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ON THE LAW OF ITERATED LOGARITHM FOR THE MAXIMUM SCHEME IN IDEAL BANACH SPACES

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We obtain asymptotic estimates in the law of iterated logarithm for the extreme values of a sequence of independent random variables in Banach spaces.

1. Introduction

Let $\xi,\ \xi_1,\ \xi_2, \ldots$ be a sequence of independent random variables with a distribution function $F(x)$. Assume that F has a positive derivative $F'(x)$ for all sufficiently large x, i.e., there exists a number x_0 such that

$$F'(x) > 0 \quad \forall x \in [x_0; +\infty]. \tag{1}$$

We set

$$z_n = \max_{1 \le i \le n} \xi_i.$$

The law of iterated logarithm for the maximum scheme in the one-dimensional case was studied in [1–3]. It is known (see, e.g., [1, 4]) that the asymptotic properties of the sequence (z_n) are closely connected with the asymptotic behavior of the functions

$$f(x) = \frac{1 - F(x)}{F'(x)}, \qquad g(x) = f(x) \ln\ln \left\{ \frac{1}{1 - F(x)} \right\}.$$

Thus, in [1], the following asymptotic relations were obtained for independent random variables [almost surely (a.s.)]:

$$\limsup_{n\to\infty} \frac{z_n - a_n}{f(a_n) \ln\ln n} = 1, \tag{2}$$

$$\liminf_{n\to\infty} \frac{z_n - a_n}{f(a_n) \ln\ln n} = 0, \tag{3}$$

where

$$a_n = F^{-1}\left(1 - \frac{1}{n}\right),$$

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$$F^{-1}(y) = \inf\{x:\ F(x) \geq y\} \quad \text{is inverse to} \quad F(x),$$

provided that

$$\lim_{x \to \infty} g'(x) = 0. \tag{4}$$

In [3], equality (3) is corrected as follows:

$$\liminf_{n \to \infty} \frac{z_n - a_n}{f(a_n) \ln\ln\ln n} = -1.$$

In the present paper, we generalize the law of iterated logarithm (2), (3) to the case of ideal Banach spaces.

2. Asymptotic Estimates for the Maximum Scheme in Ideal Banach Spaces

We now present several basic definitions:

Definition 1 [5, p. 1]. *A partially ordered Banach space B with norm $\|\cdot\|$ over a field of real numbers is called a Banach lattice if the following conditions are satisfied:*

(a) $x \leq y \Rightarrow x + z \leq y + z\ \forall x, y, z \in B$;

(b) $ax \geq 0$ *for* $a \geq 0,\ x \geq 0,\ x \in B,\ a \in \mathbf{R}^1$;

(c) for any $x, y \in B$, *there exist the least upper bound* $\sup(x, y)$ *and the greatest lower bound* $\inf(x, y)$;

(d) $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|\ \forall x \in B$, *where* $|x| = \sup(x, -x)$.

As an important example of Banach lattice, we can mention an ideal Banach space. This is a Banach space E of (classes of) measurable functions in a measurable space (T, Λ, μ), where μ is a σ-finite, σ-additive measure for which $|x| \leq |y|$ almost everywhere and, in addition, the fact that y belongs to E implies that x belongs to E and $\|x\| \leq \|y\|$. The notion of ideal Banach space is similar to the notion of the Köthe function space presented in [5].

In a Banach lattice, parallel with the convergence in norm, it is also possible to consider the order convergence (o-convergence).

Definition 2 [7, p. 365]. *A sequence of elements (x_n) of a Banach lattice B is called o-convergent to an element x:*

$$x = o - \lim_{n \to \infty} x_n$$

if there exists a sequence $(v_n) \in B$ such that

$$v_n \downarrow 0, \qquad |x - x_n| < v_n,$$

i.e.,

$$v_1 \geq v_2 \geq \ldots, \qquad \inf_{n \geq 1} v_n = 0.$$

For elements $x_1, x_2, \dots, x_n$ of the Banach lattice $B,$ we assume that

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \sup\left(\sum_{i=1}^{n} a_i x_i:\ \sum_{i=1}^{n} |a_i|^{p'} \le 1\right),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad p, p' > 1, \quad (a_1, \dots, a_n) \in \mathbf{R}^n.$$

We say that a Banach lattice E is σ-complete if, for any order-bounded sequence $x_n \subset E,$ its upper $\sup_{n\ge 1} x_n$ and lower $\inf_{n\ge 1} x_n$ bounds in the lattice E exist.

For a σ-complete lattice $E,$ we define the upper and lower bounds of a bounded sequence as follows:

$$\limsup_{n\to\infty} x_n = \inf_m \left(\sup_{n\ge m} x_n\right),$$

$$\liminf_{n\to\infty} x_n = \sup_m \left(\inf_{n\ge m} x_n\right).$$

It is also known that [5, 7]

$$\limsup_{n\to\infty} x_n = o - \lim_{m\to\infty} \left(\sup_{n\ge m} x_n\right),$$

$$\liminf_{n\to\infty} x_n = o - \lim_{m\to\infty} \left(\inf_{n\ge m} x_n\right).$$

Definition 3. *Let $1 \le q < \infty$. A Banach lattice B is called q-concave if there exists a constant $D_{(q)} = D_{(q)}(B)$ such that, for any $n \in N$ and any elements $(x_i)_1^n \subset B$,*

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \right\|.$$

For an ideal Banach space, the operation $\left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q}$ has the ordinary pointwise meaning.

Let E be an ideal Banach space with the norm $\|\cdot\|$ and modulus $|\cdot|,$ let X be a random element defined in the probability space (Ω, A, P) with values in $E,$ and let X_i be independent copies of $X.$ We assume that

$$X = \{X(t), t \in T\}$$

is a random process given on the parametric set T and its trajectories belong to E almost surely.

Let

$$Z_n = \max_{1\le i\le n} X_i.$$

Suppose that

$$X(\omega, t):\ \Omega \times T \to \mathbf{R}$$

can be represented in the form

$$X(\omega, t) = \sigma(t)\widetilde{X}(\omega, t), \tag{5}$$

where

$$\mathfrak{S}X = \big(\sigma(t), t \in T\big) \in E$$

and, for all $t \in T,$ the random variables $\widetilde{X}(\omega, t)$ have the same distribution in a certain random variable $\xi,$ i.e.,

$$\mathbf{P}(\widetilde{X}(\omega, t) < s) = \mathbf{P}(\xi < s) = F(s) \quad \forall t \in T, \quad s \in R.$$

Definition 4. *We say that a random element X satisfies the law of iterated logarithm for the extreme values if the equalities*

$$\limsup_{n\to\infty} \frac{Z_n - a_n\mathfrak{S}X}{f(a_n)\ln\ln n} = \mathfrak{S}X, \tag{6}$$

$$\liminf_{n\to\infty} \frac{Z_n - a_n\mathfrak{S}X}{f(a_n)\ln\ln n} = 0 \tag{7}$$

are true almost surely.

Theorem 1. *Suppose that X is a random element in a q-concave ideal Banach space E $(1 \leq q < \infty)$ that can be represented in the form (5) and X_n are its independent identically distributed copies with absolutely continuous distribution function $F(x)$ satisfying condition (1). Moreover, the function*

$$g(x) = f(x)\ln\ln\left\{\frac{1}{1 - F(x)}\right\}$$

satisfies condition (4). Let

$$\mu(x) = \ln\left(\frac{1}{1 - F(x)}\right) \quad \forall x \in [x_0; +\infty]. \tag{8}$$

Then the following assertions are true:

(i) if there exists $t_0 \in R$ such that the function $\mu'(t)$ increases on the segment $[t_0; +\infty],$ then equality (6) is true;

(ii) if there exists $t_0 \in R$ such that $F(t_0) = 0,$ $F(t) > 0$ $\forall t > t_0,$ and the function $\mu'(t)$ decreases on the segment $[t_0; +\infty],$ then equality (7) is true.

Proof. In the proof, we use the following auxiliary statements established in [6]:

Lemma 1. *Suppose that, for a sequence (ξ_n) of independent identically distributed random variables with distribution function $F(x),$ condition (1) is satisfied. Let*

$$V_1 = \sup_{n>n_0} \frac{z_n - a_n}{f(a_n)\ln\ln n} \quad (n_0\colon\, a_n \geq x_0,\ \forall n > n_0).$$

If $\mu(t)$ is given by relation (8) and there exists $t_0 \in R$ such that the function $\mu'(t)$ increases on the segment $[t_0; +\infty]$, then there exist positive constants C_3 and C_4 such that

$$\mathbf{P}(V_1 > x) \le C_3 e^{-C_4 x} \quad \forall x \in [t_0^*; +\infty], \tag{9}$$

where

$$t_0^* = \max\{x_0; t_0\}$$

and, in particular,

$$\mathbf{E}\, e^{\varepsilon V_1} < \infty, \tag{10}$$

if $0 < \varepsilon < C_4$ and there exists γ such that

$$F(x) = 0 \quad \forall x \in [-\infty; \gamma].$$

Lemma 2. *Suppose that, for the sequence (ξ_n) of independent identically distributed random variables with distribution function $F(x)$, condition (1) is satisfied. Let*

$$V_2 = \sup_{n > n_0} \frac{a_n - z_n}{f(a_n) \ln \ln n}.$$

If $\mu(t)$ is given by relation (8), there exists $t_0 \in R$ such that $F(t_0) = 0,\ F(t) > 0\ \forall t > t_0$, and the function $\mu'(t)$ decreases on the segment $[t_0; +\infty]$, then there exist positive constants C_5 and C_6 such that

$$\mathbf{P}(V_2 > x) \le C_5 e^{-C_6 x} \quad \forall x \in [t_0^{**}; +\infty],$$

where

$$t_0^{**} = \max\{x_0; 3\}$$

and, in particular,

$$\mathbf{E}\, e^{\varepsilon V_2} < \infty$$

for $0 < \varepsilon < C_4$.

It is known [5, p. 83] that a q-concave Banach lattice has a lower q-estimate. Hence, its norm is σ-complete and σ-order continuous. Therefore, the norm of a q-concave ideal Banach space is absolutely continuous, and the corresponding ideal Banach space on the measurable space (T, Λ, μ) with σ-finite measure μ and absolutely continuous norm is separable. To simplify our presentation, we assume that $\mu(T) = 1$.

In a q-concave ideal Banach space, the norm is order-continuous, i.e.,

$$x_n \downarrow 0 \quad \Rightarrow \quad \|x_n\| \to 0,$$

and, hence,

$$x(t) = o - \lim_{n\to\infty} x_n(t) \quad \Rightarrow \quad \lim_{n\to\infty} \big\|x(t) - x_n(t)\big\| = 0. \tag{11}$$

For the validity of equalities (11), it suffices to show that the condition [7, p. 364]

$$\mu\Big(t \in T:\ x(t) = \lim_{n\to\infty} x_n(t)\Big) = 1 \tag{12}$$

is satisfied and there exists $y(t) \in E$ such that

$$\mu\big(t \in T:\ |x_n(t)| \le y(t)\big) = 1$$

for $n \ge 1$.

We now verify equality (6) [equality (7) is proved in a similar way]. We set

$$U_m(t) = \sup_{n\ge m} \frac{Z_n - a_n\sigma(t)}{f(a_n)\ln\ln n} \quad (m:\ a_n \ge x_0,\ \forall n \ge m)$$

and show that

$$o - \lim_{m\to\infty} U_m = \mathfrak{S} \quad \text{a.s.},$$

which is equivalent to equality (6).

Further, we show that the sequence $U_m(t)$ satisfies condition (12). We recall that

$$Z_n = \max_{1\le k\le n} X_k,$$

and set

$$\widetilde{Z}_n(t) = \max_{1\le k\le n} \widetilde{X}_n(t).$$

According to equality (2), for any $t \in T$,

$$\limsup_{n\to\infty} \frac{\widetilde{Z}_n(t) - a_n}{f(a_n)\ln\ln n} = 1 \quad \text{a.s.}$$

Thus, for any $t \in T$,

$$\limsup_{n\to\infty} \frac{Z_n(t) - a_n\sigma(t)}{f(a_n)\ln\ln n} = \sigma(t) \quad \text{a.s.} \tag{13}$$

It follows from relation (13) that, for any $t \in T$,

$$\lim_{m\to\infty} U_m(t) = \sigma(t) \quad \text{a.s.}$$

By the Fubini theorem, we get

$$\mu\left(t \in T:\ \lim_{m\to\infty} U_m(t) = \sigma(t)\right) = 1 \quad \text{a.s,}$$

i.e., condition (12) is satisfied for $x_n(t) = U_n(t)$ and $x_n(t) = \sigma(t)$.

Since

$$\sum_{k}^{\infty}\big(1-F(a_k)\big)=\sum_{k}^{\infty}1-P(\widetilde{X}_1<a_k)=\sum_{k}^{\infty}\frac{1}{k}=+\infty,$$

it is known that (see [8, p. 190])

$$\mathbf{P}\big(Z_n(t)\geq a_n\sigma(t)\ \text{ i.o.}\big)=\mathbf{P}\big(\tilde{Z}_n(t)\geq a_n\ \text{ i.o.}\big)=1,$$

where i.o. means infinitely often.

Thus,

$$\mathbf{P}\big(U_m(t)\geq 0\big)=1\quad \forall t\in T,$$

and, therefore,

$$\mu\big(t\in T\colon\, U_m(t)\geq 0\big)=1\quad \text{a.s.}$$

It is clear that, for $k>m$,

$$\mu\big(t\in T\colon\, U_k(t)\leq U(t)\big)=1\quad \text{a.s,} \tag{14}$$

where

$$U(t)=\sup_{n\geq m}\frac{Z_n-a_n\sigma(t)}{f(a_n)\ln\ln n}.$$

It remains to prove the inequality

$$\mathbf{E}\,\|U\|^q<\infty \tag{15}$$

for a given q-concave ideal Banach space. To prove (15), we use estimate (10) and the following well-known estimate from [9]:

$$\big(\mathbf{E}\,\|Y(t)\|^q\big)^{1/q}\leq D_q\Big\|\big(\mathbf{E}|Y(t)|^q\big)^{1/q}\Big\|. \tag{16}$$

Estimate (16) is true for any random element $Y(t)$ of a q-concave ideal Banach space with $1\leq q$.

Thus, we obtain

$$\big(\mathbf{E}\|U\|^q\big)^{1/q}\leq D_q\Big\|\big(\mathbf{E}|U(t)|^q\big)^{1/q}\Big\|$$

$$=D_q\left\|\left[\mathbf{E}\left(\sigma(t)\sup_{n\geq m}\frac{\widetilde{Z}_n-a_n}{f(a_n)\ln\ln n}\right)^q\right]^{1/q}\right\|$$

$$=D_q\left\|\sigma(t)\left(\mathbf{E}\left(\sup_{n\geq m}\frac{\widetilde{Z}_n-a_n}{f(a_n)\ln\ln n}\right)^q\right)^{1/q}\right\|.$$

It follows from estimate (10) that

$$\left(\mathbf{E}\left(\sup_{n\geq m}\frac{\widetilde{Z}_n - a_n}{f(a_n)\ln\ln n}\right)^q\right)^{1/q} = C_q < \infty.$$

Hence,

$$\left(\mathbf{E}\|U\|^q\right)^{1/q} \leq D_q C_q \|\sigma(t)\| < \infty,$$

i.e., inequality (15) holds together with inequalities (14) and (6). Note that $a_n \geq x_0$ for $n \geq m$ and, therefore,

$$f(a_n) > 0 \quad \forall n \geq m.$$

Lemma 2 is proved.

We now present several distributions satisfying the corresponding theorem.

The distribution function

$$F_1(x) = 1 - e^{-x^\alpha}, \quad x \in [0; +\infty],$$

satisfies equality (6) for $\alpha > 1$ and condition (7) for $\alpha < 1$.

The distribution function

$$F_2(x) = 1 - e^{-\lambda x}, \quad x \in [0; +\infty],$$

simultaneously satisfies equalities (6) and (7) for $\lambda = 1$.

If ξ is a standard normal random variable with distribution function

$$\Phi(x) = \int_{-\infty}^{x} \varphi(s)\, ds$$

and density

$$\varphi(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}},$$

then

$$a_n = \Phi^{-1}\left(1 - \frac{1}{n}\right) = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi) + o(1)}{2(2\ln n)^{1/2}},$$

$$\psi'(x) = \frac{\varphi(x)}{1 - \Phi(x)} = c(x)x,$$

where

$$c(x) \to 1 \quad \text{as} \quad x \to \infty.$$

Since $\psi'(x)$ is increasing, equality (6) is true.

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