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BOJANOV–NAIDENOV PROBLEM FOR FUNCTIONS WITH ASYMMETRIC RESTRICTIONS FOR THE HIGHER DERIVATIVE

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For given $r \in \mathbf{N},\ p, \alpha, \beta, \mu > 0,$ we solve the extreme problems

$$\int\limits_a^b x_{\pm}^q(t)dt \to \sup, \quad q \geq p,$$

in the set of pairs (x, I) of functions $x \in L_{\infty}^r$ and intervals $I = [a, b] \subset \mathbf{R}$ satisfying the inequalities $-\beta \leq x^{(r)}(t) \leq \alpha$ for almost all $t \in \mathbf{R}$, the conditions $L(x_{\pm})_p \leq L\big((\varphi_{\lambda,r}^{\alpha,\beta})_{\pm}\big)_p$, and the corresponding condition $\mu\left(\mathrm{supp}_{[a,b]} x_+\right) \leq \mu$ or $\mu\left(\mathrm{supp}_{[a,b]} x_-\right) \leq \mu$, where

$$L(x)_p := \sup\left\{\|x\|_{L_p[a,b]} :\, a, b \in \mathbf{R},\ |x(t)| > 0,\ t \in (a, b)\right\},$$

$\mathrm{supp}_{[a,b]} x_{\pm} := \{t \in [a, b] :\, x_{\pm}(t) > 0\}$, and $\varphi_{\lambda,r}^{\alpha,\beta}$ is an asymmetric $(2\pi/\lambda)$-periodic Euler spline of order r. As a consequence, we solve the same extreme problems for the intermediate derivatives $x_{\pm}^{(k)}$, $k = 1, \ldots, r-1$, with $q \geq 1$.

1. Introduction

Consider the spaces $L_p,\ 0 < p \leq \infty$, of all measurable functions $x \colon \mathbf{R} \to \mathbf{R}$ such that $\|x\|_p < \infty$, where

$$\|x\|_p := \begin{cases} \left(\displaystyle\int_{-\infty}^{+\infty} |x\,(t)|^p\, dt\right)^{1/p} & \text{for} \quad 0 < p < \infty, \\ \operatorname*{vrai\,sup}_{t \in \mathbf{R}} |x\,(t)| & \text{for} \quad p = \infty. \end{cases}$$

For $r \in \mathbf{N}$ and $p, s \in (0, \infty]$, by $L_{p,s}^r$ we denote the space of all functions $x \in L_p$ with locally absolutely continuous derivatives up to the order $(r-1)$, inclusively, such that $x^{(r)} \in L_s$. We write L_{∞}^r instead of $L_{\infty,\infty}^r$.

It is known (see, e.g., [1, p. 47]) that the problem of finding the exact constant C in the Kolmogorov–Nagy-type inequality

$$\left\|x^{(k)}\right\|_q \leq C\, \|x\|_p^{\alpha} \left\|x^{(r)}\right\|_s^{1-\alpha} \tag{1.1}$$

in a class of functions $x \in L_{p,s}^r$, where

$$\alpha = \frac{r - k + 1/q - 1/s}{r + 1/p - 1/s},$$

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and the parameters $q,\ p,\ s \geq 1,\ r \in \mathbf{N},\ k \in \mathbf{N}_0 := \mathbf{N}\bigcup\{0\},\ k < r,$ satisfy the condition $\alpha \leq (r-k)/r,$ is equivalent to the extreme problem

$$\left\|x^{(k)}\right\|_q \to \sup \tag{1.2}$$

in a class of functions $x \in L^r_{p,s}$ satisfying the restrictions

$$\left\|x^{(r)}\right\|_s \leq A_r, \qquad \|x\|_p \leq A_0, \tag{1.3}$$

where A_0 and A_r are given positive numbers.

Despite a great number of works available in this research field (see the references in [1–3]), the exact constant C in inequality (1.1) is known for all $k, r \in \mathbf{N},\ k < r,$ only in a few cases. Therefore, it is of interest to analyze the following modification of problem (1.2) with restrictions (1.3) considered by Bojanov and Naidenov [4]: For any segment $[a, b] \subset \mathbf{R}$, they solved the problem

$$\int_a^b \Phi\left(\left|x^{(k)}(t)\right|\right) dt \to \sup, \quad k = 1, \ldots, r-1,$$

in the class of functions $x \in L^r_\infty$ satisfying conditions (1.3) with $p = s = \infty$, where Φ is a function continuously differentiable in $[0, \infty)$, positive in $(0, \infty)$, and such that $\Phi(t)/t$ does not decrease and $\Phi(0) = 0$.

By W we denote a class of continuous nonnegative and convex functions Φ given on $[0, \infty)$ and such that $\Phi(0) = 0$. For $p > 0$, we set [5]

$$L(x)_p := \sup\left\{\left(\int_a^b |x\,(t)|^p\, dt\right)^{1/p} :\ a, b \in \mathbf{R},\ |x(t)| > 0,\ t \in (a, b)\right\}. \tag{1.4}$$

The following modification of the Bojanov–Naidenov problem was solved in [6]:

$$\int_a^b \Phi\left(|x(t)|^p\right) dt \to \sup, \qquad \Phi \in W, \quad p > 0, \tag{1.5}$$

in a class of functions $x \in L^r_\infty$ satisfying the restrictions

$$\left\|x^{(r)}\right\|_\infty \leq A_r, \quad L(x)_p \leq A_0. \tag{1.6}$$

As a consequence, we obtained the solution of the problem

$$\int_a^b \Phi\left(\left|x^{(k)}(t)\right|\right) dt \to \sup, \qquad \Phi \in W, \quad k = 1, \ldots, r-1, \tag{1.7}$$

in the class of all functions $x \in L^r_\infty$ satisfying conditions (1.6). The results presented in [6] were generalized in [7].

The sharp inequalities of the form (1.1) with asymmetric restrictions imposed on the higher derivative and related extreme problems with these restrictions were considered in [8–11].

Let $r \in \mathbf{N}$ and $\alpha,\ \beta > 0$. By $\varphi_r^{\alpha,\beta}(t)$ we denote the rth 2π-periodic integral (with zero mean value over the period) of a 2π-periodic function $\varphi_0^{\alpha,\beta}(t)$ defined on $[0, 2\pi]$ as follows:

$$\varphi_0^{\alpha,\beta}(t) := \begin{cases} \alpha & \text{for} \quad t \in [0, 2\pi\beta/(\alpha+\beta)], \\ \\ -\beta & \text{for} \quad t \in [2\pi\beta/(\alpha+\beta), 2\pi]. \end{cases}$$

For $\lambda > 0,$ we set $\varphi_{\lambda,r}^{\alpha,\beta}(t) := \lambda^{-r}\varphi_r^{\alpha,\beta}(\lambda t)$. Further, let

$$W^r_{\infty,\alpha,\beta} := \left\{ x \in L^r_\infty : \left\| \alpha^{-1} x_+^{(r)} + \beta^{-1} x_-^{(r)} \right\|_\infty \le 1 \right\}.$$

Consider a class

$$L_r(p, \alpha, \beta, \lambda) := \left\{ x \in W^r_{\infty,\alpha,\beta} \colon\ L(x_\pm)_p \le L\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_\pm \Big)_p \right\},$$

where $x_\pm(t) := \max\{x_\pm(t), 0\}$.

In the present paper, we solve the problem (Theorem 1)

$$\int\limits_a^b \Phi\big(x_\pm^p(t)\big)\,dt \to \sup, \qquad \Phi \in W, \quad p > 0, \tag{1.8}$$

in the set of pairs (x, I) of functions $x \in L_r(p, \alpha, \beta, \lambda)$ and segments $I = [a, b]$ satisfying the condition

$$\mu\big(\operatorname{supp}_{[a,b]} x_\pm\big) \le \mu, \quad \mu > 0.$$

We also solve the problem (Theorem 2)

$$\int\limits_a^b \Phi\big(x_\pm^{(k)}(t)\big)\,dt \to \sup, \qquad \Phi \in W, \quad k = 1, \ldots, r-1, \tag{1.9}$$

in the set of pairs (x, I) of functions $x \in L_r(p, \alpha, \beta, \lambda)$ and segments $I = [a, b]$ satisfying the condition

$$\mu\Big(\operatorname{supp}_{[a,b]} x_\pm^{(k)}\Big) \le \mu, \quad \mu > 0,$$

where

$$\operatorname{supp}_{[a,b]} x := \{t \in [a, b] \colon\ |x(t)| > 0\}.$$

Note that, in the symmetric case (i.e., for $\alpha = \beta$), problems (1.8) and (1.9) were solved in [12].

2. Auxiliary Statements

Lemma 1 [11]. *Suppose that* $r \in \mathbf{N}$ *and* $p,\ \alpha,\ \beta > 0$. *If, for the function* $x \in W^r_{\infty,\alpha,\beta}$, *the number* $\lambda > 0$ *is chosen so that*

$$L(x_\pm)_p \le L\Big(\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm\Big)_p,$$

where the quantity $L(x)_p$ *is given by equality (1.4), then*

$$\|x_\pm\|_\infty \le \Big\|\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm\Big\|_\infty.$$

Lemma 2 [11]. *Suppose that* $k, r \in \mathbf{N},\ k < r$, *and* $p,\ \alpha,\ \beta > 0$. *If, for the function* $x \in W^r_{\infty,\alpha,\beta}$, *the number* $\lambda > 0$ *is chosen so that*

$$L(x_\pm)_p \le L\Big(\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm\Big)_p,$$

then, for any $q \ge 1$,

$$L(x^{(k)}_\pm)_q \le L\Big(\big(\varphi^{\alpha,\beta}_{\lambda,r-k}\big)_\pm\Big)_q.$$

Corollary 1. *Let* $r \in \mathbf{N}$ *and let* $\alpha,\ \beta > 0$. *If the function* $x \in W^r_{\infty,\alpha,\beta}$ *satisfies the condition* $L(x)_p < \infty$ *with some* $p > 0$ *and* $|x(t)| > 0$ *for* $t \in (a, b)$, *where either* $a = -\infty$ *or* $b = +\infty$, *then* $x(t) \to 0$ *as* $t \to -\infty$ *or* $t \to +\infty$.

Under the conditions of Corollary 1, we assume that $x(-\infty) = 0$ or $x(+\infty) = 0$.

For a function x summable on the segment $[a, b]$, by $r(x, t)$ we denote the permutation of the function $|x|$ (see, e.g., [13], Sec. 1.3). We also assume that $r(x, t) = 0$ for $t > b - a$.

Lemma 3. *Suppose that* $r \in \mathbf{N},\ p > 0,\ \alpha,\ \beta > 0$, *and for a function* $x \in W^r_{\infty,\alpha,\beta}$, *the number* $\lambda > 0$ *is chosen so that the conditions*

$$L(x_\pm)_p \le L\Big(\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm\Big)_p, \tag{2.1}$$

are satisfied; here, $L(x)_p$ *is given by equality (1.4).*

If a (finite or infinite) interval $(a_\pm, b_\pm) \subset \mathbf{R}$ *and a segment* $[A_\pm, B_\pm] \subset \mathbf{R}$ *are such that*

$$x_\pm(a_\pm) = x_\pm(b_\pm) = 0, \qquad x_\pm(t) > 0, \quad t \in (a_\pm, b_\pm), \tag{2.2}$$

and

$$\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm(A_\pm) = \big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm(B_\pm) = 0, \qquad \big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm(t) > 0, \quad t \in (A_\pm, B_\pm), \tag{2.3}$$

then, for any $\xi > 0$ *and an arbitrary function* $\Phi \in W$, *the following inequalities are true:*

$$\int\limits_{a_\pm}^{a_\pm+\xi} \Phi\big(\overline{x}^p_\pm(t)\big)\,dt \le \int\limits_{A_\pm}^{A_\pm+\xi} \Phi\Big(\big(\overline{\varphi}^{\alpha,\beta}_{\lambda,r}\big)^p_\pm(t)\Big)\,dt \tag{2.4}$$

and

$$\int_{b_\pm-\xi}^{b_\pm} \Phi\big(\overline{x}^p_\pm(t)\big)dt \le \int_{B_\pm-\xi}^{B_\pm} \Phi\Big(\big(\overline{\varphi}^{\alpha,\beta}_{\lambda,r}\big)^p_\pm(t)\Big)\,dt, \tag{2.5}$$

where $\overline{x}_\pm$ *is the restriction of the function* $x_\pm$ *to* $(a_\pm, b_\pm)$ *and* $\big(\overline{\varphi}^{\alpha,\beta}_{\lambda,r}\big)_\pm$ *is the restriction of* $\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm$ *to* $[A_\pm, B_\pm]$*; moreover, the functions* $\overline{x}_\pm$ *and* $\big(\overline{\varphi}^{\alpha,\beta}_{\lambda,r}\big)_\pm$ *are set equal to zero outside these intervals.*

In addition, if

$$b_\pm - a_\pm \le B_\pm - A_\pm, \tag{2.6}$$

then, for any segment $[\alpha_\pm, \beta_\pm] \subset [A_\pm, B_\pm]$ *such that*

$$\beta_\pm - \alpha_\pm = b_\pm - a_\pm, \tag{2.7}$$

the inequality

$$\int_{a_\pm}^{b_\pm} \Phi\big(x^p_\pm(t)\big)\,dt \le \int_{\alpha_\pm}^{\beta_\pm} \Phi\Big(\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)^p_\pm(t)\Big)\,dt, \quad \Phi \in W, \tag{2.8}$$

is true.

Proof. We fix a function x and intervals $(a_\pm, b_\pm)$ and $[A_\pm,\ B_\pm]$ satisfying the conditions of Lemma 3. It is necessary to prove inequality (2.4) [inequality (2.5) is proved in a similar way].

We first prove the inequality

$$\int_0^{\xi} r^p(\overline{x}_\pm, t)\,dt \le \int_0^{\xi} r^p(\overline{\varphi}_\pm, t)\,dt, \quad \xi > 0, \tag{2.9}$$

where, for the sake of brevity, we set $\overline{\varphi}_\pm := \big(\overline{\varphi}^{\alpha,\beta}_{\lambda,r}\big)_\pm$. First, we show that the difference

$$\delta_\pm(t) := r(\overline{x}_\pm, t) - r(\overline{\varphi}_\pm, t)$$

changes its sign on $[0,\infty)$ (from minus to plus) at most once. To prove this, we note that

$$\delta_\pm(0) \le \|x_\pm\|_\infty - \big\|\big(\varphi^{\alpha,\beta}_{\lambda,r}\big)_\pm\big\|_\infty \le 0 \tag{2.10}$$

by Lemma 1. By virtue of this inequality and relations (2.2) and (2.3), for any $z_\pm \in (0, \|\overline{x}_\pm\|_\infty)$, there exist points

$$t^\pm_i \in (a_\pm, b_\pm), \quad i = 1,\dots,m, \quad m \ge 2,$$

$$y^\pm_j \in (A_\pm, B_\pm), \quad j = 1, 2,$$

such that

$$z_{\pm} = \overline{x}_{\pm}(t_i^{\pm}) = \overline{\varphi}_{\pm}(y_j^{\pm}), \qquad \overline{\varphi}'_{\pm}(y_1^{\pm}) > 0, \qquad \overline{\varphi}'_{\pm}(y_2^{\pm}) < 0. \tag{2.11}$$

In this case, $\left|x'_{\pm}(t_i^{\pm})\right| \neq 0,\ i = 1, \ldots, m,$ for almost all $z_{\pm} \in \left(0, \|\overline{x}_{\pm}\|_{\infty}\right)$ and, in addition, among the points $t_i^{\pm}$, there exist at least one point $t_{i_1}^{\pm}$ and one point $t_{i_2}^{\pm}$ for which

$$\overline{x}'_{\pm}\left(t_{i_1}^{\pm}\right) > 0, \qquad \overline{x}'_{\pm}(t_{i_2}^{\pm}) < 0. \tag{2.12}$$

By virtue of the inclusion $x \in W^r_{\infty,\alpha,\beta}$ and inequality (2.10), all conditions of the Hörmander comparison theorem are satisfied [8] (see also [1, p. 96]). According to this theorem, for the points $t_i^{\pm}$ and $y_j^{\pm}$ satisfying conditions (2.11) and (2.12), we get

$$\left|\overline{x}'_{\pm}\left(t_{i_1}^{\pm}\right)\right| \leq \left|\overline{\varphi}'_{\pm}(y_1^{\pm})\right|, \qquad \left|\overline{x}'_{\pm}(t_{i_2}^{\pm})\right| \leq \left|\overline{\varphi}'_{\pm}(y_2^{\pm})\right|.$$

Hence, if the points $\theta_1^{\pm}, \theta_2^{\pm} > 0$ are chosen so that

$$z_{\pm} = r\left(\overline{x}_{\pm}, \theta_1^{\pm}\right) = r\left(\overline{\varphi}_{\pm}, \theta_2^{\pm}\right),$$

then, by the theorem on the derivative of permutation (see, e.g., [13], Proposition 1.3.2), we find

$$\left|r'(\overline{x}_{\pm}, \theta_1^{\pm})\right| = \left[\sum_{i=1}^{m} |\overline{x}'_{\pm}(t_i^{\pm})|^{-1}\right]^{-1}$$

$$\leq \left[\sum_{j=1}^{2} |\overline{\varphi}'_{\pm}(y_j^{\pm})|^{-1}\right]^{-1} = \left|r'(\overline{\varphi}_{\pm}, \theta_2^{\pm})\right|.$$

This means that the difference

$$\delta^{\pm}(t) := r(\overline{x}_{\pm}, t) - r(\overline{\varphi}_{\pm}, t)$$

changes its sign on $[0, \infty)$ (from minus to plus) at most once. The same is true for the difference

$$\delta_p^{\pm}(t) := r^p(\overline{x}_{\pm}, t) - r^p(\overline{\varphi}_{\pm}, t).$$

Consider the integral

$$I_p^{\pm}(\xi) := \int_0^{\xi} \delta_p^{\pm}(t)\, dt, \quad \xi \geq 0.$$

It is clear that $I_p^{\pm}(0) = 0$ and, by virtue of condition (2.1), for $\xi \geq \max\{b_{\pm} - a_{\pm}, B_{\pm} - A_{\pm}\}$, we get

$$I_p^{\pm}(\xi) \leq L(x_{\pm})_p - L\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}\right)_p \leq 0.$$

Furthermore, the derivative $(I_p^{\pm})'(t) = \delta^{\pm}(t)$ changes its sign on $[0,\infty)$ (from minus to plus) at most once. Therefore, $I_p^{\pm}(\xi) \le 0$ for all $\xi \ge 0$. Inequality (2.9) is proved.

By virtue of the Hardy–Littlewood–Pólya theorem (see, e.g., [13], Theorem 1.3.11), this inequality implies that

$$\int\limits_{a_\pm}^{b_\pm} \Phi\big(x_\pm^p(t)\big)\,dt \le \int\limits_{A_\pm}^{B_\pm} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_\pm^p(t)\Big)\,dt, \quad \Phi \in W. \tag{2.13}$$

We now prove inequality (2.4). Passing to the shifts of the functions $\overline{x}$ and $\big(\overline{\varphi}_{\lambda,r}^{\alpha,\beta}\big)$, we can write

$$a_\pm = A_\pm = 0. \tag{2.14}$$

Thus, by the Hörmander comparison theorem, the difference

$$\Delta^{\pm}(t) := \overline{x}_\pm(t) - \overline{\varphi}_\pm(t)$$

changes its sign on $[0,\infty)$ (from minus to plus) at most once. Since the functions $f(t) = t^p$ and $\Phi \in W$ are monotonically increasing, the same is true for the difference

$$\Delta_\Phi^{\pm}(t) := \Phi\big(\overline{x}_\pm^p(t)\big) - \Phi\big(\overline{\varphi}_\pm^p(t)\big), \quad t > 0.$$

We set

$$I_\Phi^{\pm}(\xi) := \int\limits_0^{\xi} \Delta_\Phi^{\pm}(t)\,dt, \quad \xi \ge 0.$$

It is clear that $I_\Phi^{\pm}(0) = 0$. Taking into account inequality (2.13) and assumption (2.14), we get

$$I_\Phi^{\pm}(\xi) \le \int\limits_{a_\pm}^{b_\pm} \Phi\big(x_\pm^p(t)\big)\,dt - \int\limits_{A_\pm}^{B_\pm} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_\pm^p(t)\Big)\,dt \le 0$$

for

$$\xi \ge \max\{b_\pm - a_\pm, B_\pm - A_\pm\}.$$

In addition, the derivative $\big(I_\Phi^{\pm}\big)'(t) = \Delta_\Phi^{\pm}(t)$ changes its sign on $[0,\infty)$ (from minus to plus) at most once. Hence,

$$I_\Phi^{\pm}(\xi) \le 0 \quad \text{for all} \quad \xi \ge 0.$$

By virtue of assumption (2.14), this is equivalent to inequality (2.4).

It remains to prove inequality (2.8) under conditions (2.6) and (2.7). Assume that the last two conditions are satisfied. Thus, shifting (if necessary) the function x, we can assume that

$$a_\pm = \alpha_\pm, \qquad b_\pm = \beta_\pm. \tag{2.15}$$

By the Hörmander comparison theorem (as indicated above, its conditions are satisfied), the inequalities

$$x_{\pm}(t) \le \left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}(t), \quad t \in [a_{\pm}, b_{\pm}],$$

are true. Hence, by virtue of assumption (2.15), we directly arrive at inequality (2.8).

Lemma 3 is proved.

In the course of the proof of Lemma 3, we have, in fact, established inequality (2.13). Setting in this inequality

$$\Phi(t) = t^{q/p}, \quad \text{where} \quad q \ge p,$$

and using conditions (2.2) and (2.3) and the definition of the quantity $L(x)_q$ (1.4), we arrive at the following corollary:

Corollary 2. *Under the conditions of Lemma 3, for any function $\Phi \in W$, the inequality*

$$\int\limits_{a_{\pm}}^{b_{\pm}} \Phi\left(x_{\pm}^{p}(t)\right) dt \le \int\limits_{A_{\pm}}^{B_{\pm}} \Phi\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}^{p}(t)\right) dt = \int\limits_{0}^{2\pi/\lambda} \Phi\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}^{p}(t)\right) dt$$

is true. In particular, for any $q \ge p$,

$$L(x_{\pm})_q \le L\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}\right)_q.$$

We set

$$d_r^{\pm} := \mu\left(\operatorname{supp}_{[0,\, 2\pi/\lambda]}\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}\right), \tag{2.16}$$

where the set $\operatorname{supp}_{[a,\, b]} x_{\pm}$ is given by equality (1.8). Note that

$$d_r^{+} + d_r^{-} = 2\pi/\lambda.$$

Moreover, for odd r, $d_r^{+} = d_r^{-}$.

Lemma 4. *Let $r \in \mathbf{N}$ and $p, \alpha, \beta > 0$. Assume that, for a function $x \in W_{\infty,\alpha,\beta}^{r}$, the number $\lambda > 0$ is chosen to guarantee that the following conditions are satisfied:*

$$L(x_{\pm})_p \le L\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}\right)_p, \tag{2.17}$$

where the quantity $L(x)_p$ is given by equality (1.4). If the segment $[a, b] \subset \mathbf{R}$ satisfies the condition

$$\delta_{+} := \mu\left(\operatorname{supp}_{[a,b]} x_{+}\right) \le d_r^{+} \tag{2.18}$$

or the condition

$$\delta_{-} := \mu\left(\operatorname{supp}_{[a,b]} x_{-}\right) \le d_r^{-}, \tag{2.19}$$

then, for any function $\Phi \in W$, *either the inequality*

$$\int\limits_a^b \Phi\big(x_+^p(t)\big)\,dt \le \int\limits_{m^+-\Theta_1^+}^{m^++\Theta_2^+} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\,dt \tag{2.20}$$

or, respectively, the inequality

$$\int\limits_a^b \Phi\big(x_-^p(t)\big)dt \le \int\limits_{m^--\Theta_1^-}^{m^-+\Theta_2^-} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_-^p(t)\Big)\,dt \tag{2.21}$$

is true. Here, $m^\pm$ *are points of local maxima of the function* $\big(\varphi_{\lambda,r}^{\alpha,\beta}(t)\big)_\pm$, *and the quantities* $\Theta_1^\pm,\ \Theta_2^\pm > 0$ *are such that*

$$\varphi_{\lambda,r}^{\alpha,\beta}\big(m^\pm - \Theta_1^\pm\big) = \varphi_{\lambda,r}^{\alpha,\beta}\big(m^\pm + \Theta_2^\pm\big). \tag{2.22}$$

In addition,

$$\Theta_1^\pm + \Theta_2^\pm = \delta_\pm. \tag{2.23}$$

Note that $\Theta_1^\pm = \Theta_2^\pm$ for even r.

Proof. We fix a function $x \in W_{\infty,\alpha,\beta}^r$ and a segment $[a,b]$ satisfying the conditions of Lemma 4. We prove inequality (2.20) under condition (2.18) [inequality (2.21) under condition (2.19) is proved similarly]. Assume that

$$x_+(a) > 0, \qquad x_+(b) > 0 \tag{2.24}$$

[if at least one of these inequalities is not true, then the proof of inequality (2.20) is simplified].

If the function x does not have zeros in (a,b), then, by Corollary 1 of Lemma 2, there exists an (finite or infinite) interval (c,d) such that $(a,b) \subset (c,d)$ and, in addition,

$$x_+(c) = x_+(d) = 0, \qquad x_+(t) > 0, \quad t \in (c,d).$$

By $\overline{x}_+$ we denote the restriction of x_+ to (c,d) and by $\overline{\varphi}_+$ we denote the restriction of $\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+$ to $[0, 2\pi/\lambda]$. Repeating the reasoning used in the proof of inequality (2.9), we obtain

$$\int\limits_0^\xi r^p(\overline{x}_+, t)\,dt \le \int\limits_0^\xi r^p(\overline{\varphi}_+, t)\,dt, \quad \xi > 0.$$

By virtue of the Hardy–Littlewood–Pólya theorem (see, e.g., [13], Theorem 1.3.11), we get

$$\int\limits_0^\xi \Phi\big(r^p(\overline{x}_+, t)\big)\,dt \le \int\limits_0^\xi \Phi\big(r^p(\overline{\varphi}_+, t)\big)\,dt, \qquad \Phi \in W, \quad \xi > 0.$$

Hence,

$$\int\limits_a^b \Phi\big((\overline{x}_+^p(t)\big)\,dt = \int\limits_0^{b-a} \Phi\big(r^p(\overline{x}_+,t)\big)dt \le \int\limits_0^{b-a} \Phi\big(r^p(\overline{\varphi}_+(t)\big)\,dt.$$

In the case where x does not have zeros in (a,b), inequality (2.20) follows from the obvious equality

$$\int\limits_0^{b-a} \Phi\big(r^p(\overline{\varphi}_+(t)\big)\,dt = \int\limits_{m^+-\Theta_1^+}^{m^++\Theta_2^+} \Phi\Big((\varphi_{\lambda,r}^{\alpha,\beta})_+^p(t)\Big)\,dt,$$

where m^+ is the point of local maximum of the spline $\big(\varphi_{\lambda,r}^{\alpha,\beta}(t)\big)_+$ and $\Theta_1^+,\Theta_2^+ > 0$ satisfy conditions (2.22) and (2.23); moreover, $\delta_+ = b-a$.

We now assume that x have zeros in (a,b) and set

$$a' := \inf\{t\in(a,b):\ x_+(t)=0\}, \qquad b' := \sup\{t\in(a,b):\ x_+(t)=0\}.$$

Thus, by virtue of (2.24), the support $\mathrm{supp}_{[a,b]}x_+$ has the form

$$\mathrm{supp}_{[a,b]}x_+ = (a,a')\bigcup(b',b)\bigcup\bigcup_k(a_k,b_k), \tag{2.25}$$

where $(a_k,b_k)\subset(a',b')$. Moreover,

$$x_+(a_k)=x_+(b_k)=0, \qquad x_+(t)>0, \quad t\in(a_k,b_k)$$

(note that the set of these intervals (a_k,b_k) may be empty). By virtue of relation (2.18), assumption (2.24), and the definitions of the quantities a' and b', we get

$$\delta_+ = (a'-a)+(b-b')+\sum_k(b_k-a_k) \le d_r^+. \tag{2.26}$$

Let A_+ and B_+ be two neighboring zeros of the spline $\varphi_{\lambda,r}^{\alpha,\beta}$ and, in addition, $\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+(t)>0$ for $t\in(A_+,B_+)$. By virtue of Corollary 1, there exist (finite or infinite) intervals (α',a') and (b',β') such that

$$x_+(\alpha')=x_+(a')=0, \qquad x_+(t)>0, \quad t\in(\alpha',a'),$$

and

$$x_+(b')=x_+(\beta')=0, \qquad x_+(t)>0, \quad t\in(b',\beta').$$

Applying inequalities (2.4) and (2.5) to the intervals (α',a') and (b',β') and the segment $[A_+,B_+]$, we find

$$\int\limits_{b'}^{b} \Phi\big(x_+^p(t)\big)\,dt \le \int\limits_{A_+}^{A_++\xi} \Phi\Big((\varphi_{\lambda,r}^{\alpha,\beta})_+^p(t)\Big)\,dt, \quad \xi=b-b', \tag{2.27}$$

and

$$\int_{a}^{a'} \Phi\big(x_+^p(t)\big)\, dt \le \int_{B_+-\eta}^{B_+} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt, \quad \eta = a' - a, \tag{2.28}$$

$\big[$by virtue of (2.26), in inequality (2.4), $\overline{x}_+$ can be replaced with x_+ and $\big(\overline{\varphi}_{\lambda,r}^{\alpha,\beta}\big)_+$ can be replaced with $\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+\big]$. In view of (2.26), there exist mutually disjoint intervals (α_k, β_k) such that

$$(\alpha_k, \beta_k) \subset (A_+ + \xi,\ B_+ - \eta), \qquad \beta_k - \alpha_k = b_k - a_k.$$

By virtue of relation (2.8), the inequality

$$\int_{a_k}^{b_k} \Phi\big(x_+^p(t)\big)\, dt \le \int_{\alpha_k}^{\beta_k} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt \tag{2.29}$$

is true for these intervals. Finding the sum of estimates (2.27)–(2.29) and taking into account (2.25), we get

$$\int_{a}^{b} \Phi\big(x_+^p(t)\big)\, dt = \int_{a}^{a'} \Phi\big(x_+^p(t)\big)\, dt + \int_{b'}^{b} \Phi\big(x_+^p(t)\big)\, dt + \sum_k \int_{a_k}^{b_k} \Phi\big(x_+^p(t)\big)\, dt$$

$$\le \int_{A_+}^{A_++\xi} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt + \int_{B_+-\eta}^{B_+} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt$$

$$+ \sum_k \int_{\alpha_k}^{\beta_k} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt.$$

Since $\beta_k - \alpha_k = b_k - a_k$, by virtue of (2.26), we conclude that

$$\xi + \eta + \sum_k (\beta_k - \alpha_k) = \delta_+.$$

Hence, the sum of the integrals on the right-hand side of the obtained estimate does not exceed

$$\int_{0}^{\delta_+} r\Big(\Phi\Big(\big(\overline{\varphi}_{\lambda,r}^{\alpha,\beta}\big)_+^p, t\Big)\Big)\, dt = \int_{m^+-\Theta_1^+}^{m^++\Theta_2^+} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\, dt,$$

where $\big(\overline{\varphi}_{\lambda,r}^{\alpha,\beta}\big)_+$ is the restriction of $\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+$ to $[A_+, B_+]$, m^+ is the point of local maximum of the function $\big(\varphi_{\lambda,r}^{\alpha,\beta}(t)\big)_+$, and $\Theta_1^+,\ \Theta_2^+ > 0$ satisfy relations (2.22) and (2.23). Inequality (2.20) is proved.

Lemma 4 is proved.

Corollary 3. *If, under the conditions of Lemma 4, the inequality*

$$\mu\Big(\mathrm{supp}_{[a,b]}x_+\Big) \le d_r^+$$

is satisfied, then the inequality

$$\int\limits_a^b \Phi\big(x_+^p(t)\big)\,dt \le \int\limits_0^{2\pi/\lambda} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_+^p(t)\Big)\,dt \tag{2.30}$$

holds. At the same time, under the condition

$$\mu\Big(\mathrm{supp}_{[a,b]}x_-\Big) \le d_r^-,$$

the following inequality is true:

$$\int\limits_a^b \Phi\big(x_-^p(t)\big)dt \le \int\limits_0^{2\pi/\lambda} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_-^p(t)\Big)\,dt.$$

3. Main Results

Let $r \in \mathbf{N}$ and let $p,\ \alpha,\ \beta,\ \lambda > 0$. Recall that

$$L_r(p,\alpha,\beta,\lambda) := \left\{x \in W^r_{\infty,\alpha,\beta}\colon\ L(x_\pm)_p \le L\left(\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_\pm\right)_p\right\}, \tag{3.1}$$

where the quantity $L(x)_p$ is given by equality (1.4). We fix a number $\mu > 0$ and introduce a set of pairs (x, I) of functions x and segments $I = [a,b]$ by the equality

$$L_r^\pm(p,\alpha,\beta,\lambda,\mu) := \big\{(x,I)\colon\ x \in L_r(p,\alpha,\beta,\lambda),\ \mathrm{supp}_{[a,b]}x_\pm \le \mu\big\}. \tag{3.2}$$

We also recall that

$$d_r^\pm := \mu\Big(\mathrm{supp}_{[0,\,2\pi/\lambda]}\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_\pm\Big). \tag{3.3}$$

It is clear that $d_r^+ + d_r^- = 2\pi/\lambda$ and, moreover, $d_r^+ = d_r^-$ for odd r. We represent the number μ either in terms of d_r^+ or in terms of d_r^- as follows:

$$\mu = n_\pm d_r^\pm + \Theta_1^\pm + \Theta_2^\pm, \quad n_\pm \in \mathbf{N}\bigcup\{0\}, \qquad \Theta_1^\pm,\ \Theta_2^\pm,\ \Theta_1^\pm + \Theta_2^\pm \in [0,\ d_r^\pm); \tag{3.4}$$

in addition, $\Theta_1^\pm = \Theta_2^\pm$ for even r.

Note that if the numbers $\tau^\pm \in \mathbf{R}$ and the segment $[A, B]$ are such that

$$B - A = n_\pm\frac{2\pi}{\lambda} + \Theta_1^\pm + \Theta_2^\pm, \tag{3.5}$$

$$\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}(A+\Theta_1^{\pm}+\tau^{\pm}) = \left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}(B-\Theta_2^{\pm}+\tau^{\pm}) = \left\|\left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_{\pm}\right\|_{\infty}, \tag{3.6}$$

then

$$\left(\varphi_{\lambda,r}^{\alpha,\beta}(\cdot+\tau^{\pm}), [A,B]\right) \in L_r^{\pm}(p,\alpha,\beta,\lambda,\mu).$$

Theorem 1. *Suppose that $r \in \mathbf{N}$ and $p,\ \alpha,\ \beta,\ \lambda,\ \mu > 0$. Then, for any function $\Phi \in W$,*

$$\sup\left\{\int_a^b \Phi\big(x_{\pm}^p(t)\big)\,dt : (x,I) \in L_r^{\pm}(p,\alpha,\beta,\lambda,\mu)\right\} = \int_A^B \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_{\pm}^p(t+\tau^{\pm})\Big)\,dt,$$

where the sets $L_r^{\pm}(p,\alpha,\beta,\lambda,\mu)$, the numbers $\tau^{\pm}$, and the segment $[A,B]$ are given by relations (3.1)–(3.6).

Proof. We fix a pair

$$(x,I) \in L_r^{\pm}(p,\alpha,\beta,\lambda,\mu)$$

and prove the theorem for x_+ (for x_-, the proof is similar). We first establish the inequality

$$I := \int_a^b \Phi\big(x_+^p(t)\big)\,dt \le \int_A^B \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_{+}^p(t+\tau^{+})\Big)\,dt := I(\mu). \tag{3.7}$$

First, we consider the case where $\operatorname{supp}_{[a,\,b]} x_+ = \mu$. Since μ admits representation (3.4), we represent the segment $[a,b]$ as follows:

$$[a,b] = \bigcup_{k=1}^{n_+} [\alpha_k,\beta_k] \bigcup [\alpha,\beta].$$

Moreover, the intervals (α_k,β_k) and (α,β) are mutually disjoint and, in addition,

$$\mu\big(\operatorname{supp}_{[\alpha_k,\,\beta_k]} x_+\big) = d_r^+, \qquad \mu\big(\operatorname{supp}_{[\alpha,\beta]} x_+\big) = \Theta_1^+ + \Theta_2^+.$$

Thus, we get

$$\int_a^b \Phi\big(x_+^p(t)\big)\,dt = \sum_{k=1}^{n_+} \int_{\alpha_k}^{\beta_k} \Phi\big(x_+^p(t)\big)\,dt + \int_{\alpha}^{\beta} \Phi\big(x_+^p(t)\big)\,dt.$$

To estimate the integrals on the right-hand side of the last equality, we use inequalities (2.30) and (2.20) and obtain

$$\int_a^b \Phi\big(x_+^p(t)\big)\,dt \le n_+ \int_0^{2\pi/\lambda} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_{+}^p(t)\Big)\,dt + \int_{m^+-\Theta_1^+}^{m^++\Theta_2^+} \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_{+}^p(t)\Big)\,dt$$

$$= \int_A^B \Phi\Big(\big(\varphi_{\lambda,r}^{\alpha,\beta}\big)_{+}^p(t+\tau^{+})\Big)\,dt,$$

where m^+ is the point of maximum of the spline $\varphi^{\alpha,\beta}_{\lambda,r}$. The last equality in this chain follows from (3.5) and (3.6). Thus, inequality (3.7) is proved in the case where $\mathrm{supp}_{[a,\,b]}x_+ = \mu$.

Now let

$$\mu_1 := \mathrm{supp}_{[a,\,b]}x_+ < \mu^+.$$

Note that the number μ is uniquely represented in the form (3.4) (in terms of d_r^+) and, hence, the number uniquely (to within translations) specifies the segment $[A, B]$ and the number τ^+. Therefore, the integral $I(\mu)$ on the right-hand side of (3.7) is uniquely determined by the number μ. It is clear that $I(\mu)$ is a strictly increasing function of μ. Repeating the reasoning used in the previous case, we arrive at the following estimate for the integral I on the left-hand side of (3.7):

$$I \le I(\mu_1) < I(\mu).$$

Hence, inequality (3.7) is completely proved. It remains to note that, for the function

$$x(\cdot) = \varphi^{\alpha,\beta}_{\lambda,r}(\cdot + \tau^+)$$

and the segment $[A, B]$ given by relations (3.5) and (3.6), inequality (3.7) turns into the equality.

Theorem 1 is proved.

Let $k,\ r \in \mathbf{N},\ k < r$ and let $p,\ \alpha,\ \beta,\ \lambda > 0$. By virtue of Lemma 2, if $x \in L_r(p, \alpha, \beta, \lambda)$ [this set is defined by equality (3.1)], then $x^{(k)} \in L_{r-k}(q, \alpha, \beta, \lambda)$ for any $q \ge 1$.

We fix a number $\mu > 0$ and introduce a set of pairs (x, I) of functions x and segments $I = [a, b]$ by the equality

$$L^{\pm}_{r,k}(p, \alpha, \beta, \lambda, \mu) := \Big\{(x, I)\colon\ x \in L_r(p, \alpha, \beta, \lambda),\ \mathrm{supp}_{[a,b]}x_{\pm}^{(k)} \le \mu\Big\}. \tag{3.8}$$

Further, we represent the number μ in terms of d^+_{r-k} or d^-_{r-k} as follows:

$$\mu = n_{\pm}d^{\pm}_{r-k} + \Theta_1^{\pm} + \Theta_2^{\pm},$$
$$n_{\pm} \in \mathbf{N}\bigcup\{0\}, \qquad \Theta_1^{\pm},\ \Theta_2^{\pm},\ \Theta_1^{\pm} + \Theta_2^{\pm} \in [0,\ d^{\pm}_{r-k}). \tag{3.9}$$

Moreover, $\Theta_1^{\pm} = \Theta_2^{\pm}$ for even $r - k$ and the quantities $d_r^{\pm}$ are given by relation (3.3).

Finally, we choose numbers $\tau^{\pm} \in \mathbf{R}$ and a segment $[A,\ B]$ such that

$$B - A = n_{\pm}\frac{2\pi}{\lambda} + \Theta_1^{\pm} + \Theta_2^{\pm}, \tag{3.10}$$

$$\big(\varphi^{\alpha,\beta}_{\lambda,r-k}\big)_{\pm}(A + \Theta_1^{\pm} + \tau^{\pm}) = \big(\varphi^{\alpha,\beta}_{\lambda,r-k}\big)_{\pm}(B - \Theta_2^{\pm} + \tau^{\pm}) = \left\|\Big(\varphi^{\alpha,\beta}_{\lambda,r-k}\Big)_{\pm}\right\|_{\infty}. \tag{3.11}$$

This enables us to conclude that

$$\Big(\varphi^{\alpha,\beta}_{\lambda,r}(\cdot + \tau^{\pm}), [A, B]\Big) \in L^{\pm}_{r,k}(p, \alpha, \beta, \lambda, \mu).$$

Theorem 2. *Suppose that* $k,\ r \in \mathbf{N},\ k < r,$ *and* $p,\ \alpha,\ \beta,\ \lambda,\ \mu > 0.$ *Then, for any function* $\Phi \in W,$

$$\sup\left\{\int\limits_a^b \Phi\Big(x_\pm^{(k)}(t)\Big)\,dt\colon\ (x,I) \in L_{r,k}^\pm(p,\alpha,\beta,\lambda,\mu)\right\} = \int\limits_A^B \Phi\left(\left(\varphi_{\lambda,r-k}^{\alpha,\beta}\right)_\pm (t+\tau^\pm)\right)dt,$$

where the sets $L_{r,k}^\pm(p,\alpha,\beta,\lambda,\mu)$*, the numbers* $\tau^\pm$*, and the segment* $[A,B]$ *are given by relations (3.1) and (3.8)–(3.11).*

Proof. By virtue of Lemma 2, the following implication is true:

$$x \in L_r(p,\alpha,\beta,\lambda) \Rightarrow x^{(k)} \in L_{r-k}(1,\alpha,\beta,\lambda).$$

This yields

$$(x,I) \in L_{r,k}^\pm(p,\alpha,\beta,\lambda,\mu) \Rightarrow (x^{(k)},I) \in L_{r-k}^\pm(1,\alpha,\beta,\lambda,\mu).$$

Applying Theorem 1 to the class $L_{r-k}^\pm(1,\alpha,\beta,\lambda,\mu),$ we arrive at the assertion of Theorem 2.

Theorem 2 is proved.

Setting $\Phi(t) = t^{q/p}$ in Theorem 1 and $\Phi(t) = t^q$ in Theorem 2, we get the following corollary:

Corollary 4. *Let* $r \in \mathbf{N}$ *and let* $p,\ \alpha,\ \beta,\ \lambda,\ \mu > 0.$ *Then, for any* $q \geq p,$

$$\sup\left\{\int\limits_a^b x_\pm^q(t)dt\colon\ (x,I) \in L_r^\pm(p,\alpha,\beta,\lambda,\mu)\right\} = \int\limits_A^B \left(\varphi_{\lambda,r}^{\alpha,\beta}\right)_\pm^q (t+\tau^\pm)\,dt,$$

where the sets $L_r^\pm(p,\alpha,\beta,\lambda,\mu)$*, the numbers* $\tau^\pm$*, and the segment* $[A,B]$ *are given by relations (3.1)–(3.6).*

Furthermore, for any $k \in \mathbf{N},\ k < r,$ *and* $q \geq 1,$

$$\sup\left\{\int\limits_a^b \left(x_\pm^{(k)}(t)\right)^q dt\colon\ (x,I) \in L_{r,k}^\pm(p,\alpha,\beta,\lambda,\mu)\right\} = \int\limits_A^B \left(\left(\varphi_{\lambda,r-k}^{\alpha,\beta}\right)_\pm^q (t+\tau^\pm)\right)dt,$$

where the sets $L_{r,k}^\pm(p,\alpha,\beta,\lambda,\mu)$*, the numbers* $\tau^\pm$*, and the segment* $[A,B]$ *are given by relations (3.1) and (3.8)–(3.11).*

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