functional-discrete method for the solution of operator equations and eigenvalue problems. These equations also appear in the context of supersymmetric Casimir operators for the di-spin algebra and in the solution of square operator equations, such as $A^2u = f$ (e.g., of the biharmonic equation).

1. Introduction

The present paper is the second part of our paper published in the previous issue of the journal. In this part, we study the resonant equations with differential operators specifying classical orthogonal polynomials on infinite intervals, namely, the Hermite and the Laguerre orthogonal polynomials. We use Algorithm 3.1 from Part I (see [4]) to obtain particular solutions of the corresponding resonant equations of the first and of the second kind and obtain explicit formulas for the general solutions of the corresponding inhomogeneous resonant differential equations.

2. Resonant Equation of the Hermite Type

2.1. Hermite Resonant Equation of the First Kind. In this section, we consider the following Hermite-type resonant gather:

$$\exp(x^2)\frac{d}{dx}\left[\exp\left(-x^2\right)\frac{du(x)}{dx}\right] + 2nu(x) = H_n(x), \tag{2.1}$$

where $H_n(x)$ is a Hermite polynomial satisfying the homogeneous differential equation. The Hermite polynomial $H_n(x)$ can be represented via the hypergeometric function

$$H_\nu(x) = \frac{2^\nu\sqrt{\pi}\left(\dfrac{1-\nu}{2}\right)_{\left[\frac{n}{2}\right]+1}}{\Gamma\left(\left[\dfrac{n}{2}\right]+1+\dfrac{1-\nu}{2}\right)}\,{}_1F_1\left[-\frac{\nu}{2};\frac{1}{2};x^2\right]$$

$$-\frac{2^{\nu+1}\sqrt{\pi}\left(\dfrac{-\nu}{2}\right)_{\left[\frac{n}{2}\right]+1}}{\Gamma\left(\left[\dfrac{n}{2}\right]+1-\dfrac{\nu}{2}\right)}\,x\;{}_1F_1\left[\frac{1-\nu}{2};\frac{3}{2};x^2\right] \tag{2.2}$$

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with $\nu = n$ (see [6, p. 147]). The general solution of the homogeneous equation (2.1) is given by the formula

$$u(x) = c_1 H_n(x) + c_2 h_n(x),$$

where

$$h_n(x) = -\int_{\infty}^{\infty} \frac{\exp\left(-t^2\right) H_n(t)}{t-z}\, dt, \quad n = 0, 1, 2, \ldots, \quad x \in \mathbb{C} \setminus (-\infty, \infty),$$

is an Hermite functions of the second kind satisfying the recurrence equation for Hermite polynomials. This function can also be expressed via the confluent hypergeometric function in the following way [5]:

$$h_{2n}(x) = (-1)^n 2^{n+1} (2n)!! x {}_1F_1\left(-\frac{2n-1}{2}; \frac{3}{2}; x^2\right)$$

$$= \left[p_{2n}(x)\exp(x^2) + \sqrt{\pi} H_{2n}(x) \text{erfi}(x)\right], \quad n = 1, 2, \ldots,$$

$$h_{2n+1}(x) = (-1)^{n+1} (2n)!! 2^{n+1}_1 F_1\left(-\frac{2n+1}{2}; \frac{1}{2}; x^2\right)$$

$$= \left[p_{2n+1}(x)\exp(x^2) + \sqrt{\pi} H_{2n+1}(x) \text{erfi}(x)\right], \quad n = 0, 1, \ldots.$$

These formulas were obtained by using Maple to solve the Hermite differential equation. They satisfy the difference equation

$$p_{n+1}(x) = 2x p_n(x) - 2n p_{n-1}(x), \quad n = 1, 2, \ldots, \qquad p_0(x) = 0, \qquad p_1(x) = -2. \tag{2.3}$$

The formulas for the odd and the even indices can be combined into the following formula:

$$h_n(x) = (-1)^{\left[\frac{n+1}{2}\right]} 2^{\left[\frac{n}{2}\right]+1} \left(2\left[\frac{n}{2}\right]\right)!!\ x^{2\left\{\frac{n+1}{2}\right\}} \times {}_1F_1\left(-\frac{n}{2} + \left\{\frac{n+1}{2}\right\}; \frac{1}{2} + 2\left\{\frac{n+1}{2}\right\}; x^2\right), \quad n = 0, 1, \ldots, \tag{2.4}$$

where $[x]$ and $\{x\}$ denote the integral and fractional parts of a real number x.

The last expression can be transformed into

$$h_n(x) = H_n(x)\sqrt{\pi}\, \text{erfi}(x) + p_n(x)\exp(x^2),$$

where the polynomials $p_n(x)$ satisfy the recurrence equation (2.3).

We use Theorem 3.1 of [4] to find a particular solution of the inhomogeneous equation. First, we consider the case $n = 0$, i.e., differentiate representation (2.2) with respect to ν, i.e.,

$$\tilde{u}_0(x) = -\frac{1}{2} \frac{d}{d\nu}\ {}_1F_1\left(\frac{-\nu}{2}; \frac{1}{2}; x^2\right)\bigg|_{\nu=0}.$$

Then we set $\nu = 0$ and omit some terms satisfying the homogeneous equation

$$u_0(x) = \frac{1}{4}\sum_{p=1}^{\infty} \frac{x^{2p}}{p\left(\frac{1}{2}\right)_p} = \frac{\sqrt{\pi}}{2}\int\limits_0^x \exp\left(t^2\right)\left[1 - \mathrm{erfc}(t)\right] dt. \tag{2.5}$$

Similarly, in order to get $u_1(x)$ we set $n = 1$ in (2.2), differentiate with respect to ν, substitute $\nu = 1$, and omit some terms satisfying the homogeneous differential equation. Thus, by using Maple, we obtain

$$\begin{aligned} u_1(x) &= -x\,\frac{d}{d\nu}\,{}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; x^2\right)\Big|_{\nu=1} \\ &= \frac{1}{2}\,x\sum_{p=1}^{\infty} \frac{x^{2p}}{p\left(\frac{3}{2}\right)_p} = \frac{x}{2}\int\limits_0^x \frac{1}{t^2}\left\{\sqrt{\pi}\exp\left(t^2\right)[1 - \mathrm{erfc}(t)] - 2t\right\} dt, \end{aligned} \tag{2.6}$$

where $\mathrm{erfc}(x)$ is the imaginary error function [2].

It is easy to see that this way of getting particular solutions is very cumbersome. In what follows, we show that Algorithm 3.1 from Part I [4] provides a more comfortable way.

Actually, we differentiate the recurrence relation for the Hermite polynomials

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

with respect to n and then apply Theorem 3.1 from [4]. As a result, we get the following recursion:

$$u_{n+1}(x) = 2xu_n(x) - 2nu_{n-1}(x) + H_{n-1}(x), \quad n = 1, 2, \dots. \tag{2.7}$$

By using (2.5), (2.6), we obtain the following expressions as particular solutions:

$$\chi_0(x) = E(x) = \frac{\sqrt{\pi}}{2}\int\limits_0^x \mathrm{erf}(t)\exp\left(t^2\right)\, dt,$$

$$\chi_1(x) = \sum_{p=0}^{1}(-1)^p C_1^p H_{1-p}(x)\frac{d^p}{dx^p}\,E(x) + x.$$

Further, we use the following ansatzes:

$$\begin{aligned} u_0(x) &= \chi_0(x) + c_0, \\ u_1(x) &= \chi_1(x) + c_1 x \end{aligned} \tag{2.8}$$

with undetermined coefficients c_0 and c_1 for the initial values of Algorithm 3.1 from Part I (see [4]). Substituting these expressions in the recurrence equation (2.7) with $n = 1$ and choosing the coefficients guaranteeing that

$u_2(x)$ satisfies the resonant equation, we conclude that c_0 can be arbitrary, whereas c_1 must satisfy the equation

$$4 + 4c_1 = 0,$$

i.e., $c_1 = -1$. Note that if we choose $c_0 = 0$, then we arrive at a representation

$$u_n(x) = \sum_{p=0}^{n} (-1)^p C_n^p H_{n-p}(x) \frac{d^p}{dx^p} E(x),$$
$$E(x) = \frac{\sqrt{\pi}}{2} \int_0^x \operatorname{erf}(t) \exp\left(t^2\right) dt. \tag{2.9}$$

For $n = 0, 1, 2$ this representation was obtained in [2].

Thus, we have constructed $u_k(x)$, $k = 0, 1, 2$, which are particular solutions of the Hermite resonant equation of the first kind. The next theorem shows that this is true for all $n = 0, 1, 2, \ldots$.

Theorem 2.1. *The functions* $u_k(x)$, $k = 3, 4, \ldots$, *obtained by the recursion (2.7) with the initial conditions* $u_k(x)$, $k = 0, 1$, *given by (2.8) with* $c_0 = 0$ *and* $c_1 = -1$, *satisfy the resonant Hermite differential equation of the first kind.*

Proof. We prove this assertion by induction.

Assume that all $u_p(x)$, $p = 0, 1, \ldots, n$, satisfy the resonant Hermite differential equation of the first kind (2.1). Applying the Hermite differential operator

$$\mathcal{A}_{n+1} = \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2(n+1),$$

to the recurrence equation (2.7), by the assumption of induction, we obtain

$$\mathcal{A}_{n+1} u_{n+1}(x) = H_{n+1}(x) + \left[4 \frac{du_n(x)}{dx} - 8n\, u_{n-1}(x) + 4\, H_{n-1}(x) \right]. \tag{2.10}$$

Further, we use the classical relation (see, e.g., [3], Sec. 10.13)

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x).$$

Differentiating this equality with respect to n and using Theorem 3.1 in [4], we get

$$-2 \frac{du_n(x)}{dx} = -4nu_{n-1}(x) + 2H_{n-1}(x),$$

which shows that the square bracket in (2.10) is equal to zero.

Theorem 2.1 is proved.

Remark 2.1. Despite their beauty, relations (2.9) are uncomfortable for practical calculations because these calculations require differentiation. From this point of view, our recurrence algorithm is more comfortable and can be easily realized by using a computer-algebra tool, such as Maple.

Further, the general solution of the resonant equation (2.1) is given by the formula

$$u(x) = c_1 H_n(x) + c_2 h_n(x) + u_n(x), \tag{2.11}$$

where c_1 and c_2 are arbitrary constants.

2.2. The Hermite Resonant Equation of the Second Kind. Consider a resonant equation

$$\exp(x^2)\frac{d}{dx}\left[\exp\left(-x^2\right)\frac{du_n(x)}{dx}\right] + 2nu_n(x) = h_n(x), \tag{2.12}$$

where $h_n(x)$ are the Hermite functions of the second kind defined by (2.4).

In view of Theorem 3.1 in [4], we get a particular solution of the Hermite resonant equation (2.12) of the second kind in the form

$$u_n(x) = (-1)^{\left[\frac{n+1}{2}\right]}2^{\left[\frac{n}{2}\right]+1}\left(2\left[\frac{n}{2}\right]\right)!!\left[\frac{\left(-\frac{n-1}{2}\right)_n}{\left(\frac{1}{2}\right)^2_{\left[\frac{n}{2}\right]}}\,x\,\frac{\partial}{\partial\nu}\,{}_1F_1\left(-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2};x^2\right)\right.$$
$$\left.-\frac{\left(-\frac{n}{2}+1\right)_{n-1}}{\left(\frac{1}{2}\right)^2_{\left[\frac{n}{2}\right]}}\,\frac{\partial}{\partial\nu}\,{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};x^2\right)\right]_{\nu=n}, \tag{2.13}$$

where $(a)_{-1} = 0$. The general solution of (2.12) has the form (2.11).

To obtain a recursive algorithm for particular solutions, we differentiate the recurrence equation for Hermite functions of the second kind with respect to n and obtain

$$u_{n+1}(x) = 2xu_n(x) - 2nu_{n-1}(x) + h_{n-1}(x), \quad n = 1, 2, \ldots. \tag{2.14}$$

From (2.13), we extract the following particular solutions for $n = 0, 1$:

$$\chi_0(x) = \sqrt{\pi}\int_0^x\int_0^t \operatorname{erfi}(\xi)\exp\left(-\xi^2\right)d\xi\,\exp\left(t^2\right)\,dt,$$

$$\chi_1(x) = \left(-2\int_0^x\left(\sqrt{\pi}\operatorname{erfi}(\xi)\xi - \exp\left(\xi^2\right)\right)^2\exp\left(-\xi^2\right)\,d\xi\,x\right.$$
$$\left.+2\int_0^x\left(\sqrt{\pi}\operatorname{erfi}(\xi)\xi\exp\left(-\xi^2\right)-1\right)\xi\,d\xi\right)\left(\sqrt{\pi}\operatorname{erfi}(x)x - \exp(x^2)\right). \tag{2.15}$$

These solutions are modified to the initial values of recursion (2.14) in order to guarantee that $u_2(x)$ satisfies the differential equation. To this end, we use the ansatzes

$$u_0(x) = \chi_0(x) + c_0 h_0(x), \qquad u_1(x) = \chi_1(x) + c_1 h_1(x) \tag{2.16}$$

with undetermined coefficients c_1 and c_2. Substituting these relations in (2.14) with $n = 1$ we demand that $u_2(x)$ must satisfy the resonant differential equation and obtain the following formulas for arbitrary constants and the particular solution $u_2(x)$:

$$c_0 = \frac{3}{8}, \qquad c_1 = \frac{1}{4},$$

$$\begin{aligned} u_2(x) &= 2xu_1(x) - 2u_0(x) + h_0(x) \\ &= -2\chi_0(x) + 2x\chi_1(x) + \left(x^2 + \frac{1}{4}\right)\sqrt{\pi}\,\mathrm{erfi}(x) - x\exp(x^2). \end{aligned}$$

Thus, we get a particular solution of the form

$$\begin{aligned} u_n(x) &= -2H_{n-2}(x)\chi_0(x) + p_n(x)\chi_1(x) \\ &\quad + q_n(x)\sqrt{\pi}\mathrm{erfi}(x) + v_n(x)\exp(x^2), \end{aligned} \tag{2.17}$$

where the polynomials $p_n(x)$ satisfy the recurrence equation for the Hermite polynomials with the initial conditions $p_0(x) = 0$ and $p_1(x) = 1$, the polynomials $q_n(x)$ solve the initial-value problem

$$q_{n+1}(x) = 2xq_n(x) - 2nq_{n-1}(x) + H_{n-1}(x), \quad n = 1, 2, \ldots,$$

$$q_0(x) = \frac{3}{8}, \qquad q_1(x) = \frac{x}{2},$$

and the polynomials $v_n(x)$ solve the discrete problem

$$v_{n+1}(x) = 2xv_n(x) - 2nv_{n-1}(x) + p_{n-1}(x), \quad n = 1, 2, \ldots,$$

$$v_0(x) = 0, \qquad v_1(x) = -\frac{1}{2}.$$

The particular solutions $u_n(x)$ of the Hermite resonant equation of the second kind satisfy the resonant differential equation for $n = 0, 1, 2$ by construction. The next theorem shows that this is the case for all $n = 0, 1, 2, \ldots$.

Theorem 2.2. *The functions $u_k(x)$ obtained by using recursion (2.17) with the initial conditions $u_k(x)$, $k = 0, 1,$ given by (2.16) satisfy the resonant Hermite differential equation of the second kind for all $k = 3, 4, \ldots$.*

The proof is completely analogous to the proof of Theorem 2.1 if we take into account the fact that the Hermite functions of the second kind (that are not polynomials!) satisfy the same recurrence equation as the Hermite polynomials and the same differentiation formula.

3. Resonant Equation of the Laguerre Type

3.1. The Laguerre Resonant Equation of the First Kind. In this section, we consider the following equation of the Laguerre type:

$$x\,\frac{d^2u(x)}{dx^2}+(1+\alpha-x)\,\frac{du(x)}{dx}+n\,u(x)=L_n^\alpha(x), \tag{3.1}$$

where

$$L_n^\alpha(x)=\frac{(\alpha+1)_n}{n!}\,\Phi(-n,\alpha+1,x)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$$

is the Laguerre polynomial satisfying the homogeneous differential equation corresponding to (3.1). This polynomial can be represented in terms of the confluent hypergeometric function (i.e., in terms of the solution of a confluent hypergeometric equation, which is a degenerate form of the hypergeometric differential equation corresponding to the case where two out of three regular singularities merge into an irregular singularity) [1, p. 189] [relation (14)]. Since the Laguerre polynomial resolves the homogeneous equation, the inhomogeneous equation (3.1) is resonant.

The second linear independent solution of the homogeneous differential equation is the Laguerre function of the second kind [7, p. 16, 20]. If we solve the corresponding differential equation by using Maple, then we get the following representation for the Laguerre function of the second kind with noninteger α:

$$l_n^\alpha(x)=x^{-\alpha}\,{}_1F_1(-n-\alpha;-\alpha+1;x)=\Gamma(1-\alpha,-x)L_n^\alpha(x)-(-x)^{-\alpha}p_n^\alpha(x)\exp(x),$$

$$p_{n+1}^\alpha(x)=\frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x)-(n+\alpha)p_{n-1}^\alpha(x)\right],\qquad n=1,2,\ldots, \tag{3.2}$$

$$p_0^\alpha(x)=1,\qquad p_1^\alpha(x)=1-x,$$

where

$$\Gamma(a,z)=\int_z^\infty e^{-t}t^{a-1}dt$$

is the incomplete Gamma function. For nonnegative integer $\alpha,$ we find

$$l_n^\alpha(x)=\mathrm{Ei}_1(-x)L_n^\alpha(x)-(-x)^{-\alpha}p_n^\alpha(x)\exp(x),$$

$$p_1^\alpha(x)=-x^\alpha+\sum_{p=1}^{\alpha}(p-1)!(\alpha-p+1)_+x^{\alpha-p},\qquad (y)_+=\begin{cases}y, & y>0,\\ 0, & y\le 0,\end{cases}$$

$$p_0^\alpha(x)=x^{\alpha-1}+x^\alpha\left[U(2,2,-x)+(-1)^\alpha\alpha!U(1+\alpha,1+\alpha,-x)\right]=\sum_{p=1}^{\alpha}x^{\alpha-p}(p-1)!,$$

where

$$\mathrm{Ei}_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt, \qquad |\mathrm{Arg}(z)| < \pi,$$

is the exponential integral and $U(a,b,z)$ is Kummer's function of the second kind. This function is a solution of Kummer's differential equation

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0.$$

The other linear independent solution is Kummer's function of the first kind M defined, e.g., by a generalized hypergeometric series:

$$M(a,b,z) = \sum_{n=0}^\infty \frac{a_{(n)} z^n}{b_{(n)} n!} = {}_1F_1(a;b;z),$$

where $a_{(0)} = 1,\ a_{(n)} = a(a+1)(a+2)\dots(a+n-1)$ is the Pochhammer symbol. Note that Kummer's function of the second kind can be represented in the form

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}\, M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b} M(a+1-b, 2-b, z).$$

Let

$$f^\alpha(z,x) = \sum_{n=0}^\infty z^n p_n^\alpha(x)$$

be the generating function for the polynomials $p_n^\alpha(x)$ with both noninteger and nonnegative integer $\alpha.$ Thus, multiplying the second equation (3.2) by z^n and finding the sum over $n,$ we arrive at the Cauchy problem

$$(1-z)^2 \frac{\partial}{\partial z} f^\alpha(z,x) = [\alpha + 1 - x - z(1+a)]\, f^\alpha(z,x)$$

$$+ \left(1 + 2z^2\right) p_1^\alpha(x) - (\alpha+1-x) p_0^\alpha(x), \quad f^\alpha(0,x) = p_0^\alpha(x),$$

with the following solution:

$$f^\alpha(z,x) = \sum_{n=0}^\infty z^n p_n^\alpha(x) = (1-z)^{-\alpha-1} \exp\left(\frac{x}{z-1}\right)$$

$$\times \left\{ -\int_0^z \left(-\left(2t^2+1\right) p_1^\alpha(x) + (\alpha+1-x) p_0^\alpha(x)\right)(t-1)^{\alpha-1} \exp\left(-\frac{x}{t-1}\right) dt \right.$$

$$\left. + (-1)^{-\alpha-1} p_0^\alpha(x) \exp(x) \right\}.$$

In particular, for $\alpha = 0,$ we find

$$f^0(z,x) = (1-z)^{-1}(3-x)\exp\left(\frac{xz}{z-1}\right) + (1-z)^{-1}(x-1)(x-3)\exp\left(\frac{x}{z-1}\right) \times \left\{-\mathrm{Ei}_1(-x) + \mathrm{Ei}_1\left(-\frac{x}{z-1}\right)\right\} + z - x + 3,$$

for $\alpha = 1,$ we get

$$f^1(z,x) = (1-z)^{-2}\left(x^2-5x+2\right)(1-x)\,\frac{1}{3}\,\exp\left(\frac{xz}{z-1}\right) + (1-z)^{-2}\left(x^3-7x^2+12x-3\right)\frac{x}{3}\,\exp\left(\frac{x}{z-1}\right)\left\{-\mathrm{Ei}_1(-x)+\mathrm{Ei}_1\left(-\frac{x}{z-1}\right)\right\} - \frac{1}{3(z-1)}\left[x^3-(z+6)x^2+\left(2z^2+3z+7\right)x-2z^2-2z+1\right].$$

Moreover, for $\alpha = 1/2,$ we can write

$$f^{1/2}(z,x) = -(z-1)^{-3/2}\exp\left(\frac{xz}{z-1}\right) \times \left\{\frac{\exp(-x)}{2}\int\limits_0^z \frac{\left[1+4(x-1)t^2\right]}{\sqrt{t-1}}\,\exp\left(\frac{x}{t-1}\right)dt + 1\right\}.$$

The general solution of the homogeneous Laguerre differential equation is given by the formula

$$u(x) = c_1 L_n^{\alpha}(x) + c_2 l_n^{\alpha}(x)$$

with arbitrary constants c_1 and c_2. By Theorem 3.1 in [4], a particular solution of the Laguerre resonant differential equation of the first kind

$$x\,\frac{d^2u_\nu(x)}{dx^2} + (1-x)\,\frac{du_\nu(x)}{dx} + \nu u_\nu(x) = L_\nu(x) \equiv \Phi(-\nu,1,x)$$

is given by

$$u_\nu(x) = \frac{d}{d\nu}\,\Phi(-\nu,1,x) = -\sum_{k=1}^{\infty}\frac{x^k}{(k!)^2}\,(-\nu)_k\sum_{i=0}^{p-1}\frac{1}{-\nu+i},$$

where $\Phi(a, c; x)$ is the confluent hypergeometric function [1] (Ch. 6). In this relation, we replace $\nu \in \mathbb{R}$ with $n \in \mathbb{N}$ and obtain the resonant Laguerre equation and the corresponding particular solution:

$$u_n(x) = \sum_{k=1}^{n} \frac{x^k}{(k!)^2} \frac{d(-\nu)_k}{d\nu}\Big|_{\nu=n} - (-1)^n n! \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \frac{1}{\prod_{i=0}^{n}(k-i)} = u_{n,1}(x) + u_{n,2}(x). \qquad (3.3)$$

By using the relation

$$\frac{1}{\prod_{i=0}^{n}(k-i)} = \sum_{i=0}^{n} \frac{a_i^{(n)}}{k-n+i}, \quad a_i^{(n)} = \frac{(-1)^i}{i!(n-i)!}, \qquad (3.4)$$

we transform the sums $u_{n,1}(x)$ and $u_{n,2}(x)$ so that it becomes possible to compute them in the closed form, e.g., by the Maple software. As a result, we get

$$u_0(x) = -\left[\mathrm{Ei}_1(-x) + \ln(-x) + \gamma\right],$$

$$u_1(x) = -L_1(x)u_0(x) - \exp(x) + 1 + x,$$

$$u_2(x) = -L_2(x)u_0(x) + \frac{1}{2}(x-3)\exp(x) - \frac{3}{4}x^2 + x + \frac{3}{2},$$

$$u_3(x) = -L_3(x)u_0(x) + \frac{1}{6}\left(-x^2 + 8x - 11\right)\exp(x) + \frac{11}{36}x^3 - \frac{7}{4}x^2 + \frac{1}{2}x + \frac{11}{6},$$

$$u_4(x) = -L_4(x)u_0(x) + \frac{1}{24}\left(x^3 - 15x^2 + 58x - 50\right)\exp(x)$$

$$-\frac{25}{288}x^4 + \frac{19}{18}x^3 - \frac{7}{4}x^2 - \frac{1}{3}x + \frac{25}{12},$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant. With an aim to get the sum $u_{n,2}(x)$ in the closed form, we note that

$$v_i^{(n)}(x) = \sum_{p=n+1}^{\infty} \frac{x^p}{p!(p-i)} = -\frac{x^i}{i!}w(x) - \frac{\exp(x)}{i!}\sum_{p=0}^{i-1} p! x^{i-1-p}$$

$$-\sum_{p=0}^{n-i-1} \frac{x^{n-p}}{(n-p)!(n-p-i)} + \frac{x^i}{i!}\sum_{p=0}^{i-1}\frac{1}{p+1} + \sum_{p=0}^{i-1}\frac{x^p}{p!(i-p)},$$

$$w(x) = \mathrm{Ei}_1(-x) + \ln(-x) + \gamma. \qquad (3.5)$$

Then it follows from (3.3)–(3.5) that

$$u_{n,2}(x) = -(-1)^n n! \sum_{i=0}^{n} a_i^{(n)} v_{n-i}^{(n)}(x) = \sum_{i=0}^{n} (-1)^{n+1-i} C_n^i v_{n-i}^{(n)}(x)$$

$$= L_n(x)w(x) - \exp(x)\sum_{p=0}^{n-1} x^p \sum_{i=0}^{n-p-1} \frac{(-1)^{n+i} n!(n-i-1-p)!}{i![(n-i)!]^2}$$

$$+ \sum_{p=0}^{n} x^p \sum_{i=0}^{n} \frac{(-1)^{n+1-i} n!}{i!(n-i)!}\, b_{p,i},$$

where

$$b_{p,i} = \begin{cases} \dfrac{1}{p!(i-p)}, & p \neq i, \\ \dfrac{1}{i!}\sum_{t=0}^{i-1} \dfrac{1}{t+1}, & p = i. \end{cases}$$

The technique presented above for $\alpha = 0$ is even more cumbersome for $\alpha \neq 0$. This is why, in what follows, we use our recursive algorithm for finding particular solutions in order to be able to get the general solution of the Laguerre resonant equation (3.1) in the form

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x),$$

with arbitrary constants c_1 and c_2.

Differentiating the recurrence equation for the Laguerre polynomials with respect to n and using Theorem 3.1 in [4], we get the following recurrence formula for the particular solutions:

$$u_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1}\, u_n^\alpha(x) - \frac{n+\alpha}{n+1}\, u_{n-1}^\alpha(x)$$

$$+ \frac{\alpha-1-x}{(n+1)^2}\, L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2}\, L_{n-1}^\alpha(x), \quad n = 1, 2, \ldots, \tag{3.6}$$

with the corresponding initial conditions. Thus, for $\alpha = 1$ we get

$$u_0^1(x) = \frac{1}{x} - \ln(x),$$

$$u_1^1(x) = (2-x)u_0^1(x) - x - \frac{1}{x},$$

and the following representation for the particular solution of the resonant equation:

$$u_n(x) = L_n^1(x)u_0(x) + q_n(x)\left(x + \frac{1}{x}\right) + v_n(x).$$

Here, the polynomial $q_n(x)$ satisfies the recurrence equation for the Laguerre polynomials with the initial conditions

$$v_0(x) = 0, \qquad v_1(x) = -1.$$

The polynomial $v_n(x)$ solves the difference problem

$$v_{n+1}^1(x) = \frac{2n+2-x}{n+1}\, v_n^1(x) - v_{n-1}^1(x) - \frac{x}{(n+1)^2}\, L_n^1(x), \quad n = 1, 2, \ldots,$$
$$v_0^1(x) = 0, \qquad v_1^1(x) = 0. \tag{3.7}$$

For any α, by Theorem 3.1 in [4], we obtain a particular solution

$$u_n^\alpha(x) = -\frac{d}{d\nu}\, L_\nu^\alpha(x)\bigg|_{\nu=n} = -\Phi(-n, \alpha+1; x)\frac{d}{d\nu}\, \frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)\Gamma(\nu+1)}\bigg|_{\nu=n}$$
$$- \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\, \frac{d}{d\nu}\, \Phi(-\nu, \alpha+1; x)\bigg|_{\nu=n},$$

whence we get the following particular solutions for $n = 0, 1$:

$$\chi_0(x) = \frac{x}{\alpha+1}\ {}_2F_2(1, 1; 2, 2+\alpha; x),$$

$$\chi_1(x) = x\ {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{x^2}{\alpha+2}\ {}_2F_2(1, 1; 2, 3+\alpha; x).$$

With an aim to obtain the solutions of the resonant differential equation from the recurrence formula, we use the following ansatzes:

$$u_0^\alpha(x) = \chi_0(x) + c_0,$$

$$u_1^\alpha(x) = \chi_1(x) + c_1 L_1^\alpha(x)$$

with undetermined coefficients c_0 and c_1. Substituting these expressions in (3.7) and demanding that the particular solution $u_1^\alpha(x)$ must satisfy the resonant differential equation, we find

$$c_0 = -\frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)}, \qquad c_1 = -\frac{\alpha}{2(\alpha+2)}.$$

In this case, the initial values of the recursive algorithm for finding the particular solutions take the form

$$u_0^\alpha(x) = \frac{x}{\alpha+1}\ {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)},$$
$$u_1^\alpha(x) = x\ {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{x^2}{\alpha+2}\ {}_2F_2(1, 1; 2, 3+\alpha; x) - \frac{\alpha}{2(\alpha+2)}\, L_1^\alpha(x). \tag{3.8}$$

The next assertion shows that the functions $u_n^\alpha(x)$ generated by recursion (3.6) with the initial values (3.8) satisfy the Laguerre resonant differential equation of the first kind for all $n = 0, 1, 2, \ldots$.

Theorem 3.1. *The functions $u_n^\alpha(x)$ generated by the recursive algorithm (3.6) with the initial values (3.8) are particular solutions of the Laguerre resonant differential equation of the first kind for all $n = 0, 1, 2, \ldots$.*

Proof. We prove this assertion by induction. First, we note that the functions $u_n^\alpha(x)$ for $n = 0, 1, 2$ are particular solution due to their construction. We assume that all functions $u_p^\alpha(x),\ p = 0, 1, \ldots, n,$ are particular solutions and prove that, in this case, $u_{n+1}^\alpha(x)$ is also a particular solution.

Actually, by applying the Laguerre differential operator

$$\mathcal{A}_{n+1}^\alpha = x\,\frac{d^2}{dx^2} + (\alpha + 1 - x)\,\frac{d}{dx} + n + 1$$

to both sides of (3.6) and using the induction hypothesis, we can write

$$\mathcal{A}_{n+1}^\alpha u_{n+1}^\alpha(x) = L_{n+1}^\alpha(x) + \frac{2}{n+1}\left[n u_n^\alpha(x) - x\,\frac{du_n^\alpha(x)}{dx} - (n+\alpha)u_{n-1}^\alpha(x)\right]$$
$$- \frac{2}{n+1}\left[L_n^\alpha(x) - L_{n-1}^\alpha(x)\right]. \tag{3.9}$$

Further, we use the relation (see, e.g., [8], Sec. 10.12)

$$x\,\frac{d\,L_n^\alpha(x)}{dx} = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x).$$

Differentiating this relation with respect to n and using Theorem 3.1 of [4], we conclude that both square brackets in (3.9) are equal to zero and, hence, the assertion is proved.

The general representation of the particular solutions has the form

$$u_n^\alpha(x) = p_n^\alpha(x)\ {}_2F_2(1,1;2,2+\alpha;x)$$
$$+\, q_n^\alpha(x)\ {}_2F_2(1,1;2,3+\alpha;x) + v_n^\alpha(x), \quad n = 2, 3, \ldots,$$

where the polynomials $p_n^\alpha(x),\ q_n^\alpha(x)$ satisfy the classical Laguerre recurrence equation with the initial conditions

$$p_0^\alpha(x) = \frac{x}{\alpha+1}, \qquad p_1^\alpha(x) = x,$$

$$q_0^\alpha(x) = 0, \qquad q_1^\alpha(x) = -\frac{x^2}{\alpha+2},$$

respectively. The polynomials $v_n^\alpha(x)$ satisfy the inhomogeneous recurrence equation

$$v_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1}\,v_n^\alpha(x) - \frac{n+\alpha}{n+1}\,v_{n-1}^\alpha(x)$$
$$+ \frac{\alpha-1-x}{(n+1)^2}\,L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2}\,L_{n-1}^\alpha(x), \quad n = 1, 2, \ldots,$$

with the initial conditions

$$v_0^\alpha(x) = -\frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)},$$

$$v_1^\alpha(x) = -\frac{\alpha(\alpha+1-x)}{2(\alpha+2)}.$$

3.2. The Laguerre Resonant Equation of the First Kind (Revisited). In this section, we again consider the resonant Laguerre differential equation of the first type (3.1) and show that its particular solutions can be represented in terms only of elementary functions.

We know that one of linear independent solutions of the homogeneous differential equation is the Laguerre function of the second kind [7, p. 16, 20]. Solving the corresponding differential equation by using Maple, we arrive at the following representation of the Laguerre function of the second kind for noninteger α:

$$l_n^\alpha(x) = x^{-\alpha}\ {}_1F_1(-n-\alpha, -\alpha+1; x)$$

$$= \Gamma(1-\alpha, -x)L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x)\exp(x),$$

$$p_{n+1}^\alpha(x) = \frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x) - (n+\alpha)p_{n-1}^\alpha(x)\right], \quad n = 1, 2, \ldots,$$

$$p_0^\alpha(x) = 0, \qquad p_1^\alpha(x) = 1 - x.$$

For nonnegative natural $\alpha \in \mathbb{N}$ we obtain

$$l_n^\alpha(x) = \mathrm{Ei}_1(-x)L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x)\exp(x),$$

$$p_{-1}^\alpha(x) = (\alpha-1)!, \tag{3.10}$$

$$p_0^\alpha(x) = x^{\alpha-1} + x^\alpha\left[U(2,2,-x) + (-1)^\alpha \alpha! U(1+\alpha, 1+\alpha, -x)\right].$$

Note that the function in the second initial condition in (3.10) solves the following difference initial-value problem:

$$p_0^\alpha(x) = x p_0^{\alpha-1}(x) + (\alpha-1)!, \qquad \alpha = 1, 2, \ldots, \quad p_0^0(x) = 0.$$

By Theorem 3.1 in [4], we can represent the particular solutions of the Laguerre resonant equation of the first kind as follows:

$$u_n(x) = \frac{(-1)^{n+1}}{n!}\frac{\partial}{\partial\nu}\, U(-\nu, 1+\alpha, -x)\Big|_{n=\nu}, \quad n = 0, 1\ldots.$$

This representation gives the particular solutions

$$\chi_0^\alpha(x) = u_0(x) = -\ln(x) + \sum_{p=0}^{\alpha-1}\frac{(\alpha-p)_{p+1}}{(p+1)x^{p+1}},$$

$$\chi_1^\alpha(x) = u_1(x) = -L_1^\alpha(x)\ln(x) + \sum_{p=0}^{\alpha} \frac{k_p(\alpha)}{x^p},$$

where

$$k_{p+1}(\alpha) = p\sum_{i=1}^{\alpha-1} k_p(i), \quad p = 1, 2, \ldots, \alpha-1,$$

$$k_1(\alpha) = \frac{\alpha(\alpha+1)}{2}, \qquad k_0(\alpha) = -\alpha-2, \quad \alpha = 2, 3, \ldots.$$

In the first step of Algorithm 3.1 of [4] we use the following ansatzes:

$$u_0^\alpha(x) = \chi_0^\alpha(x) + c_0 L_0^\alpha(x) + d_0 L_1^\alpha(x),$$

$$u_1^\alpha(x) = \chi_1^\alpha(x) + c_1 L_0^\alpha(x) + d_1 L_1^\alpha(x)$$

with undetermined coefficients $c_0,\ d_0,\ c_1$, and d_1, substitute these expressions in (3.6) with $n = 1$, determine $u_2^\alpha(x)$, and choose $c_0,\ d_0,\ c_1$, and d_1 for which $u_2^\alpha(x)$ satisfies the resonant differential equation. As a result, we obtain $d_0 = 0,\ d_1 = 0$, and $c_1 = 1 + c_0$.

We can now prove that

$$u_n^\alpha(x) = -L_n^\alpha(x)\ln(x) + \frac{p_n^\alpha(x)}{x^\alpha},$$

where the polynomials $p_n^\alpha(x)$ satisfy the recurrence equation

$$p_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1}\, p_n^\alpha(x) - \frac{n+\alpha}{n+1}\, p_{n-1}^\alpha(x)$$
$$+ \frac{\alpha-1-x}{(n+1)^2}\, L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2}\, L_{n-1}^\alpha(x), \quad n = 1, 2, \ldots,$$

with the initial conditions

$$p_0^\alpha(x) = \sum_{p=0}^{\alpha-1} \frac{x^{\alpha-p-1}(\alpha-p)_{p+1}}{p+1} + c_0 x^\alpha,$$

$$p_1^\alpha(x) = \sum_{p=0}^{\alpha} x^{\alpha-p} k_p(\alpha) + (1+c_0)x^\alpha L_1^\alpha(x).$$

3.3. The Laguerre Resonant Equation of the Second Kind. In this section, we consider the resonant equation

$$x\,\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\,\frac{du(x)}{dx} + nu(x) = l_n^\alpha(x), \tag{3.11}$$

where $l_n^\alpha(x)$ is the Laguerre function of the second kind given by (3.2).

By virtue of Theorem 3.1 in [4], the relation

$$u_n(x) = -\frac{d}{d\nu}\, l_\nu^\alpha(x)|_{\nu=n} \tag{3.12}$$

specifies a particular solution of (3.11). Hence, its general solution is given by the formula

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x).$$

The application of relation (3.12) for arbitrary n is quite cumbersome. Therefore, we use Algorithm 3.1 from [4], where, for the sake of simplicity, we set $\alpha = 0$. Solving the differential equation (3.11) with the help of Maple for $n = 0$ and $n = 1$, we find

$$\begin{aligned} \chi_0(x) &= -\int_1^x \frac{\exp(t)}{t} \int_1^t \mathrm{Ei}_1(-\xi)\exp(-\xi)d\xi\, dt, \\ \chi_1(x) &= \big[(1-x)\mathrm{Ei}_1(-x) - \exp(x)\big]\int_1^x \big[1 + \mathrm{Ei}_1(-\xi)(-1+\xi)\exp(-\xi)\big](-1+\xi)d\xi \\ &\quad + \int_1^x \exp(-\xi)\big[\mathrm{Ei}_1(-\xi)(-1+\xi) + \exp(-\xi)\big]^2 d\xi(-1+x). \end{aligned} \tag{3.13}$$

As the ansatzes for the initial values of our algorithm, we use

$$\begin{aligned} u_0^0(x) &= \chi_0(x) + c_0\mathrm{Ei}_1(-x) + d_0, \\ u_1^0(x) &= \chi_1(x) + c_1 l_1^0(x) + d_1 L_1^0(x) \end{aligned} \tag{3.14}$$

with undetermined constants $c_0,\ d_0,\ c_1$, and d_1. Differentiating the recurrence equation for the Laguerre functions of the second kind with respect to n with regard for (3.12), we obtain the following recurrence relation for the particular solutions:

$$\begin{aligned} u_{n+1}^0(x) = {}& \frac{2n+1-x}{n+1}\, u_n^0(x) - \frac{n}{n+1}\, u_{n-1}^0(x) \\ & - \frac{1+x}{(n+1)^2}\, l_n^0(x) + \frac{1}{(n+1)^2}\, l_{n-1}^0(x). \end{aligned} \tag{3.15}$$

We substitute (3.14) to this equation with $n = 1$ and demand that the obtained function $u_2^0(x)$ must satisfy the resonant differential equation (3.11) with $n = 2$. As a result, we obtain

$$\begin{gathered} c_0 = -\mathrm{Ei}_1(-1)\exp(-1) - 1, \qquad d_0 = -\big[\mathrm{Ei}_1(-1)\exp(-1/2) + \exp(1/2)\big]^2, \\ c_1 = 0, \qquad d_1 = 0. \end{gathered} \tag{3.16}$$

By analogy with Theorem 3.1, we can prove the following assertion:

Theorem 3.2. *The functions $u_n^0(x)$ generated by the recursive algorithm (3.15) with the initial values (3.14) and constants given by (3.16) are particular solutions of the Laguerre resonant differential equation of the second kind for all $n = 0, 1, 2, \ldots$.*

As a result of substitution in (3.15), it can be proved that the following representation is true:

$$u_n^0(x) = p_n^0(x)\chi_1(x) + q_n^0(x)\chi_0(x) + v_n^0(x)\,\mathrm{Ei}_1(-x) + w_n^0(x)\exp(x) + q_n^0(x)d_0, \tag{3.17}$$

where the polynomials $p_n^0(x)$ and $q_n^0(x)$ satisfy the recurrence relation for the Laguerre polynomials with the initial conditions

$$p_0^0(x) = 0, \qquad p_1^0(x) = 1, \qquad q_0^0(x) = 1, \qquad q_1^0(x) = 0.$$

The polynomials $w_n^0(x)$ satisfy the inhomogeneous recurrence relation for the Laguerre polynomials

$$w_{n+1}^0(x) = \frac{2n+1-x}{n+1}\,w_n^0(x) - \frac{n}{n+1}\,w_{n-1}^0(x) - \frac{1+x}{(n+1)^2}\,p_n^0(x) + \frac{1}{(n+1)^2}\,p_{n-1}^0(x), \quad n = 1, 2, \ldots,$$

with the initial conditions

$$w_1^0(x) = 0, \qquad w_2^0(x) = \frac{x+1}{4}.$$

Here, $p_n^0(x)$ are the same polynomials as in (3.17).

The polynomials $v_n^0(x)$ solve the following discrete initial-value problem:

$$v_{n+1}^0(x) = \frac{2n+1-x}{n+1}\,v_n^0(x) - \frac{n}{n+1}\,v_{n-1}^0(x) - \frac{1+x}{(n+1)^2}\,L_n^0(x) + \frac{1}{(n+1)^2}\,L_{n-1}^0(x), \quad n = 1, 2, \ldots,$$

$$v_1^0(x) = 0, \qquad v_2^0(x) = \frac{x^2 - 2c_0}{4}.$$

We now present some particular solutions of the Laguerre resonant equation of the second kind obtained by using our algorithm:

$$u_0^0(x) = \chi_0(x) + c_0\,\mathrm{Ei}_1(-x) + d_0, \qquad u_1^0(x) = \chi_1(x),$$

$$u_2^0(x) = -\frac{x-3}{2}\,\chi_1(x) - \frac{1}{2}\,\chi_0(x) + \frac{x^2-2c_0}{4}\,\mathrm{Ei}_1(-x) - \frac{x^2-1}{8}\,\exp(x) - \frac{1}{2}\,d_0,$$

$$u_3^0(x) = \left(\frac{1}{6}\,x^2 - \frac{4}{3}\,x + \frac{11}{6}\right)\chi_1(x) + \left(\frac{1}{6}\,x - \frac{5}{6}\right)\chi_0(x)$$

$$+ \left(-\frac{5}{36}\,x^3 + \frac{7}{12}\,x^2 + \frac{c_0}{6}\,x - \frac{5c_0}{6}\right)\mathrm{Ei}_1(-x)$$

$$+ \left(\frac{1}{24}\,x^3 - \frac{11}{72}\,x^2 - \frac{23}{72}\,x - \frac{1}{72}\right)\exp(x) + \left(\frac{1}{6}\,x - \frac{5}{6}\right)d_0,$$

where c_0 and d_0 are given by relation (3.16) and χ_0 and χ_1 are given by relation (3.13).

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