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GEOMETRIC PROPERTIES OF METRIC SPACES

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We study some problems of geometrization of arbitrary metric spaces. In particular, we analyze the notions of straight and flat placements of points in these spaces. We continue the investigations of Kagan devoted to the detailed analysis of the notion of rectilinearity based on four groups of postulates. Our results are based on the notion of angular characteristics of three points of the space proposed by Alexandrov. We establish the conditions under which the set of points of an arbitrary metric space satisfies all five postulates of the first group of Kagan's placement postulates. The relationship between the rectilinear and flat placements of points in the metric space is investigated. Examples of placements of this kind based on linear functions in some classical spaces are presented. The presented results are obtained without using the property of completeness of the space and can be used for the discrete calculations and structuring of specific metric spaces.

1. Introduction

The present paper is devoted to the problems of "geometrization" of an arbitrary metric space, i.e., to the introduction of notions similar to the principal classical geometric notions in these spaces: line, straight line, angle, and plane. As a specific feature of the present paper, we can mention the fact that we do not use the notion of limit transition in analyzing the posed problems and, hence, the notion of completeness of the space, which necessarily appears in the construction of a complete analog of the Euclidean geometry in an arbitrary metric space. In our opinion, this approach enables one to use the accumulated results in finite metric spaces.

The notion of metric space is one of the central notions of mathematics. Parallel with metric spaces, the researchers also perform extensive investigations of their special classes and modifications with numerous applications in various fields of contemporary mathematics. In this connection, we especially mention ultrametric or non-Archimedean spaces (e.g., in [1], the notion of ultrametric is considered for free groups) and fuzzy metric spaces (see, e.g., [2], where a fuzzy metrization of the space of probability measures is constructed).

The unique numerical characteristic of an arbitrary metric space (X, ρ) is the distance $\rho(x, y)$ between arbitrary elements (points) x and y of this space. This partially explains significant difficulties encountered in its geometrization because the introduction of analogs of the principal geometric notions of the Euclidean geometry (straight line, angle, and plane) inevitably requires the property of completeness of the space.

In our opinion, in any metric space, in some cases (e.g., in the case of a space with finite or countable number of points), it is possible to introduce the notions of angle, parallelism, and perpendicularity without using the requirement of completeness of the space if we do not try to create a complete analog of the Euclidean geometry. In a similar way, Kagan considered the notion of "rectilinear placement" of points in a metric space and "rectilinear image." Following Aleksandrov [3, p. 36], as a characteristic of these notions and properties, it is possible to take one of the numerical characteristics of plane angle in the Euclidean geometry. In this case, we can introduce the notion of "flat placement" of points of metric space as an analog of a plane in the Euclidean geometry.

In [4, pp. 260–297], Kagan constructed the axiomatic theory of Euclidean straight line and proposed four groups of postulates: the placement postulates I_{1-5}, the structure postulates II_{1-3}, the congruence postulates III_{1-7},

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the Archimedean postulate IV_1, and the Cantor postulate IV_2. In [5, p. 29], for the investigation of the notion of rectilinearity in an arbitrary metric space, we introduced the notion of angle formed by three points of the space (as an ordered triple of these points) and the notion of angular characteristic. The indicated characteristic is based on the cosine formula. In [6, pp. 11, 12; 7, pp. 42, 43], on the basis of the notions of angle and angular characteristic, we introduced the notion of flat placement of points of an arbitrary metric space by equating an analog of the determinant of the Gram matrix of a system of unit vectors to zero.

In the present paper, we prove some statements announced in [6], introduce the notion of rectilinear ordering of points in a metric space, and show that, under the condition of rectilinear ordering of points of a certain set in an arbitrary metric space, it satisfies the Kagan postulates I_{1-5}.

The notion of rectilinear ordering of points in the metric space was studied in detail in [4]. In the form used in the present paper, this notion is encountered in [8, p. 527].

The aim of the present paper is to develop an algorithm for the construction of ordinary geometric objects and notions of the Euclidean and non-Euclidean geometries in a metric space, which would enable us to introduce a structure in this space.

2. Preliminary Information

We now present definitions introduced in our previous works with slight modifications performed for better understanding of our subsequent reasoning.

In what follows, we assume that all points of a space are different, i.e., consider only positive values of the metric of the space. We say that a collection of three points $a,\ b,$ and c of the space forms a triangle and denote it by $\triangle\ (a,b,c)$. These points are called vertices and the pairs of points $(a,b),\ (b,c),$ and (a,c) are called the sides of the triangle.

Definition 1. *Let $a,\ b,$ and c be arbitrary points of a metric space (X,ρ). An ordered triple (a,b,c) of these points is called an angle with vertex at the point b and denoted by $\angle(a,b,c)$. Moreover, the pairs of points (a,b) and (b,c) are called the sides of the angle (see [5, p. 28]).*

Definition 2. *Let $a,\ b,$ and c be arbitrary points of the metric space (X,ρ). The characteristic of the angle $\angle(a,b,c)$ or the angular characteristic is defined as a real number $\varphi(a,b,c)$ given by the formula*

$$\varphi(a,b,c)=\frac{\rho^2(a,b)+\rho^2(b,c)-\rho^2(a,c)}{2\rho(a,b)\rho(b,c)} \tag{1}$$

(see [3, p. 36; 5, p. 29]).

A metric space (X,ρ) with the notions of angle and its characteristic introduced by Definitions 1 and 2, respectively, is called a metric space with angular characteristic and denoted by $\varPi$.

Definition 3. *We say that the points $a,\ b,$ and c of the space $\varPi$ are rectilinearly placed if the equality*

$$\varphi^2(a,b,c)=1 \tag{2}$$

holds for at least one of these points (e.g., for the point b) (see [5, p. 29]).

Definition 4. *We say that a set of points of the space $\varPi$ is rectilinearly placed if any three points of this set are rectilinearly placed (see [7, p. 527]).*

Equality (2) is equivalent to the equality $\varphi(a, b, c) = \pm 1$. Moreover, for $\varphi(a, b, c) = -1$, it is natural to say that the point b "lies between" the points a and c (or is an interior point for them) and to call the angle $\angle(a, b, c)$ "straight." At the same time, for $\varphi(b, a, c) = 1$, it is natural to say that the point a "lies beyond" the points b and c (or is extreme for these points) and the angle $\angle(b, a, c)$ is "equal to zero."

By using equality (1), we easily conclude that the equality $\varphi(a, b, c) = -1$ is equivalent to the equality

$$\rho(a, c) = \rho(a, b) + \rho(b, c)$$

and the equality $\varphi(a, b, c) = 1$ is equivalent to the set of two equalities

$$\left[\begin{array}{l}\rho(a, b) = \rho(a, c) + \rho(b, c), \\ \rho(b, c) = \rho(a, c) + \rho(a, b).\end{array}\right.$$

By analogy with the Euclidean geometry, by using equality (1), we can give the definition of the "right" angle $\angle(a, b, c)$ in the space Π.

Definition 5. *If the points a, b, and c of the space Π satisfy the equality $\varphi(a, b, c) = 0$, then the angle $\angle(a, b, c)$ is called right.*

Consider a given metric space (X, ρ) and any three points x_1, x_2, and x_3 of this space. For the sake of convenience, we use the notation $\rho(x_i, x_j) = \rho_{ij}$ and

$$\frac{\rho_{ij}^2 + \rho_{jk}^2 - \rho_{ik}^2}{2\rho_{ij}\rho_{jk}} = \varphi_{ijk}, \quad i, j, k = 1, 2, 3.$$

By using equality (1), we easily prove that, for any three points x_i, x_j, and x_k of the space Π, the inequalities

$$-1 \le \varphi_{ijk} \le 1$$

are true.

We now present an example of a rectilinear placement of an infinite set of points in the metric space $C_{[0;1]}$.

Example 1. Consider a set of functions $y = kx$ on the segment $[0; 1]$ as a subset of the space $C_{[0;1]}$ of functions continuous on the segment $[0; 1]$.

We show that any three different points $y_1 = k_1 x$, $y_2 = k_2 x$, and $y_3 = k_3 x$ of this set are rectilinearly placed. By Definition 4, this means that the entire set is rectilinearly placed.

Further, for definiteness, we assume that $k_1 < k_2 < k_3$. Under this assumption, we determine the distances between the points y_1, y_2, and y_3 in the metric of the space $C_{[0;1]}$:

$$\rho(f, g) = \max_{x \in [0;1]} \left| f(x) - g(x) \right|.$$

We have $\rho_{12} = k_2 - k_1$, $\rho_{13} = k_3 - k_1$, and $\rho_{23} = k_3 - k_2$. Since the equality $\rho_{13} = \rho_{12} + \rho_{23}$ is true, this means that the points y_1, y_2, and y_3 are rectilinearly placed. In view of the arbitrariness of the choice of these points, this means that the entire set of functions $y = kx$ is rectilinearly placed in the space $C_{[0;1]}$.

3. Main Results

We now formulate the results obtained in the present paper (their proofs can be found in Sec. 4).
We first establish the relationship between three angles of a triangle in the metric space.

Theorem 1. *For any points $x_1,\ x_2$, and x_3 of the space Π, the following equality is true:*

$$\begin{vmatrix} 1 & \varphi_{213} & -\varphi_{123} \\ \varphi_{213} & 1 & \varphi_{132} \\ -\varphi_{123} & \varphi_{132} & 1 \end{vmatrix} = 1 - 2\varphi_{213}\varphi_{123}\varphi_{132} - \varphi_{213}^2 - \varphi_{123}^2 - \varphi_{132}^2 = 0. \tag{3}$$

The proof of Theorem 1 is presented in Sec. 4.1.
Consider some special cases of equality (3).

Theorem 2. *In a set of three rectilinearly placed points of the space Π, one and only one point lies between the other two points and each of these two points lies beyond the other two points.*

The proof of Theorem 2 is presented in Sec. 4.2.

Lemma 1. *If four points are rectilinearly placed in the space Π, then two of these points lie between the other two points.*

The proof of Lemma 1 is presented in Sec. 4.3.
Note that the points in the formulation of Lemma 1 are not uniquely defined. On the other hand, the following statement is true:

Lemma 2. *Suppose that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly placed in the space Π and in addition, the points x_2 and x_4 lie between the points x_1 and x_3 and the point x_1 lies beyond the points x_2 and x_4. If the point x_4 lies between the points x_1 and x_3, then it lies either between the points x_1 and x_2 or between the points x_2 and x_3.*

The proof of Lemma 2 is presented in Sec. 4.3.
We now establish conditions under which the set of points of an arbitrary metric space satisfies all five postulates from the first group of the Kagan placement postulates. We now show that the set of rectilinearly placed points in the space Π satisfies the placement postulates I_{1-4} from [4]. The indicated postulates from [4] with insignificant modifications in the statements and notation take the form:

I_1. *If a point b lies between the points a and c, then it also lies between the points c and a (see [4, p. 260]).*

The validity of this postulate for the space Π follows from the symmetry of equality (1): $\varphi(a,b,c) = \varphi(c,b,a)$.

I_2. *Among any three points $a,\ b$, and c, at least one point lies between the other two points (see [4, p. 260]).*

This postulate is a simple corollary of Theorem 2.

I_3. *If a point b lies between the points a and c, then the point c does not lie between the points a and b (see [4, p. 260]).*

This postulate also follows from Theorem 2.

I_4. *If the point b lies between the points a and c and the point d lies between the points a and b, then the point d lies between the points a and c (see [4, p. 260]).*

This postulate is true in any metric space. Indeed, assume that the point b lies between the points a and c. This means that the equality $\rho(a,c) = \rho(a,b) + \rho(b,c)$ is true. If the point d lies between the points a and b, then this means that the equality $\rho(a,b) = \rho(a,d) + \rho(d,b)$ is true. Substituting this equality in the right-hand side of the previous equality, we obtain

$$\rho(a,c) = \rho(a,b) + \rho(b,c) = \rho(a,d) + \rho(d,b) + \rho(b,c).$$

By using the triangle inequality, we get $\rho(d,b)+\rho(b,c) \geq \rho(d,c)$. Hence, the inequality $\rho(a,c) \geq \rho(a,d)+\rho(d,c)$ is true. By the triangle inequality, this is possible only in the case where the equality $\rho(a,c) = \rho(a,d) + \rho(d,c)$ is true. Thus, the point d lies between the points a and c.

By Lemma 2, the following postulate holds for four points of the space Π satisfying the conditions of the lemma:

I_5. *If a point b on a straight line lies between points a and c and a point d different from b also lies between the points a and c, then at least one of the following two placements occurs: either the point d lies between the points a and b or the point d lies between the points b and c (see [4, p. 260]).*

Kagan did not separate the set of points of the metric space satisfying the placement postulates I_{1-5}. Actually, he added the structure postulates L_{II} to the placement postulates and selected the set L_{II} of points satisfying both these groups of postulates [4, p. 265]. Thus, it is quite natural to present the following definitions:

Definition 6. *Assume that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly placed in the space Π and the points x_2 and x_4 lie between the points x_1 and x_3. If the point x_1 lies beyond the points x_2 and x_4, then we say that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered and the point x_1 is extreme for these points.*

For any rectilinearly placed set of points of the space Π, we introduce the definition of its rectilinear ordering.

Definition 7. *If any four points of a rectilinearly placed set of points of the space Π are rectilinearly ordered, then this set is called rectilinearly ordered and denoted by L_{I}.*

Summarizing the results obtained above, we conclude that all Kagan placement postulates I_{1-5} are true in the set L_{I}.

We now establish an analytic criterion for the rectilinear ordering of four points of the space Π.

Lemma 3. *In order that the points $x_1,\ x_2,\ x_3$, and x_4 of the space Π be rectilinearly ordered, it is necessary and sufficient that the equality*

$$\varphi_{213}\varphi_{214}\varphi_{314} = 1 \tag{4}$$

be true for at least one of these points (e.g., for the point x_1),.

The proof of Lemma 3 is presented in Sec. 4.5.

It is possible to establish the condition under which two points in a set of four rectilinearly ordered points are extreme.

Lemma 4. *Suppose that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered in the space Π and the points x_2 and x_4 lie between the points x_1 and x_3. If the point x_1 is extreme, then the point x_3 is also extreme for the points $x_1,\ x_2,\ x_3$, and x_4.*

The proof of Lemma 4 is presented in Sec. 4.6.

It is worth noting that the rectilinear placement of a set of points does not always imply that this set contains an extreme point. The following example illustrates this assertion:

Example 2. Consider a metric space R_0^2 of ordered groups of two real numbers $a(a_1, a_2)$ such that the distance between the elements $a(a_1, a_2)$ and $b(b_1, b_2)$ of these groups is given by the formula

$$\rho(a, b) = \max\left(|a_1 - b_1|, |a_2 - b_2|\right).$$

In the space R_0^2, we take four points: $a(1,0),\ b(0,1),\ c(-1,0)$, and $d(0,-1)$ and determine the distances between these points in the metric of the space: $\rho(a,b) = 1,\ \rho(a,c) = 2,\ \rho(a,d) = 1,\ \rho(b,c) = 1,\ \rho(b,d) = 2$, and $\rho(c,d) = 1$.

The obtained values imply that any three points in this set of points are rectilinearly ordered. Indeed, by using relation (1), we determine all angular characteristics as follows:

$$\varphi(b,a,c) = 1, \qquad \varphi(b,a,d) = -1, \qquad \varphi(c,a,d) = 1,$$

$$\varphi(a,b,c) = -1, \qquad \varphi(a,b,d) = 1, \qquad \varphi(c,b,d) = 1, \qquad \varphi(a,c,b) = 1, \qquad \varphi(a,c,d) = 1,$$

$$\varphi(b,c,d) = -1, \qquad \varphi(a,d,b) = 1, \qquad \varphi(a,d,c) = -1, \qquad \varphi(b,d,c) = 1.$$

In all cases, equality (2) is true. Hence, by Definitions 3 and 4, the points $a,\ b,\ c$, and d are rectilinearly ordered. However, this set of points does not contain extreme points because each of these points lies between two other points from this group.

We now consider a generalization of the notion of rectilinear placement of points in the space Π introduced by the author in [6, pp. 11, 12; 7, p. 42].

Definition 8. *We say that the points $x_1,\ x_2,\ x_3$, and x_4 of the space Π are flatly placed if, for at least one of these points (e.g., for the point x_1), the equality*

$$\begin{vmatrix} 1 & \varphi_{213} & \varphi_{214} \\ \varphi_{213} & 1 & \varphi_{314} \\ \varphi_{214} & \varphi_{314} & 1 \end{vmatrix} = 1 + 2\varphi_{213}\varphi_{214}\varphi_{314} - \varphi_{213}^2 - \varphi_{214}^2 - \varphi_{314}^2 = 0 \tag{5}$$

is true.

In the Euclidean geometry, equality (5) means that the volume of the tetrahedron with vertices at the points $x_1,\ x_2,\ x_3$, and x_4 is equal to zero [9, p. 61].

For points of an arbitrary set in the space Π, it is natural to present the definition of their "flat placement."

Definition 9. *We say that a set of points of the space Π is flatly placed if any four points from this set are flatly placed (see [6, p. 12; 7, p. 43]).*

In the space Π, the relationships between the rectilinear and flat placements of points are more complicated than in the Euclidean geometry in which these relationships are established by postulates. However, in the metric space, we can use the properties of the set of real numbers.

We now establish the following relation in the analytic form:

Lemma 5. *In order that the rectilinearly placed points $x_1,\ x_2,\ x_3$, and x_4 in the space Π be flatly placed in this space, it is necessary and sufficient that equality (4) be true for at least one of these points (e.g., for the point x_1).*

The proof of Lemma 5 is presented in Sec. 4.7.

Combining Lemmas 3 and 5 with Corollary 2, we arrive at the following relationship between the rectilinear and flat placements of points in the space Π :

Theorem 3. *A rectilinearly placed set of points of the space Π is flatly placed in this space if and only if it is rectilinearly ordered.*

The proof of Theorem 3 is presented in Sec. 4.8.

In order to construct flatly placed sets of points of the space Π on the basis of three rectilinearly placed points, we consider the notion of adjacency of two angles introduced in [6, p. 11; 10, p. 65].

In the Euclidean geometry, two adjacent angles complement each other to a straight angle. In the space Π, this is not always true.

Example 3. In the space $C_{[0;1]}$, we consider the points $y_1 = 0,\ y_2 = 1,\ y_3 = x$, and $y_4 = -x$. In the metric of the space $C_{[0;1]}$, the distances between these points are as follows: $\rho_{12} = 1,\ \rho_{13} = 1,\ \rho_{14} = 1,\ \rho_{23} = 1,$ $\rho_{24} = 2$, and $\rho_{34} = 2$.

As follows from the obtained values, the points $y_1,\ y_3$, and y_4 are rectilinearly placed. Moreover, the point y_1 lies between the other two points. In addition, the points $y_1,\ y_2$, and y_4 are also rectilinearly placed and, as in the previous case, the point y_1 lies between the other two points.

Thus, the angles $\angle(y_3, y_1, y_4)$ and $\angle(y_2, y_1, y_3)$ complement each other to the straight angle $\angle(y_2, y_1, y_4)$.

We now determine angular characteristics of these angles: $\varphi(y_3, y_1, y_4) = -1$ and $\varphi(y_2, y_1, y_3) = 0.5$. Hence, a nonzero angle complements a straight angle to an angle, which is also straight.

Thus, Example 3 shows that the definition of adjacent angles in the space Π should be based on their angular characteristics.

In [7, p. 43; 10, p. 65], it was shown that the equality $\varphi_{124} = -\varphi_{324}$ is a necessary and sufficient condition for the flat placement of the points $x_1,\ x_2,\ x_3$, and x_4 from the space Π such that the point x_2 lies between the points x_1 and x_3. This equality is also true for the adjacent angles in the Euclidean geometry. Therefore, it should be chosen for the definition of adjacency of two angles in the space Π.

Definition 10. *Assume that the points x_1, x_2, and x_3 of the space Π are rectilinearly placed and, in addition, that the angle $\angle(x_1, x_2, x_3)$ is straight. If the point x_4 of this space is such that the equality*

$$\varphi_{124} = -\varphi_{324} \tag{6}$$

is true, then the angles $\angle(x_1, x_2, x_4)$ and $\angle(x_3, x_2, x_4)$ are called adjacent.

It is worth noting that Definition 10 also covers the case where one of the angles is right. In this case, by Definition 10, the angle adjacent to it is also right because equality (6) is true for the values $\varphi_{124} = \varphi_{324} = 0$.

Flatly placed sets of points in the space Π can be also constructed according to three points from this space that form a right angle. Indeed, if the angle $\angle(x_2, x_1, x_3)$ is right, i.e., the equality $\varphi_{213} = 0$ is true, then the point x_4 flatly placed with the points $x_1, x_2,$ and x_3 can be found from equality (5). Substituting the value $\varphi_{213} = 0$ in this equality, we arrive at the equality

$$\varphi_{214}^2 + \varphi_{314}^2 = 1. \tag{7}$$

Equality (7) can be regarded as an analog of trigonometric unit in the Euclidean geometry and, hence, can be used for the construction of flatly placed sets.

4. Proofs of the Results

4.1. Proof of Theorem 1. We compute the expression

$$\begin{vmatrix} 1 & \varphi_{213} & -\varphi_{123} \\ \varphi_{213} & 1 & \varphi_{132} \\ -\varphi_{123} & \varphi_{132} & 1 \end{vmatrix} = 1 - 2\varphi_{213}\varphi_{123}\varphi_{132} - \varphi_{213}^2 - \varphi_{123}^2 - \varphi_{132}^2$$

$$= 1 - 2\frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2}{2\rho_{12}\rho_{13}} \frac{\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2}{2\rho_{12}\rho_{23}} \frac{\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2}{2\rho_{13}\rho_{23}}$$

$$- \left(\frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2}{2\rho_{12}\rho_{13}}\right)^2 - \left(\frac{\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2}{2\rho_{12}\rho_{23}}\right)^2 - \left(\frac{\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2}{2\rho_{13}\rho_{23}}\right)^2$$

$$= 1 - \frac{(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)}{4\rho_{12}^2\rho_{13}^2\rho_{23}^2}$$

$$- \frac{\rho_{23}^2(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)^2 + \rho_{13}^2(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)^2 + \rho_{12}^2(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)^2}{4\rho_{12}^2\rho_{13}^2\rho_{23}^2}$$

$$= 1 - \frac{1}{4\rho_{12}^2\rho_{13}^2\rho_{23}^2}\Big((\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)$$

$$+ \rho_{23}^2(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)^2 + \rho_{13}^2(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)^2 + \rho_{12}^2(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)^2\Big).$$

We now find the expression in parentheses:

$$(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)$$

$$+ \rho_{23}^2(\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)^2 + \rho_{13}^2(\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2)^2 + \rho_{12}^2(\rho_{13}^2 + \rho_{23}^2 - \rho_{12}^2)^2$$

$$= (\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)(\rho_{12}^2\rho_{13}^2 + \rho_{12}^2\rho_{23}^2 - \rho_{12}^4 + \rho_{13}^2\rho_{23}^2 + \rho_{23}^4 - \rho_{12}^2\rho_{23}^2 - \rho_{13}^4$$

$$- \rho_{13}^2\rho_{23}^2 + \rho_{12}^2\rho_{13}^2) + \rho_{12}^4\rho_{23}^2 + \rho_{13}^4\rho_{23}^2 + \rho_{23}^6 + 2\rho_{12}^2\rho_{13}^2\rho_{23}^2 - 2\rho_{12}^2\rho_{23}^4$$

$$
\begin{aligned}
&- 2\rho_{13}^2\rho_{23}^4 + \rho_{12}^4\rho_{13}^2 + \rho_{13}^2\rho_{23}^4 + \rho_{13}^6 + 2\rho_{12}^2\rho_{13}^2\rho_{23}^2 - 2\rho_{12}^2\rho_{13}^4 - 2\rho_{13}^4\rho_{23}^2 \\
&+ \rho_{12}^2\rho_{13}^4 + \rho_{12}^2\rho_{23}^4 + \rho_{12}^6 + 2\rho_{12}^2\rho_{13}^2\rho_{23}^2 - 2\rho_{12}^4\rho_{13}^2 - 2\rho_{12}^4\rho_{23}^2 \\
= {}& (\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)(2\rho_{12}^2\rho_{13}^2 - \rho_{12}^4 - \rho_{13}^4 + \rho_{23}^4) - \rho_{12}^4\rho_{13}^2 - \rho_{12}^4\rho_{23}^2 \\
&+ \rho_{12}^6 - \rho_{12}^2\rho_{13}^4 - \rho_{13}^4\rho_{23}^2 + \rho_{13}^6 - \rho_{12}^2\rho_{23}^4 - \rho_{13}^2\rho_{23}^4 + \rho_{23}^6 + 6\rho_{12}^2\rho_{13}^2\rho_{23}^2 \\
= {}& 2\rho_{12}^4\rho_{13}^2 - \rho_{12}^6 - \rho_{12}^2\rho_{13}^4 + \rho_{12}^2\rho_{23}^4 + 2\rho_{12}^2\rho_{13}^4 - \rho_{12}^4\rho_{13}^2 - \rho_{13}^6 + \rho_{13}^2\rho_{23}^4 \\
&- 2\rho_{12}^2\rho_{13}^2\rho_{23}^2 + \rho_{12}^4\rho_{23}^2 + \rho_{13}^4\rho_{23}^2 - \rho_{23}^6 - \rho_{12}^4\rho_{13}^2 - \rho_{12}^4\rho_{23}^2 + \rho_{12}^6 \\
&- \rho_{12}^2\rho_{13}^4 - \rho_{13}^4\rho_{23}^2 + \rho_{13}^6 - \rho_{12}^2\rho_{23}^4 - \rho_{13}^2\rho_{23}^4 + \rho_{23}^6 + 6\rho_{12}^2\rho_{13}^2\rho_{23}^2 \\
= {}& 4\rho_{12}^2\rho_{13}^2\rho_{23}^2.
\end{aligned}
$$

Finally, we arrive at the equality

$$1 - 2\varphi_{213}\varphi_{123}\varphi_{132} - \varphi_{213}^2 - \varphi_{123}^2 - \varphi_{132}^2 = 0.$$

Hence, equality (3) is proved.

Theorem 1 is proved.

The proof of equality (3) is quite simple for an ordinary triangle in the Euclidean geometry. Indeed, if we denote $\varphi_{213} = \cos\angle A$, $\varphi_{123} = \cos\angle B$, and $\varphi_{132} = \cos\angle C$, where $\angle A$, $\angle B$, and $\angle C$ are the interior angles of the triangle $\triangle\, ABC$, then, in view of the equality

$$\cos\angle C = \cos\big(\pi - (\angle A + \angle B)\big) = -\cos(\angle A + \angle B),$$

the left-hand side of equality (3) takes the form

$$
\begin{aligned}
&1 + 2\cos\angle A\cos\angle B\cos(\angle A + \angle B) - \cos^2\angle A - \cos^2\angle B - \cos^2(\angle A + \angle B) \\
&\quad = 1 + 2\cos\angle A\cos\angle B(\cos\angle A\cos\angle B - \sin\angle A\sin\angle B) \\
&\qquad - \cos^2\angle A - \cos^2\angle B - (\cos\angle A\cos\angle B - \sin\angle A\sin\angle B)^2 \\
&\quad = 1 + 2\cos^2\angle A\cos^2\angle B - 2\cos\angle A\cos\angle B\sin\angle A\sin\angle B \\
&\qquad - \cos^2\angle A - \cos^2\angle B - \cos^2\angle A\cos^2\angle B \\
&\quad + 2\cos\angle A\cos\angle B\sin\angle A\sin\angle B - \sin^2\angle A\sin^2\angle B \\
&\quad = 1 - \cos^2\angle A - \cos^2\angle B + \cos^2\angle A\cos^2\angle B - \sin^2\angle A\sin^2\angle B
\end{aligned}
$$

$$= 1 - \cos^2 \angle A - \cos^2 \angle B + \cos^2 \angle A \cos^2 \angle B - (1 - \cos^2 \angle A)(1 - \cos^2 \angle B)$$

$$= 1 - \cos^2 \angle A - \cos^2 \angle B + \cos^2 \angle A \cos^2 \angle B$$

$$- 1 + \cos^2 \angle A + \cos^2 \angle B - \cos^2 \angle A \cos^2 \angle B = 0.$$

Hence, the validity of equality (3) is a necessary condition for the existence of a triangle with given angular characteristics in the metric space.

By using equality (3), we can deduce a relationship between the angular characteristics in the case where one of them is equal to zero, i.e., in the case where one of three angles is right.

Corollary 1. *If, for arbitrary points $x_1, x_2,$ and x_3 of the space Π, the angle $\angle(x_1, x_2, x_3)$ is right, then the equality*

$$\varphi_{213}^2 + \varphi_{132}^2 = 1 \tag{8}$$

is true.

Proof. If the angle $\angle(x_1, x_2, x_3)$ is right, then the equality $\varphi_{123} = 0$ is true. Substituting this value in equality (3), we get

$$\varphi_{213}^2 + \varphi_{132}^2 - 1 = 0 \quad \text{or} \quad \varphi_{213}^2 + \varphi_{132}^2 = 1.$$

Equality (8), like equality (7), can be regarded as an analog of trigonometric unit in the Euclidean geometry.

4.2. Proof of Theorem 2. Assume that one of the angles of triangle, e.g., the angle $\angle(x_1, x_2, x_3)$, is straight, i.e., the equality $\varphi_{123} = -1$ is true. Substituting this value in equality (1), we get

$$\varphi_{213}^2 + (-1)^2 + \varphi_{132}^2 + 2\varphi_{213}(-1)\varphi_{132} - 1 = 0, \qquad \varphi_{213}^2 + \varphi_{132}^2 - 2\varphi_{213}\varphi_{132} = 0,$$

$$(\varphi_{213} - \varphi_{132})^2 = 0, \qquad \varphi_{213} = \varphi_{132}.$$

The obtained equality can be improved. By using the equality

$$\varphi_{123} = \frac{\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2}{2\rho_{12}\rho_{23}} = -1,$$

we obtain

$$\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2 = -2\rho_{12}\rho_{23}, \qquad \rho_{12}^2 + 2\rho_{12}\rho_{23} + \rho_{23}^2 = \rho_{13}^2,$$

$$(\rho_{12} + \rho_{23})^2 = \rho_{13}^2, \qquad \rho_{12} + \rho_{23} = \rho_{13}.$$

By virtue of the last equality, we get the following angular characteristic:

$$\varphi_{213} = \frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2}{2\rho_{12}\rho_{13}} = \frac{\rho_{12}^2 + (\rho_{12} + \rho_{23})^2 - \rho_{23}^2}{2\rho_{12}(\rho_{12} + \rho_{23})}$$

$$= \frac{\rho_{12}^2 + \rho_{12}^2 + 2\rho_{12}\rho_{23} + \rho_{23}^2 - \rho_{23}^2}{2\rho_{12}(\rho_{12} + \rho_{23})}$$

$$= \frac{2\rho_{12}^2 + 2\rho_{12}\rho_{23}}{2\rho_{12}(\rho_{12} + \rho_{23})} = \frac{2\rho_{12}(\rho_{12} + \rho_{23})}{2\rho_{12}(\rho_{12} + \rho_{23})} = 1.$$

Hence, $\varphi_{132} = \varphi_{213} = 1$.

We now assume that one of the angles of triangle, e.g., the angle $\angle(x_1, x_2, x_3)$, is equal to zero, i.e., the equality $\varphi_{123} = 1$ is true. Substituting this value in equality (3), we obtain

$$\varphi_{213}^2 + 1^2 + \varphi_{132}^2 + 2\varphi_{213}\varphi_{132} - 1 = 0, \qquad \varphi_{213}^2 + \varphi_{132}^2 + 2\varphi_{213}\varphi_{132} = 0,$$

$$(\varphi_{213} + \varphi_{132})^2 = 0, \qquad \varphi_{213} = -\varphi_{132}.$$

As in the previous case, we can improve this equality. By using the equality

$$\varphi_{123} = \frac{\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2}{2\rho_{12}\rho_{23}} = 1,$$

we get

$$\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2 = 2\rho_{12}\rho_{23}, \qquad \rho_{12}^2 - 2\rho_{12}\rho_{23} + \rho_{23}^2 = \rho_{13}^2, \qquad (\rho_{12} - \rho_{23})^2 = \rho_{13}^2.$$

This yields the set of equalities

$$\left[\begin{array}{l} \rho_{12} - \rho_{23} = \rho_{13}, \\ \rho_{12} - \rho_{23} = -\rho_{13}; \end{array}\right. \qquad \left[\begin{array}{l} \rho_{12} = \rho_{13} + \rho_{23}, \\ \rho_{23} = \rho_{12} + \rho_{13}. \end{array}\right.$$

By using the first of these equalities, we get the angular characteristic

$$\varphi_{213} = \frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2}{2\rho_{12}\rho_{13}} = \frac{(\rho_{13} + \rho_{23})^2 + \rho_{13}^2 - \rho_{23}^2}{2(\rho_{13} + \rho_{23})\rho_{13}}$$

$$= \frac{\rho_{13}^2 + 2\rho_{13}\rho_{23} + \rho_{23}^2 + \rho_{13}^2 - \rho_{23}^2}{2(\rho_{13} + \rho_{23})\rho_{13}}$$

$$= \frac{2\rho_{13}^2 + 2\rho_{13}\rho_{23}}{2(\rho_{13} + \rho_{23})\rho_{13}} = \frac{2\rho_{13}(\rho_{13} + \rho_{23})}{2(\rho_{13} + \rho_{23})\rho_{13}} = 1.$$

The value of the angular characteristic φ_{132} is determined from the equality obtained above

$$\varphi_{132} = -\varphi_{213} = -1.$$

By using the second equality established above, we get the angular characteristic

$$\varphi_{213} = \frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2}{2\rho_{12}\rho_{13}} = \frac{\rho_{12}^2 + \rho_{13}^2 - (\rho_{12} + \rho_{13})^2}{2\rho_{12}\rho_{13}}$$

$$= \frac{\rho_{12}^2 + \rho_{13}^2 - \rho_{12}^2 - 2\rho_{12}\rho_{13} - \rho_{13}^2}{2\rho_{12}\rho_{13}} = \frac{-2\rho_{12}\rho_{13}}{2\rho_{12}\rho_{13}} = -1.$$

The angular characteristic φ_{132} is obtained from the equality $\varphi_{132} = -\varphi_{213} = 1$.

By using the values obtained above, we conclude that only one point among three rectilinearly placed points lies between the other two points (and is interior for these points) and each of these two points lies outside the remaining two points (and is extreme for these points).

Theorem 2 is proved.

4.3. Proof of Lemma 1. Assume that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly placed in the space Π. Then the points x_1, x_2, and x_3 are also rectilinearly placed. By Theorem 2, only one point in this collection is located between the other two points. Thus, we assume that the point x_2 lies between the points x_1 and x_3. Then the equality $\rho_{13} = \rho_{12} + \rho_{23}$ is true. If the point x_4 also lies between these points, then the assertion of Lemma 1 is true.

Assume that the point x_4 lies beyond the points x_1 and x_3. Since these three points are also rectilinearly placed, the following set of equalities is true:

$$\left[\begin{array}{l} \rho_{14} = \rho_{13} + \rho_{34}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{array}\right. \tag{9}$$

Suppose that the first equality in (9) is true, i.e., the point x_3 lies between the points x_1 and x_4. Then we arrive at the system of equalities

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \rho_{14} = \rho_{13} + \rho_{34}. \end{cases}$$

Thus, we successively get

$$\rho_{14} = \rho_{13} + \rho_{34} = (\rho_{12} + \rho_{23}) + \rho_{34} = \rho_{12} + (\rho_{23} + \rho_{34}) \geq \rho_{12} + \rho_{24}.$$

In view of the triangle inequality, this inequality may be true only in the case where the equality $\rho_{14} = \rho_{12}+\rho_{24}$ holds, which means that the point x_2, as the point x_3, lies between the points x_1 and x_4.

Now let the second equality in (9) be true, i.e., the point x_1 lies between the points x_3 and x_4. In this case, we get the system of equalities

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{cases}$$

Thus, we successively obtain

$$\rho_{34} = \rho_{13} + \rho_{14} = (\rho_{12} + \rho_{23}) + \rho_{14} = \rho_{23} + (\rho_{12} + \rho_{14}) \geq \rho_{23} + \rho_{24}.$$

In view of the triangle inequality, this inequality is true only in the case where the equality $\rho_{34} = \rho_{23} + \rho_{24}$ holds but this means that the point x_2, just as the point x_1, lies between the points x_3 and x_4.

Since we have considered all possible cases of placement of the points, the assertion of the lemma is true.

4.4. Proof of Lemma 2. Under the condition of the lemma, the point x_2 lies between the points x_1 and x_3. Hence, the equality $\rho_{13} = \rho_{12} + \rho_{23}$ is true. In view of the condition that the point x_1 lies beyond the points x_2 and x_4, we arrive at the set of equalities

$$\left[\begin{aligned}&\rho_{12} = \rho_{14} + \rho_{24},\\ &\rho_{14} = \rho_{12} + \rho_{24}.\end{aligned}\right. \tag{10}$$

The first equality in (10) means that the point x_4 lies between the points x_1 and x_2. Therefore, in this case, the assertion of the lemma is true.

We consider the second equality in (10) and assume that the point x_4 also lies between the points x_1 and x_3, i.e., the equality $\rho_{13} = \rho_{14} + \rho_{34}$ is true. Thus, we get the system of equalities

$$\begin{cases}\rho_{13} = \rho_{12} + \rho_{23},\\ \rho_{13} = \rho_{14} + \rho_{34},\\ \rho_{14} = \rho_{12} + \rho_{24}.\end{cases}$$

By using this system, we successively get

$$\rho_{23} = \rho_{13} - \rho_{12} = (\rho_{14} + \rho_{34}) - \rho_{12} = (\rho_{14} - \rho_{12}) + \rho_{34} = \rho_{24} + \rho_{34}.$$

This equality means that the point x_4 lies between the points x_2 and x_3. Hence, in this case, the assertion of the lemma is also true.

4.5. Proof of Lemma 3. If the points x_1, x_2, x_3, and x_4 are rectilinearly placed in the space Π and the points x_2 and x_4 lie between the points x_1 and x_3, then the equalities $\rho_{13} = \rho_{12} + \rho_{23}$ and $\rho_{13} = \rho_{14} + \rho_{34}$ are true. In addition, if the point x_1 lies beyond the points x_2 and x_4, then, by Lemma 2, the point x_4 lies either between the points x_1 and x_2 or between the points x_2 and x_3. Similarly, the point x_2 lies either between the points x_1 and x_4 or between the points x_3 and x_4. In each of these cases, the equalities $\varphi_{213} = 1$, $\varphi_{214} = 1$, and $\varphi_{314} = 1$ are true and, hence, equality (4) is valid.

Now let the points x_1, x_2, x_3, and x_4 of the space Π satisfy equality (4). Since the modulus of the angular characteristic does not exceed one, equality (4) can be true only for the values $\varphi_{213} = \pm 1$, $\varphi_{214} = \pm 1$, and $\varphi_{314} = \pm 1$, i.e., only in the following cases:

1. $\varphi_{213} = 1,\ \varphi_{214} = 1,\ \varphi_{314} = 1;$
2. $\varphi_{213} = 1,\ \varphi_{214} = -1,\ \varphi_{314} = -1;$
3. $\varphi_{213} = -1,\ \varphi_{214} = 1,\ \varphi_{314} = -1;$
4. $\varphi_{213} = -1,\ \varphi_{214} = -1,\ \varphi_{314} = 1.$

By virtue of Definition 4, to prove the lemma, it suffices to show that the points x_2, x_3, and x_4 are rectilinearly placed. According to Definition 3, to this end, it is sufficient to prove the equality $\varphi_{234}^2 = 1$. We successively consider the indicated four cases.

1. Let the equalities $\varphi_{213} = 1$, $\varphi_{214} = 1$, and $\varphi_{314} = 1$ be simultaneously true. This is equivalent to the following systems of equalities:

$$\begin{cases} \left[\begin{array}{l} \rho_{12} = \rho_{13} + \rho_{23}, \\ \rho_{13} = \rho_{12} + \rho_{23}, \end{array}\right. \\ \left[\begin{array}{l} \rho_{12} = \rho_{14} + \rho_{24}, \\ \rho_{14} = \rho_{12} + \rho_{24}, \end{array}\right. \\ \left[\begin{array}{l} \rho_{13} = \rho_{14} + \rho_{34}, \\ \rho_{14} = \rho_{13} + \rho_{34}; \end{array}\right. \end{cases} \quad \begin{cases} \left[\begin{array}{l} \rho_{23} = \rho_{12} - \rho_{13}, \\ \rho_{23} = -(\rho_{12} - \rho_{13}), \end{array}\right. \\ \left[\begin{array}{l} \rho_{24} = \rho_{12} - \rho_{14}, \\ \rho_{24} = -(\rho_{12} - \rho_{14}), \end{array}\right. \\ \left[\begin{array}{l} \rho_{34} = \rho_{13} - \rho_{14}, \\ \rho_{34} = -(\rho_{13} - \rho_{14}); \end{array}\right. \end{cases} \quad \begin{cases} \rho_{23}^2 = (\rho_{12} - \rho_{13})^2, \\ \rho_{24}^2 = (\rho_{12} - \rho_{14})^2, \\ \rho_{34}^2 = (\rho_{13} - \rho_{14})^2. \end{cases}$$

Further, for different cases of these systems, we find the angular characteristic

$$\varphi_{234} = \frac{\rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2}{2\rho_{23}\rho_{34}}.$$

In all these cases, it suffices to show that the equality

$$\rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2 = \pm 2\rho_{23}\rho_{34}, \tag{11}$$

and, hence, the equality $\varphi_{234}^2 = 1$ are true.

Substituting the obtained values in the left-hand side of equality (11), we obtain

$$\begin{aligned} \rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2 &= (\rho_{12} - \rho_{13})^2 + (\rho_{13} - \rho_{14})^2 - (\rho_{12} - \rho_{14})^2 \\ &= \rho_{12}^2 - 2\rho_{12}\rho_{13} + \rho_{13}^2 + \rho_{13}^2 - 2\rho_{13}\rho_{14} + \rho_{14}^2 - \rho_{12}^2 + 2\rho_{12}\rho_{14} - \rho_{14}^2 \\ &= 2\rho_{13}^2 - 2\rho_{12}\rho_{13} - 2\rho_{13}\rho_{14} + 2\rho_{12}\rho_{14} = 2(\rho_{13}(\rho_{13} - \rho_{12}) - \rho_{14}(\rho_{13} - \rho_{12})) \\ &= 2(\rho_{13} - \rho_{12})(\rho_{13} - \rho_{14}) = \pm 2\rho_{23}\rho_{34}. \end{aligned}$$

Hence, in this case, the points x_2, x_3, and x_4 are rectilinearly placed, which, in turn, means that the points x_1, x_2, x_3, and x_4 are also rectilinearly placed.

We now show that the points x_1, x_2, x_3, and x_4 are rectilinearly ordered. To this end, we separately consider each possible case of their placement following from the presented systems of equalities.

Case (a):

$$\begin{cases} \rho_{12} = \rho_{13} + \rho_{23}, \\ \rho_{12} = \rho_{14} + \rho_{24}, \\ \left[\begin{array}{l} \rho_{13} = \rho_{14} + \rho_{34}, \\ \rho_{14} = \rho_{13} + \rho_{34}. \end{array}\right. \end{cases}$$

In this case, the points x_3 and x_4 lie between the points x_1 and x_2 and, in view of the equality $\varphi_{314} = 1$, the point x_1 lies beyond the points x_3 and x_4. Thus, by Definition 6, the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. Moreover, the point x_1 is extreme for the points $x_1,\ x_2,\ x_3$, and x_4.

Case (b):

$$\begin{cases} \left[\begin{aligned} &\rho_{12} = \rho_{13} + \rho_{23}, \\ &\rho_{13} = \rho_{12} + \rho_{23}, \end{aligned}\right. \\ \rho_{14} = \rho_{12} + \rho_{24}, \\ \rho_{14} = \rho_{13} + \rho_{34}. \end{cases}$$

In this case, the points x_2 and x_3 lie between the points x_1 and x_4. In view of the equality $\varphi_{213} = 1$, the point x_1 lies beyond the points x_2 and x_3. Thus, by Definition 6, the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. Moreover, the point x_1 is extreme for the points $x_1,\ x_2,\ x_3$, and x_4.

Case (c):

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \left[\begin{aligned} &\rho_{12} = \rho_{14} + \rho_{24}, \\ &\rho_{14} = \rho_{12} + \rho_{24}, \end{aligned}\right. \\ \rho_{13} = \rho_{14} + \rho_{34}. \end{cases}$$

In this case, the points x_2 and x_4 lie between the points x_1 and x_3. In view of the equality $\varphi_{214} = 1$, the point x_1 lies beyond the points x_2 and x_4 and, by Definition 6, the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. Moreover, the point x_1 is extreme for the points $x_1,\ x_2,\ x_3$, and x_4.

2. Let the equalities $\varphi_{213} = 1,\ \varphi_{214} = -1$, and $\varphi_{314} = -1$ be simultaneously true. This is equivalent to the system

$$\begin{cases} \left[\begin{aligned} &\rho_{12} = \rho_{13} + \rho_{23}, \\ &\rho_{13} = \rho_{12} + \rho_{23}, \end{aligned}\right. \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \rho_{34} = \rho_{13} + \rho_{14}; \end{cases} \qquad \begin{cases} \left[\begin{aligned} &\rho_{23} = \rho_{12} - \rho_{13}, \\ &\rho_{23} = -(\rho_{12} - \rho_{13}), \end{aligned}\right. \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \rho_{34} = \rho_{13} + \rho_{14}; \end{cases} \qquad \begin{cases} \rho_{23}^2 = (\rho_{12} - \rho_{13})^2, \\ \rho_{24}^2 = (\rho_{12} + \rho_{14})^2, \\ \rho_{34}^2 = (\rho_{13} + \rho_{14})^2. \end{cases}$$

Substituting the obtained values in the left-hand side of equality (11), we obtain

$$\begin{aligned} \rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2 &= (\rho_{12} - \rho_{13})^2 + (\rho_{13} + \rho_{14})^2 - (\rho_{12} + \rho_{14})^2 \\ &= \rho_{12}^2 - 2\rho_{12}\rho_{13} + \rho_{13}^2 + \rho_{13}^2 + 2\rho_{13}\rho_{14} + \rho_{14}^2 - \rho_{12}^2 - 2\rho_{12}\rho_{14} - \rho_{14}^2 \\ &= 2\rho_{13}^2 - 2\rho_{12}\rho_{13} + 2\rho_{13}\rho_{14} - 2\rho_{12}\rho_{14} \end{aligned}$$

$$= 2\big(\rho_{13}(\rho_{13} - \rho_{12}) + \rho_{14}(\rho_{13} - \rho_{12})\big)$$

$$= 2(\rho_{13} - \rho_{12})(\rho_{13} + \rho_{14}) = \pm 2\rho_{23}\rho_{34}.$$

Hence, in this case, the points $x_2,\ x_3$, and x_4 are rectilinearly placed and, therefore, the points $x_1,\ x_2,\ x_3$, and x_4 are also rectilinearly placed.

We now show that the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. To this end, we consider all possible cases of their location.

Case (a):

$$\begin{cases} \rho_{12} = \rho_{13} + \rho_{23}, \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{cases}$$

In this case, the points x_1 and x_3 lie between the points x_2 and x_4. It follows from the first equality of the system that the point x_3 lies between the points x_1 and x_2. By Theorem 2, the point x_2 lies beyond the points x_1 and x_3 and, by Definition 6, the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. Moreover, the point x_2 is extreme for the points $x_1,\ x_2,\ x_3$, and x_4.

Case (b):

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{cases}$$

In this case, the points x_1 and x_2 lie between the points x_3 and x_4. It follows from the first equality of the system that the point x_2 lies between the points x_1 and x_3. By Theorem 2, the point x_3 lies beyond the points x_1 and x_2 and, by Definition 6, the points $x_1,\ x_2,\ x_3$, and x_4 are rectilinearly ordered. Moreover, the point x_3 is extreme for the points $x_1,\ x_2,\ x_3$, and x_4.

3. Assume that the equalities $\varphi_{213} = -1,\ \varphi_{214} = 1$, and $\varphi_{314} = -1$ are simultaneously true. This is equivalent to the system

$$\begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \left[\begin{array}{l} \rho_{12} = \rho_{14} + \rho_{24}, \\ \rho_{14} = \rho_{12} + \rho_{24}, \end{array}\right. \\ \rho_{34} = \rho_{13} + \rho_{14}; \end{cases} \qquad \begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \left[\begin{array}{l} \rho_{24} = \rho_{12} - \rho_{14}, \\ \rho_{24} = -(\rho_{12} - \rho_{14}), \end{array}\right. \\ \rho_{34} = \rho_{13} + \rho_{14}; \end{cases} \qquad \begin{cases} \rho_{23}^2 = (\rho_{12} + \rho_{13})^2, \\ \rho_{24}^2 = (\rho_{12} - \rho_{14})^2, \\ \rho_{34}^2 = (\rho_{13} + \rho_{14})^2. \end{cases}$$

Substituting the obtained values in the left-hand side of equality (11), we get

$$\rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2 = (\rho_{12} + \rho_{13})^2 + (\rho_{13} + \rho_{14})^2 - (\rho_{12} - \rho_{14})^2$$

$$= \rho_{12}^2 + 2\rho_{12}\rho_{13} + \rho_{13}^2 + \rho_{13}^2 + 2\rho_{13}\rho_{14} + \rho_{14}^2 - \rho_{12}^2 + 2\rho_{12}\rho_{14} - \rho_{14}^2$$

$$= 2\rho_{13}^2 + 2\rho_{12}\rho_{13} + 2\rho_{13}\rho_{14} + 2\rho_{12}\rho_{14}$$

$$= 2(\rho_{13}(\rho_{13} + \rho_{12}) + \rho_{14}(\rho_{13} + \rho_{12}))$$

$$= 2(\rho_{13} + \rho_{12})(\rho_{13} + \rho_{14}) = 2\rho_{23}\rho_{34}.$$

In this case, the points x_2, x_3, and x_4 are also rectilinearly placed, which, in turn, implies the rectilinear placement of the points x_1, x_2, x_3, and x_4.

We now show that the points x_1, x_2, x_3, and x_4 are rectilinearly ordered. To this end, we consider all possible cases of their placement.

Case (a):

$$\begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \rho_{12} = \rho_{14} + \rho_{24}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{cases}$$

In this case, the points x_1 and x_4 lie between the points x_2 and x_3. It follows from the second equality of the system that the point x_4 lies between the points x_1 and x_2. By Theorem 2, the point x_2 lies beyond the points x_1 and x_4 and, hence, by Definition 6, the points x_1, x_2, x_3, and x_4 are rectilinearly ordered. Moreover, the point x_2 is extreme for the points x_1, x_2, x_3, and x_4.

Case (b):

$$\begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \rho_{14} = \rho_{12} + \rho_{24}, \\ \rho_{34} = \rho_{13} + \rho_{14}. \end{cases}$$

In this case, the points x_1 and x_2 lie between the points x_3 and x_4. It follows from the first equality of the system that the point x_1 lies between the points x_2 and x_3. By Theorem 2, the point x_3 lies beyond the points x_1 and x_2 and, thus, by Definition 6, the points x_1, x_2, x_3, and x_4 are rectilinearly ordered. Moreover, the point x_3 is extreme for the points x_1, x_2, x_3, and x_4.

4. Assume that the equalities $\varphi_{213} = -1$, $\varphi_{214} = -1$, and $\varphi_{314} = 1$ are simultaneously true. This is equivalent to the system

$$\begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \left[\begin{array}{l} \rho_{13} = \rho_{14} + \rho_{34}, \\ \rho_{14} = \rho_{13} + \rho_{34}; \end{array}\right. \end{cases} \qquad \begin{cases} \rho_{23} = \rho_{12} + \rho_{13}, \\ \rho_{24} = \rho_{12} + \rho_{14}, \\ \left[\begin{array}{l} \rho_{34} = \rho_{13} - \rho_{14}, \\ \rho_{34} = -(\rho_{13} - \rho_{14}); \end{array}\right. \end{cases} \qquad \begin{cases} \rho_{23}^2 = (\rho_{12} + \rho_{13})^2, \\ \rho_{24}^2 = (\rho_{12} + \rho_{14})^2, \\ \rho_{34}^2 = (\rho_{13} - \rho_{14})^2. \end{cases}$$

Substituting the obtained values in the left-hand side of equality (11), we find

$$\begin{aligned}\rho_{23}^2+\rho_{34}^2-\rho_{24}^2&=(\rho_{12}+\rho_{13})^2+(\rho_{13}-\rho_{14})^2-(\rho_{12}+\rho_{14})^2\\&=\rho_{12}^2+2\rho_{12}\rho_{13}+\rho_{13}^2+\rho_{13}^2-2\rho_{13}\rho_{14}+\rho_{14}^2-\rho_{12}^2-2\rho_{12}\rho_{14}-\rho_{14}^2\\&=2\rho_{13}^2+2\rho_{12}\rho_{13}-2\rho_{13}\rho_{14}-2\rho_{12}\rho_{14}\\&=2\big(\rho_{13}(\rho_{13}+\rho_{12})-\rho_{14}(\rho_{13}+\rho_{12})\big)\\&=2(\rho_{13}+\rho_{12})(\rho_{13}-\rho_{14})=\pm2\rho_{23}\rho_{34}.\end{aligned}$$

In this case, the points $x_2,\ x_3,$ and x_4 are also rectilinearly placed. Hence, the same is true for the points $x_1,$ $x_2,\ x_3,$ and $x_4.$

We now show that the points $x_1,\ x_2,\ x_3,$ and x_4 are rectilinearly ordered. To this end, we consider all possible cases of their arrangement.

Case (a):

$$\begin{cases}\rho_{23}=\rho_{12}+\rho_{13},\\ \rho_{24}=\rho_{12}+\rho_{14},\\ \rho_{13}=\rho_{14}+\rho_{34}.\end{cases}$$

In this case, the points x_1 and x_4 lie between the points x_2 and $x_3.$ It follows from the second equality of the system that the point x_1 lies between the points x_2 and $x_4.$ By Theorem 2, the point x_2 lies beyond the points x_1 and x_4 and, therefore, by Definition 6, the points $x_1,\ x_2,\ x_3,$ and x_4 are rectilinearly ordered. Moreover, the point x_2 is extreme for the points $x_1,\ x_2,\ x_3,$ and $x_4.$

Case (b):

$$\begin{cases}\rho_{23}=\rho_{12}+\rho_{13},\\ \rho_{24}=\rho_{12}+\rho_{14},\\ \rho_{14}=\rho_{13}+\rho_{34}.\end{cases}$$

In this case, the points x_1 and x_3 lie between the points x_2 and $x_4.$ It follows from the first equality of the system that the point x_1 lies between the points x_2 and $x_3.$ By Theorem 2, the point x_2 lies beyond the points x_1 and x_3 and, thus, by Definition 6, the points $x_1,\ x_2,\ x_3,$ and x_4 are rectilinearly ordered. Moreover, the point x_2 is extreme for the points $x_1,\ x_2,\ x_3,$ and $x_4.$

Thus, in all four possible cases, the points $x_2, x_3,$ and x_4 are rectilinearly placed, which, in turn, means that all points $x_1,\ x_2,\ x_3,$ and x_4 of the space Π are also rectilinearly placed. In addition, in all four cases, the points $x_1,\ x_2,\ x_3,$ and x_4 are rectilinearly ordered.

Lemma 3 is proved.

4.6. Proof of Lemma 4. Assume that the point x_1 is extreme for the points $x_1,\ x_2,\ x_3,$ and x_4. In this case, the equalities $\varphi_{213} = \varphi_{214} = \varphi_{314} = 1$ are true.

The equality $\varphi_{214} = 1$ is equivalent to the set of equalities (10). Since, under the condition of the lemma, the points x_2 and x_4 lie between the points x_1 and x_3, this is equivalent to the system of equalities

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \rho_{13} = \rho_{14} + \rho_{34}. \end{cases}$$

We finally arrive at the system of equalities

$$\begin{cases} \rho_{13} = \rho_{12} + \rho_{23}, \\ \rho_{13} = \rho_{14} + \rho_{34}, \\ \left[\begin{array}{l} \rho_{12} = \rho_{14} + \rho_{24}, \\ \rho_{14} = \rho_{12} + \rho_{24}. \end{array}\right. \end{cases}$$

By using the first equality in the system, we successively find

$$\rho_{23} = \rho_{13} - \rho_{12} = (\rho_{14} + \rho_{34}) - \rho_{12} = (\rho_{12} + \rho_{24}) + \rho_{34} - \rho_{12} = \rho_{24} + \rho_{34}.$$

By using the second equality in the system, we successively get

$$\rho_{34} = \rho_{13} - \rho_{14} = (\rho_{12} + \rho_{23}) - \rho_{14} = (\rho_{14} + \rho_{24}) + \rho_{23} - \rho_{14} = \rho_{23} + \rho_{24}.$$

We finally obtain the set of equalities

$$\left[\begin{array}{l} \rho_{23} = \rho_{24} + \rho_{34}, \\ \rho_{34} = \rho_{23} + \rho_{24}, \end{array}\right.$$

which is equivalent to the equalities

$$(\rho_{23} - \rho_{34})^2 = \rho_{24}^2, \qquad \rho_{23}^2 - 2\rho_{23}\rho_{34} + \rho_{34}^2 = \rho_{24}^2, \qquad \rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2 = 2\rho_{23}\rho_{34}.$$

Dividing both sides of the equality by $2\rho_{23}\rho_{34}$, we obtain

$$\varphi_{234} = \frac{\rho_{23}^2 + \rho_{34}^2 - \rho_{24}^2}{2\rho_{23}\rho_{34}} = 1.$$

This equality shows that the point x_3 lies beyond the points x_2 and x_4, i.e., it is extreme for the points x_1, $x_2,\ x_3,$ and x_4.

Lemma 4 is proved.

4.7. Proof of Lemma 5. Assume that the points $x_1,\ x_2,\ x_3,$ and x_4 in the space Π are rectilinearly and flatly placed. Their flat placement implies that equality (5) is true. By Definition 4, any three points from the set of points $x_1,\ x_2,\ x_3,$ and x_4 are rectilinearly placed. Thus, the equalities $\varphi_{213}^2 = \varphi_{214}^2 = \varphi_{314}^2 = 1$ are true. Substituting these values in equality (5), we obtain $\varphi_{213}\varphi_{214}\varphi_{314} = 1$. Hence, equality (4) is a necessary condition for the flat placement of the rectilinearly placed points $x_1,\ x_2,\ x_3,$ and x_4 in the space Π.

Now let the points $x_1,\ x_2,\ x_3,$ and x_4 satisfy equality (4). Then, by Lemma 3, these points are rectilinearly placed and the equalities $\varphi_{213}^2 = \varphi_{214}^2 = \varphi_{314}^2 = 1$ are true. Substituting these values and $\varphi_{213}\varphi_{214}\varphi_{314} = 1$ from equality (4) in the left-hand side of equality (5), we get the identity. Hence, by Definition 8, the points $x_1,\ x_2,\ x_3,$ and x_4 are flatly placed.

4.8. Proof of Theorem 3. Assume that a rectilinearly placed set of points of the space Π is also flatly placed. It is necessary to show that this set is rectilinearly ordered. To this end, we choose any four points from this set $x_1,\ x_2,\ x_3,$ and x_4. By Lemma 5, it follows from the rectilinear and flat placement of these points that equality (4) is true for one of these points (e.g., for x_1). By Lemma 3, these points are rectilinearly ordered. In view of the arbitrary choice of these points, by Definition 7, the entire set of points is rectilinearly ordered.

Now let the set of points of the space Π be rectilinearly ordered. By Definition 7, any four points $x_1,\ x_2,\ x_3,$ and x_4 of this set are rectilinearly ordered. Hence, by Lemma 3, equality (4) is true for one of these points (e.g., for x_1). Thus, by Lemma 5, the points $x_1,\ x_2,\ x_3,$ and x_4 are flatly placed. In view of the arbitrary choice of points of the set and Definition 7, the set of points is flatly placed.

Theorem 3 is proved.

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