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RESONANT EQUATIONS WITH CLASSICAL ORTHOGONAL POLYNOMIALS. I

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We study some resonant equations related to the classical orthogonal polynomials and propose an algorithm for finding their particular and general solutions in the explicit form. The algorithm is especially suitable for the computer algebra tools, such as Maple. The resonant equations form an essential part of various applications, e.g., of the efficient functional-discrete method aimed at the solution of operator equations and eigenvalue problems. These equations also appear in the context of supersymmetric Casimir operators for the di-spin algebra, as well as for the square operator equations $A^2u = f$; e.g., for the biharmonic equation.

1. Introduction

Polynomials (especially the orthogonal polynomials [8, 9, 25]) prove to be a very important and extensively used mathematical tool. One of the application fields for polynomials are differential equations. Some of them possess polynomial solutions and the solution of other equations can be approximated by polynomials. In the present paper, we study a special class of resonant differential equations with differential operators related to the classical orthogonal polynomials.

There are various definitions of resonant equations (see, e.g., [1, 2]), where a boundary-value problem is called resonant if the operator defined by the differential equation and the boundary conditions does not possess the inverse. In the present paper, we follow the definition from [6, 16, 18] and call an equation of the form $Lf = g$ with $Lg = 0$ resonant. In other words, the right-hand side of the resonant equation belongs to the kernel $K(L)$ of the operator L. These equations are of interest both from the theoretical point of view and from the practical side in various applications. Thus, in [17], the authors proposed the so-called functional-discrete method (FD-method) for the solution of the operator equations and eigenvalue problems. The method is based on the ideas of perturbation of the analyzed operator and on the homotopy idea. This approach was applied to various problems and, in particular, to eigenvalue problems in [10–13]. It was proved that the method possesses a superexponential convergence rate. An essential part of the algorithm are certain inhomogeneous equations with resonant component in a sense of the definition presented above.

A simple but profound example showing principally different behaviors of the solutions in the resonant and nonresonant cases gives the following simple differential equation (the so-called vibration equation):

$$\frac{d^2y}{dt^2} + \mu^2 y = \sin(\nu t). \tag{1.1}$$

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There exists a particular solution of the form

$$y(t) = \begin{cases} \dfrac{1}{\mu^2 - \nu^2} \sin(\nu t) & \text{for} \quad \nu \neq \pm\mu, \\ -\dfrac{t}{2\mu} \cos(\mu t) & \text{for} \quad \nu = \pm\mu, \end{cases} \tag{1.2}$$

which is resonant for $\nu = \pm\mu$ $\left(\text{in this case, the right-hand side } \sin(\mu t) \text{ solves the homogeneous equation } \dfrac{d^2 y}{dt^2} + \mu^2 y = 0\right)$ and nonresonant, otherwise. The vibration amplitude in the resonant case tends to infinity if the stimulating vibration frequency ν on the right-hand side of the equation tends to the resonant eigenfrequency μ of the system described by the differential operator on the left-hand side. The value of difference between the nonresonant and resonant solutions is discussed in [7].

The example presented above can be embedded into the following abstract framework: Assume that a system is described by an operator equation

$$Au - \lambda u = f$$

in a Hilbert space H, where the operator A is completely defined by its spectrum, i.e., by the eigenvalues λ_j, $j = 1, 2, \ldots,$ and the corresponding eigenvectors u_j, $j = 1, 2, \ldots.$ Here, λ is a parameter characterizing the system. If the right-hand side has the form $f = \alpha u_k$ for fixed α, k, i.e., f solves the equation $(A - \lambda_k)f = 0$, then the solution of the corresponding operator equation is

$$u = \frac{\alpha}{\lambda_k - \lambda} u_k.$$

We have $\|u\| \to \infty$ ($\|u\|$ can be interpreted as the amplitude in the example presented above) in the following two cases:

(i) if $\alpha \to \infty$ (the stimulating amplitude tends to infinity)

and

(ii) if the system parameter tends to an eigenvalue λ_k of the operator, i.e., $\lambda \to \lambda_k$.

The second case is called resonant and the value λ_k of the parameter λ is called the resonant value. In this case, we deal with the resonant equation in a sense of definition presented above and of the example equation (1.1). It is clear that a system may possess various resonant values of the parameter.

The resonant equations also appear when solving the quadratic operator equation

$$A^2 u = 0 \tag{1.3}$$

with a given operator A. Denoting $Au = v$, we reduce equation (1.3) to a "simpler" pair of equations $Av = 0$, $Au = v$, where the last equation is resonant.

The resonance phenomena play a very important role in the natural world and in various technical applications, e.g., in the magnetic resonance imaging (nuclear spin tomography) [23], fluid dynamics [14, 15], etc. The resonant equations also appear in the context of supersymmetric Casimir operators for the di-spin algebra (see, e.g., [6, 7] and the literature cited therein). These equations often require specific techniques for their solution and investigation [1–3]. The condition of solvability of the resonant equation in a Hilbert space is the orthogonality of the right-hand side to the kernel of the operator. The situation for other spaces is more complicated (see, e.g., [1–3]), where certain sufficient solvability conditions were proved. Thus, in [2], the following Liénard equation that describes vibrations and various dynamical systems, was considered as an illustration of the presented solvability theory:

$$\ddot{x}(t) + g(x)\dot{x}(t) + ax(t) = f(t), \qquad x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \quad t \in [0,1]. \tag{1.4}$$

Here a is a constant, the function $g\colon\ R^1 \to R^1$ is supposed to be continuous, and the solution is sought in the class of twice continuously differentiable on $[0,1]$ functions. It was shown that this problem possesses at least one solution for any function

$$f(t)\colon\ \int_0^1 f(t)\,dt = 0$$

provided that

$$|g(x)| \le b, \quad a \in R^1, \quad b + 4|a|/3 < 1. \tag{1.5}$$

A simple counterexample with

$$f(t) = \frac{4}{3\pi}\cos(2\pi t) + \sin(2\pi t), \quad g(x) \equiv 0, \quad a = 4\pi^2$$

(here, the first term is a resonant component) shows that the differential equation is resonant in the sense of our definition, conditions (1.5), due to $b = 0$ and $a = 4\pi^2$, as well as the condition

$$\int_0^1 f(t)\,dt = 0,$$

are not satisfied but there exists a set of solutions given by

$$u(t) = C\cos(2\pi t) + D\sin(2\pi t) + \frac{t}{3\pi^2}\sin(2\pi t) + \frac{1}{3\pi^2}\sin(\pi t) \quad \forall C, D \in R^1,$$

i.e., conditions (1.5) are rather coarse.

In the present paper, we consider resonant equations with differential operators of the hypergeometric type, which define classical orthogonal polynomials. The solutions of these homogeneous differential equation include the corresponding orthogonal polynomial (or the solution of the first kind) and the second linear independent solution, namely, the so-called function of the second kind. Thus, the general solution is a linear combination of both these solutions. The inhomogeneous differential equations with the corresponding orthogonal polynomial or the function of the second kind on the right-hand side are resonant equations of the first and second kinds, respectively. We need their particular solutions to write down the general solution of the inhomogeneous resonant equation.

We propose a general algorithm for finding these particular solutions in the explicit form. Thus, it becomes possible to find the general solutions of the inhomogeneous resonant equations of the first and second kinds in the explicit form. This algorithm is especially suitable for the computer-algebra tools, such as Maple, etc. In addition, it also gives a constructive proof of the existence of solutions.

The paper consists of two parts and is organized as follows: In Section 2, we show that the resonant equations are a natural part of the FD-method. The main result of Section 3 is Theorem 3.1, which gives a formula for the particular solutions of a resonant operator equation that depends on a parameter. This section also contains the description of the general Algorithm 3.1 aimed at computing the particular solutions of the inhomogeneous resonant equations with differential operators related to the classical orthogonal polynomials. Theorem 3.1 plays the crucial role in the justification of our algorithm. Each of the next two sections consists of two subsections devoted to the corresponding resonant equations of the first and second kinds with differential operators related to the classical orthogonal polynomials of the Legendre and Jacobi types. The explicit formulas are given for the general solutions of the corresponding inhomogeneous resonant differential equations. The classical orthogonal polynomials defined on the infinite intervals, namely, the Hermite and the Laguerre polynomials are the topics of Part II. With an aim to emphasize the advantages of our algorithm, we also represent particular solutions via the hypergeometric or confluent hypergeometric functions.

2. The Homotopy Based Method for the Eigenvalue Problems

We now briefly explain the ideas of perturbation and homotopy for the eigenvalue problem

$$(A+B)u_n - \lambda_n u_n = \theta \tag{2.1}$$

in a Hilbert space X with a scalar product $(\cdot,\cdot)$ and the null element θ under the assumption that the spectrum of the operator $A+B$ is discrete and we seek an eigenpair with a given fixed index n.

Let $\overline{B}$ be an approximating operator for B in a sense that the eigenvalue problem

$$(A+\overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2.2}$$

is "simpler" than problem (2.1).

Formally, a homotopy between two problems P_1 and P_2 with solutions u_1 and u_2 from some topological space X is defined as a parametric problem $P_H(t)$ with a solution $u(t)$ that continuously depends on the parameter $t \in [0,1]$ and is such that $u(0) = u_1$ and $u(1) = u_2$ (cf. http://en.wikipedia.org/wiki/Homotopy).

Following the homotopy idea, for a given eigenpair number n, we imbed our problem into the parametric family of problems

$$(A+W(t))u_n(t) - \lambda_n(t)u_n(t) = \theta, \quad t \in [0,1], \tag{2.3}$$

with

$$W(t) = \overline{B} + t\varphi(B), \quad \varphi(B) = B - \overline{B},$$

where $\overline{B}$ is an approximation of B. This family contains both the problems (2.1) and (2.2). Thus, we obviously obtain

$$u_n(0) = u_n^{(0)}, \qquad \lambda_n(0) = \lambda_n^{(0)}, \qquad u_n(1) = u_n, \qquad \lambda_n(1) = \lambda_n. \tag{2.4}$$

This suggests the idea to seek the solution of (2.3) in the form of a Taylor series

$$\lambda_n(t) = \sum_{j=0}^{\infty} \lambda_n^{(j)} t^j, \qquad u_n(t) = \sum_{j=0}^{\infty} u_n^{(j)} t^j, \tag{2.5}$$

where formally

$$\lambda_n^{(j)} = \frac{1}{j!} \frac{d^j \lambda_n(t)}{dt^j}\bigg|_{t=0}, \qquad u_n^{(j)} = \frac{1}{j!} \frac{d^j u_n(t)}{dt^j}\bigg|_{t=0}. \tag{2.6}$$

Setting $t = 1$ in (2.5), we obtain

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \qquad u_n = \sum_{j=0}^{\infty} u_n^{(j)} \tag{2.7}$$

provided that series (2.5) converge for all $t \in [0, 1]$. The truncated series

$$\overset{m}{\lambda}_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \qquad \overset{m}{u}_n = \sum_{j=0}^{\infty} u_n^{(j)} \tag{2.8}$$

represent a computational algorithm of rank m.

Relations (2.6) are not suitable for the numerical algorithm. Therefore, we need another way to compute the corrections $\lambda_n^{(j)}$ and $u_n^{(j)}$. This method is described below.

Substituting (2.5) in (2.3) and equating the coefficients of the same powers of t, we arrive at the following recurrence sequence of equations:

$$(A + \overline{B})u_n^{(j+1)} - \lambda_n^{(0)} u_n^{(j+1)} = F_n^{(j+1)}, \quad j = -1, 0, 1, \ldots, \tag{2.9}$$

with $F_n^{(0)} = 0$ and

$$\begin{aligned} F_n^{(j+1)} &= F_n^{(j+1)}\big(\lambda_n^{(0)}, \ldots, \lambda_n^{(j+1)}; u_n^{(0)}, \ldots, u_n^{(j)}\big) \\ &= -\varphi(B)u_n^{(j)} + \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} \\ &= \lambda_n^{(j+1)} u_n^{(0)} - \varphi(B)u_n^{(j)} + \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}, \quad j = -1, 0, 1, \ldots. \end{aligned} \tag{2.10}$$

For the pair $\lambda_n^{(0)}$, $u_n^{(0)}$ corresponding to the index $j = -1$, we get the so-called base eigenvalue problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2.11}$$

in a Hilbert space. It is assumed that this problem does not have multiple eigenvalues, is "simpler" than the original problem, and produces the initial data for problems (2.9), (2.10). We suppose that $u_n^{(0)}$, $n = 1, 2, \ldots$, is a basis of the corresponding Hilbert space. The case of base problems with multiple eigenvalues was studied in [10, 19, 20].

The right-hand side of each problem (2.10) contains the term $\lambda_n^{(j+1)} u_n^{(0)}$, which solves the homogeneous equation with the same operator, i.e., the solution $u_n^{(j+1)}$ of (2.10) contains a component, which is the solution of the corresponding resonant equation.

For higher indices $j \geq 0,$ problems (2.9) are solvable provided that

$$\left(F_n^{(j+1)}, u_n^{(0)}\right) = 0, \quad j = 0, 1, \ldots. \tag{2.12}$$

If we additionally suppose that (for uniqueness)

$$\left(u_n^{(j+1)}, u_n^{(0)}\right) = 0, \quad j = 0, 1, \ldots, \tag{2.13}$$

then we get

$$\lambda_n^{(j+1)} = \left(\varphi(B) u_n^{(j)}, u_n^{(0)}\right), \quad j = 0, 1, \ldots. \tag{2.14}$$

Under these conditions we obtain a particular solution

$$u_n^{(j+1)} = \sum_{p=1, p \neq n}^{\infty} \frac{\left((F_n^{(j)}, u_p^{(0)})\right)}{\lambda_p^{(0)} - \lambda_n^{(0)}} u_p^{(0)} \tag{2.15}$$

satisfying condition (2.13). The starting values $\lambda_n^{(0)},\ u_n^{(0)}$ for the recursion (2.9), (2.14) form the solution of the base problem.

The following theorem [17] gives the error estimates for the method presented above and its convergence as $m \to \infty$:

Theorem 2.1. *Let A be a closed operator in a Hilbert space H. Assume that problem (2.11) possesses a discrete spectrum of eigenvalues $0 \leq \lambda_1^{(0)} < \lambda_2^{(0)} < \ldots$ and that the corresponding eigenvectors $u_n^{(0)},\ n = 1, 2, \ldots,$ form a basis of H. Suppose that the inequality*

$$q_n = 4 M_n \|\varphi(B)\| < 1 \tag{2.16}$$

with

$$M_n = \max\left\{\frac{1}{\lambda_n^{(0)} - \lambda_{n-1}^{(0)}}, \frac{1}{\lambda_{n+1}^{(0)} - \lambda_n^{(0)}}\right\} \tag{2.17}$$

is true. Then series (2.7) converge to the solution $\lambda_n,\ u_n$ of problem (2.1) and the accuracy of algorithm (2.8) is specified by the estimates

$$\begin{gathered} \left\|u_n - \overset{m}{u}_n\right\| \leq \alpha_{m+1} \frac{q_n^{m+1}}{1 - q_n}, \\ \left\|\lambda_n - \overset{m}{\lambda}_n\right\| \leq \|\varphi(B)\| \alpha_m \frac{q_n^m}{1 - q_n}, \end{gathered} \tag{2.18}$$

where

$$\alpha_m = 2 \frac{(2m-1)!!}{(2m+21)!!}. \tag{2.19}$$

3. Representation of Particular Solutions

This section deals with particular solutions of the resonant equations. We give a representation of particular solutions to the resonant equation in a Banach space. In addition, we propose an algorithm used to compute particular solutions of the resonant equations with differential operators related to the classical orthogonal polynomials.

The following result was proved in [18].

Theorem 3.1. *Let $A\colon X \to X$ be a linear operator acting in a Banach space X, let the set $K(A) \subset X$ be the kernel of A, and let a connected set $\Sigma(A)$ in the complex plane be the spectral set of A. If $f(\lambda) \in K(A-\lambda E)$ and $\lambda \in \Sigma(A)$ is a differentiable function, then the solution of the resonant equation*

$$(A - \lambda E)u = f(\lambda) \tag{3.1}$$

can be represented as

$$u(\lambda) = \frac{df(\lambda)}{d\lambda}. \tag{3.2}$$

The proof of this theorem is based on the equivalent equation

$$(A - \lambda_0 E)\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f(\lambda)$$

with some fixed λ_0 and on passing to the limit as $\lambda \to \lambda_0$.

Now let

$$\mathcal{A}_n = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx} + \lambda_n \tag{3.3}$$

be a differential operator of hypergeometric type with a polynomial $\sigma(x)$ of degree not greater than two, a polynomial $\tau(x)$ of degree not greater than 1 and a constant λ_n and let $P_n(x)$ be a classical orthogonal polynomial satisfying the homogeneous differential equation

$$\mathcal{A}_n P_n(x) = 0 \tag{3.4}$$

(see, e.g., [5, 22, 24]). The polynomial solution $P_n(x)$ of this homogeneous differential equation is called a function of the first kind. Let $Q_n(x)$ be the second linear independent solution of the homogeneous differential equation, which is called a function of the second kind.

We now consider resonant equations of the form

$$\mathcal{A}_n u_n(x) = R_n(x). \tag{3.5}$$

In the case where $R_n(x)$ is the classical orthogonal polynomial $P_n(x)$ (the function of the first kind), the inhomogeneous differential equation (3.5) is called the resonant equation of the first kind. The inhomogeneous differential equation of the form (3.5) with the right-hand side $Q_n(x)$ instead of $R_n(x)$ is called the resonant differential equation of the second kind. Both functions $P_n(x)$ and $Q_n(x)$ satisfy the same homogeneous differential equation (3.4) and the same recurrence relations

$$R_{n+1}(x) = (\alpha_n x + \beta_n) R_n(x) - \gamma_n R_{n-1}(x), \quad n = 1, 2, \dots, \tag{3.6}$$

with some constants $\alpha_n,\ \beta_n,\ \gamma_n$ (see, e.g., [5, 21, 22, 24]).

Note that our algorithm presented below and aimed at finding particular solutions of the resonant differential equations of the first and second kind (3.5) is based on the same recurrence relation (3.6). Thus, it is valid for the resonant equations of both types and, in what follows, we use the notation $R_n(x)$ both for $P_n(x)$ and for $Q_n(x)$.

Algorithm 3.1.

1. Using Theorem 3.1, we find some particular solutions of (3.5) for $n = 0, 1$, i.e.,

$$\chi_0(x) = -\frac{1}{\lambda'(\nu)} \frac{d\,R_\nu(x)}{d\nu}\bigg|_{\nu=0}, \qquad \chi_1(x) = -\frac{1}{\lambda'(\nu)} \frac{d\,R_\nu(x)}{d\nu}\bigg|_{\nu=1}. \tag{3.7}$$

Note that here and in what follows the procedure of differentiation with respect to a natural parameter $n \in \mathbb{N}$ has the following meaning:

(i) switching to a real parameter $\nu \in \mathbb{R}$, i.e., the use of hypergeometric or confluent hypergeometric functions,

(ii) differentiation with respect to ν,

(iii) substitution of n instead of ν in the derivative.

2. The set of functions

$$\begin{aligned} u_0(x) &= \chi_0(x) + c_0 P_0(x) + d_0 Q_0(x), \\ u_1(x) &= \chi_1(x) + c_1 P_1(x) + d_1 Q_1(x) \end{aligned} \tag{3.8}$$

with arbitrary coefficients $c_0,\ c_1,\ d_0$, and d_1 represents particular solutions also of the inhomogeneous resonant equation. These coefficients can be chosen in the next step of the algorithm so that the following particular solutions $u_k(x),\ k = 2, 3, \dots,$ obtained by the recursion below satisfy the corresponding resonant equation.

3. Differentiating the recurrence equation (3.6) for R_n with respect to n, we obtain

$$\begin{aligned} u_{n+1}(x) = -\frac{1}{\lambda'(n+1)} \bigg[&-\frac{d\lambda(n)}{dn} (\alpha_n x + \beta_n)\, u_n(x) + \frac{d\lambda(n-1)}{dn} \gamma_n u_{n-1}(x) \\ &+ \left(\frac{d\alpha_n}{dn} x + \frac{d\beta_n}{dn} \right) R_n(x) - \frac{d\gamma_n}{dn} R_{n-1}(x) \bigg], \qquad n = 1, 2, \dots. \end{aligned} \tag{3.9}$$

Here, we set $n = 1$ and demand that $u_2(x)$ obtained from (3.9) and (3.8) must satisfy the resonant differential equation (3.5). From this condition, we determine the coefficients $c_0,\ c_1,\ d_0$, and d_1 and, hence, the initial values (3.8) for the recursive algorithm (3.9). By using Theorem 3.1, in what follows, we prove that, in this case, $u_n(x)$ satisfy the resonant equation for all $n = 0, 1, 2, \dots$.

4. Resonance Equation of the Legendre Type

4.1. Legendre Resonance Equation of the First Kind. We consider the following inhomogeneous equation with a Legendre differential operator on the left-hand side and a Legendre polynomial on the right-hand side:

$$\frac{d}{dx}\left[(1-x^2)\frac{du(x)}{dx}\right]+n(n+1)u(x)=P_n(x). \tag{4.1}$$

This is a resonant equation of the first kind because the Legendre polynomial $P_n(x)$ satisfies the corresponding homogeneous differential equation. The second linear independent solution of the homogeneous differential equation $Q_n(x)$ is called a Legendre function of the second kind. The general solution of the homogeneous differential equation (4.1) is given by the formula

$$u(x)=c_1P_n(x)+c_2Q_n(x),$$

where c_1 and c_2 are arbitrary constants.

The explicit expression of the Legendre function of the second kind can be represented via the hypergeometric function (see, e.g., [5], § 10.10) as follows:

$$\begin{aligned}
Q_n(x) &= Q_0(x)P_n(x)-\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{2n-4k+3}{(2k-1)(n-k+1)}P_{n-2k+1}(x)\\
&=\frac{2^n(n!)^2}{(2n+1)!(1+x)^{n+1}}F\left(n+1,n+1;2n+2;\frac{2}{1+x}\right)\\
&=(-1)^{n+1}\frac{2^n(n!)^2}{(2n+1)!(1-x)^{n+1}}F\left(n+1,n+1;2n+2;\frac{2}{1-x}\right)\\
&=\frac{1}{2}\left[F\left(n+1,n+1;2n+2;\frac{2}{1+x}\right)\right.\\
&\qquad\left.+(-1)^{n+1}\frac{2^n(n!)^2}{(2n+1)!(1-x)^{n+1}}F\left(n+1,n+1;2n+2;\frac{2}{1-x}\right)\right],\\
Q_0(x) &= \frac{1}{2}\ln\frac{x+1}{x-1}.
\end{aligned} \tag{4.2}$$

Here,

$$F(a,b;c;z)=\sum_{p=0}^{n}\frac{(a)_p(b)_pz^n}{(c)_pp!}$$

is the hypergeometric function of z, $(a)_0=1$,

$$(a)_p=\frac{\Gamma(a+p)}{\Gamma(a)}$$

is the Pochhammer symbol, and $\Gamma(x)$ is the Gamma function.

Recall that numerous well-known mathematical functions can be expressed either in terms of the hypergeometric function or as limiting cases of this function. As two typical examples, we can mention

$$\ln(1+z) = z\, F(1,1;2;-z),$$

$$(1-z)^{-a} = F(a,1;1;z).$$

The Legendre functions, as well as several orthogonal polynomials, including the Jacobi polynomials $P_n^{(\alpha,\beta)}$ and their special cases, such as the Legendre polynomials $(\alpha = 0,\ \beta = 0)$, Chebyshev polynomials $T_n(x)$ $(\alpha = -1/2,\ \beta = -1/2)$, and Gegenbauer polynomials $C_n^{\lambda}(x)$ $(\alpha = \beta = \lambda - 1/2)$, can be represented in terms of hypergeometric functions in many ways, e.g., as follows (see, e.g., [7], § 10.8):

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n} F_n\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \\ &= (-1)^n \binom{n+\beta}{n} F_n\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right), \\ P_n(x) = P_n^{(0,0)}(x) &= \frac{1}{2}\left[F_n\left(-n, n+1; 1; \frac{1-x}{2}\right)\right. \\ &\qquad \left. + (-1)^n F_n\left(-n, n+1; 1; \frac{1+x}{2}\right)\right]. \end{aligned} \tag{4.3}$$

The application of hypergeometric functions with an aim to obtain solutions of resonant equation represents a direct way used to solve the resonant equations. This way appears due to the fact that the hypergeometric differential equation

$$z(1-z)\frac{d^2u}{dz^2} + \big[c - (a+b+1)z\big]\frac{du}{dz} - abu = 0 \tag{4.4}$$

with properly chosen parameters can be transformed into the following Legendre equation [4] (§ 3.2):

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \nu(\nu+1)w = 0. \tag{4.5}$$

In view of Theorem 3.1 and (4.3), we get a particular solution of (4.1) in the form

$$u_n(x) = \frac{1}{2}\big[\tilde{u}_n(x) + (-1)^n \tilde{u}_n(-x)\big], \tag{4.6}$$

where $\{$see (4.3) and [5], § 10.8, relation (16) with $\alpha = 0$ and $\beta = 0\}$

$$\tilde{u}_n(x) = k_n \frac{d}{d\nu} P_\nu(x)|_{\nu=n} = k_n \frac{d}{d\nu} F_\nu \left(1+\nu, -\nu; 1; \frac{1-x}{2}\right)\bigg|_{\nu=n}$$

$$= k_n \left[\sum_{p=1}^{n} \frac{d}{dn} \frac{(1+n)_p(-n)_p}{(p!)^2} \left(\frac{1-x}{2}\right)^p \right.$$

$$\left. + (-1)^{n+1} n! \sum_{p=n+1}^{\infty} \frac{(1+n)_p(p-n-1)!}{(p!)^2} \left(\frac{1-x}{2}\right)^p \right] \tag{4.7}$$

with $k_n = -\dfrac{1}{2n+1}$. This formula can be transformed in the following way:

$$\tilde{u}_n(x) = k_n \left[\sum_{p=1}^{n} \frac{-1}{(p!)^2} \left[2n + p - 2n^2(n+p) \sum_{i=1}^{p-1} \frac{1}{i^2 - n^2} \right] \prod_{i=1}^{p-1} (i^2 - n^2) \left(\frac{1-x}{2}\right)^p \right.$$

$$\left. + (-1)^{n+1} \sum_{p=n+1}^{\infty} \frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} \left(\frac{1-x}{2}\right)^p \right]. \tag{4.8}$$

By using the formulas

$$\frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} = \sum_{i=0}^{n} \frac{a_{n,i}}{p-i}, \quad a_{n,i} = (-1)^{n+i} \frac{(n+i)!}{(n-i)!(i!)^2},$$

$$(-1)^n \sum_{i=0}^{n} a_{n,i} \left(\frac{1-x}{2}\right)^i \equiv F_n\left(1+n, -n; 1; \frac{1-x}{2}\right) = P_n(x), \tag{4.9}$$

the sum of the last series can be transformed as follows:

$$\sum_{p=n+1}^{\infty} \frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} \left(\frac{1-x}{2}\right)^p$$

$$= -\sum_{i=0}^{n} a_{n,i} \left(\frac{1-x}{2}\right)^i \ln\left(\frac{1+x}{2}\right) - \sum_{p=0}^{n-1} a_{n,p} \left(\frac{1-x}{2}\right)^p \sum_{i=1}^{n-p} \frac{1}{i} \left(\frac{1-x}{2}\right)^i$$

$$= (-1)^{n+1} P_n(x) \ln\left(\frac{1+x}{2}\right) - \sum_{i=1}^{n} \left(\frac{1-x}{2}\right)^i \sum_{p=0}^{i-1} \frac{a_{n,p}}{i-p}. \tag{4.10}$$

Thus, for function (4.7), we obtain

$$\tilde{u}_n(x) = -\frac{1}{2n+1} P_n(x) \ln\left(\frac{1+x}{2}\right) + \sum_{i=1}^{n} \left(\frac{1-x}{2}\right)^i b_{n,i},$$

$$b_{n,i} = -\frac{1}{2n+1} \left[\frac{1}{i!} \frac{d}{dn} (1+n)_i(-n)_i + (-1)^n \sum_{p=0}^{i-1} \frac{a_{n,p}}{i-p} \right]. \tag{4.11}$$

The direct way of getting particular solutions described above is quite cumbersome. In what follows, we propose an algorithmic procedure based on Theorem 3.1 and on the recursion relation for the corresponding orthogonal polynomials. This algorithm can be easily implemented by using the computer-algebra tools, e.g., Maple.

Actually, for $n = 0, 1,$ in view of Theorem 3.1 we get the following particular solutions from (4.6):

$$\chi_0(x) = -\frac{1}{2}\ln(1-x^2), \qquad \chi_1(x) = -\frac{x}{6}\ln(1-x^2) + \frac{11}{18}x. \tag{4.12}$$

Differentiating the recurrence relation for $P_n(x)$ with respect to $n,$ we arrive at the following recurrence equation for particular solutions:

$$u_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1}u_n(x) + \frac{n(2n-1)}{n+1}u_{n-1}(x) + \frac{x}{(n+1)^2}P_n(x) - \frac{1}{(n+1)^2}P_{n-1}(x)\right], \qquad n = 1, 2, \ldots. \tag{4.13}$$

The Legendre polynomials $P_0(x) = 1$ and $P_1(x) = x,$ as well as the Legendre functions of the second kind $Q_0(x)$ and $Q_1(x),$ satisfy the corresponding homogeneous Legendre differential equation. Hence, according to our algorithm, in view of (4.12), we can use the following ansatzes for the initial values:

$$\begin{aligned} u_0(x) &= -\frac{1}{2}\ln(1-x^2) + c_0 P_0(x) + d_0 Q_0(x), \\ u_1(x) &= -\frac{x}{6}\ln(1-x^2) + \frac{11}{18}x + c_1 P_1(x) + d_1 Q_1(x) \end{aligned} \tag{4.14}$$

with undetermined coefficients $c_0,\ c_1,\ d_0,$ and d_1. Substituting these relations in (4.13) with $n = 1,$ we demand that $u_2(x)$ must satisfy the resonant differential equation (4.1). This yields

$$\begin{aligned} d_0 = 0, \quad d_1 = 0, \\ c_0 = \frac{17}{6} + 3c_1. \end{aligned} \tag{4.15}$$

Setting, e.g., $c_0 = 0$ we conclude that

$$c_1 = -\frac{17}{18}$$

and arrive at the representations

$$\begin{aligned} u_1(x) &= -\frac{1}{6}P_1(x)\ln(1-x^2) - \frac{1}{3}x, \\ u_2(x) &= -\frac{1}{10}P_2(x)\ln(1-x^2) - \frac{7}{20}x^2 + \frac{1}{20}. \end{aligned} \tag{4.16}$$

In general, we have

$$u_n(x) = -\frac{1}{2(2n+1)}P_n(x)\ln(1-x^2) + v_n(x), \tag{4.17}$$

where $v_n(x)$ satisfies the recurrence equation

$$
\begin{gathered}
v_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1} v_n(x) + \frac{n(2n-1)}{n+1} v_{n-1}(x)\right. \\
\left. + \frac{x}{(n+1)^2} P_n(x) - \frac{1}{(n+1)^2} P_{n-1}(x)\right], \qquad n = 1, 2, \ldots, \\
v_0(x) = 0, \qquad v_1(x) = -\frac{x}{3}.
\end{gathered} \tag{4.18}
$$

This recurrence equation, together with (4.14) and (4.17) gives, e.g., the following particular solutions:

$$
\begin{gathered}
u_3(x) = -\frac{1}{14} P_3(x) \ln(1-x^2) + \frac{5}{28} x - \frac{37}{84} x^3, \\
u_4(x) = -\frac{1}{18} P_4(x) \ln(1-x^2) - \frac{7}{288} + \frac{59}{144} x^2 - \frac{533}{864} x^4.
\end{gathered} \tag{4.19}
$$

The next theorem shows that the functions $u_n(x)$ obtained by using our recursive algorithm satisfy the resonant Legendre differential equation of the first kind for all $n = 0, 1, \ldots$.

Theorem 4.1. *The functions $u_n(x)$ obtained by the recursive algorithm (4.13) satisfy the resonant Legendre differential equation of the first kind (4.1) for each $n = 0, 1, 2, \ldots$.*

Proof. For $n = 0, 1, 2,$ these functions satisfy the resonant Legendre differential equation by construction. Further, we assume that $u_p(x),\ p = 0, 1, \ldots, n,$ satisfy this differential equation and prove that this remains true for $p = n+1$. Differentiating the classical relation [7] (§ 10.10)

$$
(1-x^2)\frac{d\,P_n(x)}{dx} = n\left[P_{n-1}(x) - xP_n(x)\right] \tag{4.20}
$$

with respect to n and using Theorem 3.1, we arrive at the equation

$$
-(2n+1)(1-x^2)\frac{du_n(x)}{dx} = -n\,(2n-1)\,u_{n-1}(x) + x\,n\,(2n+1)\,u_n(x) + P_{n-1}(x) - xP_n(x). \tag{4.21}
$$

Applying the Legendre differential operator

$$
\mathcal{A}_n = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + n(n+1) \tag{4.22}
$$

to (4.13), we obtain

$$
\begin{gathered}
\mathcal{A}_{n+1} u_{n+1}(x) = P_{n+1}(x) + \frac{2(2n+1)}{(2n+3)(n+1)}\left[(2n+1)(1-x^2)\frac{du_n(x)}{dx}\right. \\
\left. - n\,(2n-1)\,u_{n-1}(x) + x\,n\,(2n+1)\,u_n(x) + P_{n-1}(x) - xP_n(x)\right].
\end{gathered} \tag{4.23}
$$

It follows from (4.21) that the expression in the square brackets is equal to zero, which proves the theorem.

Thus, the general solution of the inhomogeneous equation (4.1) can be represented in the form

$$u(x) = u_h + u_n(x) = c_1 P_n(x) + c_2 Q_n(x) + u_n(x), \tag{4.24}$$

where c_1 and c_2 are arbitrary constants.

Remark 4.1. The proof of Theorem 4.1 is based on a recurrence equation

$$xp_n(x) = \alpha_n p_{n-1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x)$$

for the corresponding polynomials $p_n(x)$ orthogonal with weight $\sigma(x)$ and the differentiation formula

$$\sigma(x) p_n'(x) = \alpha_n^{(1)} p_{n+1}(x) + \big(\beta_n^{(1)} + \gamma_n^{(1)} x\big) p_n(x)$$

[see (4.20) for the Legendre polynomials], which represents the weighted derivative of the analyzed polynomial via two neighboring polynomials (see, e.g., [21], § 9). However, the second linear independent solution of the corresponding homogeneous equation, which is called the function of the second kind (in the case analyzed above, this is the Legendre function of the second kind $Q_n(x)$, which is not polynomial!), satisfies the same recurrence equation and the same differentiation formula (see [21, p. 67]). Thus, a similar theorem for the particular solutions of the resonant equations of the second kind obtained by the corresponding recursive algorithm is also valid.

4.2. The Legendre Resonant Equation of the Second Kind. In this section, we consider equation (4.1) with the Legendre function of the second kind

$$Q_n(x) = \frac{2^n (1+x)^{-n-1} (n!)^2}{(2n+1)!} F\left(n+1, n+1; 2n+2; \frac{2}{1+x}\right)$$

as the right-hand side, i.e., we again have a resonant equation.

The general solution of this resonant equation is

$$u(x) = c_1 P_n(x) + c_2 Q_n(x) + u_n(x), \tag{4.25}$$

where the linearly independent Legendre polynomial $P_n(x)$ and the Legendre function of the second kind $Q_n(x)$ satisfy the homogeneous Legendre equation, c_1 and c_2 are arbitrary constants, and $u_n(x)$ is a particular solution of the inhomogeneous resonant equation.

Note that, in [7], a solution was obtained only for the case $n = 0$ and it was indicated that it is very difficult to obtain solutions for the other n in the closed form. However, our Theorem 3.1 allows one to get particular solutions for any n as follows:

$$u_n(x) = -\frac{1}{(2n+1)} \big[2\psi(n+1) - 2\psi(2n+2) + \ln(2) - \ln(1+x)\big] Q_n(x)$$
$$- \frac{2^n (n!)^2 (1+x)^{-n-1}}{(2n+1)(2n+1)!} \frac{d}{d\nu} \left[F\left(\nu+1, \nu+1; 2\nu+2; \frac{2}{1+x}\right)\right]_{\nu=n}, \tag{4.26}$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the logarithmic derivative of the Gamma-function. Various representations of this function can be found in [5] (§ 1.7). For $n = 0, 1$ we obtain the following particular solutions:

$$\chi_0(x) = -P_0(x)w(x),$$

$$\chi_1(x) = -\frac{1}{3}P_1(x)w(x) - \frac{1}{6}\ln\left(x^2-1\right) - \frac{2}{3},$$

$$w(x) = -\text{polylog}\left(2, \frac{2}{1+x}\right) - \frac{1}{2}\ln^2(x+1) + \frac{1}{2}\ln(x+1)\ln(x-1)$$

$$= -\text{dilog}\left(\frac{2}{1+x}\right) - \frac{1}{2}\ln^2(x+1) + \frac{1}{2}\ln(x+1)\ln(x-1), \quad x > 1, \tag{4.27}$$

where polylog stands for the so-called polylogarithm function of order s and of the argument z (Jonquière's function):

$$\text{polylog}(s, z) = \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

(the dilog or Spence's function also denoted by $\text{Li}_2(z)$ is a special case of the polylog for $s = 2$).

The other explicit representations of particular solutions can be obtained by using Algorithm 3.1. Differentiating the recurrence relation for the Legendre function of the second kind

$$Q_{\nu+1}(x) = \frac{x(2\nu+1)}{(\nu+1)}Q_\nu(x) - \frac{\nu}{(\nu+1)}Q_{\nu-1}(x) \tag{4.28}$$

with respect to ν and taking into account Theorem 3.1, we arrive at the recurrence relation

$$u_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1}u_n(x) + \frac{n(2n-1)}{n+1}u_{n-1}(x) + \frac{x}{(n+1)^2}Q_n(x) - \frac{1}{(n+1)^2}Q_{n-1}(x)\right], \quad n = 1, 2, \ldots. \tag{4.29}$$

According to our algorithm and in view of (4.27), we use the following ansatzes for the initial values:

$$\begin{aligned} u_0(x) &= -P_0(x)w(x) + c_0P_0(x) + d_0Q_0(x), \\ u_1(x) &= -\frac{1}{3}P_1(x)w(x) - \frac{1}{6}\ln\left(x^2-1\right) - \frac{2}{3} + c_1P_1(x) + d_1Q_1(x), \quad x > 1, \end{aligned} \tag{4.30}$$

with undetermined coefficients $c_0,\ c_1,\ d_0$, and d_1. After substitution in (4.29) with $n = 1$, we demand that $u_2(x)$

must satisfy the resonant differential equation of the second kind. Thus, we get

$$\begin{gathered} c_0 = 0, c_1 = 0, \\ -d_0 + 3d_1 + 1 = 0. \end{gathered} \tag{4.31}$$

Setting, e.g., $d_0 = -\frac{1}{2}$ we get

$$d_1 = -\frac{1}{2}$$

and, hence, the particular solution

$$u_2(x) = -\frac{1}{5}P_2(x)w(x) - \frac{3x}{20}\ln(x^2-1) - \frac{1}{30}\ln\left(\frac{x+1}{x-1}\right) - \frac{3x}{5} - \frac{1}{3}Q_2(x), \quad x > 1. \tag{4.32}$$

The particular solutions $u_n(x)$, $n = 3, 4, \ldots$, can be obtained using (4.29) and the initial conditions (4.30) and (4.32).

The next theorem shows that the functions $u_n(x)$ obtained according to our recursive algorithm satisfy the resonant Legendre differential equation of the second kind for all $n = 0, 1, \ldots$.

Theorem 4.2. *The functions $u_n(x)$ obtained by the recursive algorithm (4.29) satisfy the resonant Legendre differential equation of the second kind (4.1) for each $n = 0, 1, 2, \ldots$.*

The proof is completely analogous to the proof of Theorem 4.1 in view of the fact that the Legendre functions of the second kind (they are not polynomials!) satisfy the same recurrence equation and the differentiating formula (4.20) as the Legendre polynomials (see [7], § 10.10).

5. Resonance Equation of the Jacobi Type

5.1. The Jacobi Resonance Equation of the First Kind. In this section, we consider the resonant equation of the Jacobi type

$$(1-x^2)\frac{d^2u_n(x)}{dx^2} + [\beta - \alpha - (\alpha+\beta+2)x]\frac{du_n(x)}{dx} + n(n+\beta+1)u_n(x) = P_n^{(\alpha,\beta)}(x), \tag{5.1}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [5] (§ 10.8) satisfying the homogeneous differential equation. The general solution of this equation is

$$u(x) = c_1 P_n^{(\alpha,\beta)}(x) + c_2 Q_n^{(\alpha,\beta)}(x) + u_n(x), \tag{5.2}$$

where

$$\begin{aligned} Q_n^{(\alpha,\beta)}(x) &= \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(x-1)^{n+\alpha+1}(x+1)^{\beta}\Gamma(2n+\alpha+\beta+2)} \\ &\quad \times F\left(n+1, n+\alpha+1; 2n+\alpha+\beta+2; \frac{2}{1-x}\right) \end{aligned} \tag{5.3}$$

is the Jacobi function of the second kind [5] (§ 10.8), c_1 and c_2 are arbitrary constants, and $u_n(x)$ is a particular solution of the inhomogeneous equation.

By Theorem 3.1, for a particular solution, we get

$$
\begin{aligned}
u_n(x) &= -\frac{1}{2n+\alpha+\beta+1}\left[\frac{\partial}{\partial\nu}P_\nu^{(\alpha,\beta)}(x)\right]_{\nu=n} \\
&= -\frac{1}{2n+\alpha+\beta+1}\left[\frac{\partial}{\partial\nu}\frac{\Gamma(\nu+\alpha)}{\Gamma(\alpha)\Gamma(\nu+1)}F\biggl(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\biggr)\right]_{\nu=n} \\
&= -\frac{1}{2n+\alpha+\beta+1}\biggl\{\bigl[\Psi(n+\alpha)-\Psi(n+1)\bigr]P_n^{(\alpha,\beta)}(x) \\
&\qquad + \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\frac{\partial}{\partial\nu}F\biggl(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\biggr)\bigg|_{\nu=n}\biggr\} \\
&= -\frac{1}{2n+\alpha+\beta+1}\Biggl\{\bigl[\Psi(n+\alpha)-\Psi(n+1)\bigr]P_n^{(\alpha,\beta)}(x) \\
&\qquad + \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\Biggl[\sum_{p=1}^{n}\frac{d}{dn}\frac{(\alpha+\beta+1+n)_p(-n)_p}{p!(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p \\
&\qquad + (-1)^{n+1}n!\sum_{p=n+1}^{\infty}\frac{(\alpha+\beta+1+n)_p(p-n-1)!}{p!(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\Biggr]\Biggr\}.
\end{aligned}
\tag{5.4}
$$

Thus, we have the following particular solutions:

$$
\begin{aligned}
\chi_0(x) = u_0(x) &= \frac{1}{\alpha+\beta+1}\left[-\psi(\alpha)+\psi(1)+\sum_{p=1}^{\infty}\frac{(\alpha+\beta+1)_p}{p(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\right], \\
\chi_1(x) = u_1(x) &= -\frac{1}{\alpha+\beta+3}\Biggl[\bigl(\psi(\alpha+1)-\psi(2)\bigr)P_1^{(\alpha,\beta)}(x) \\
&\qquad - \alpha\frac{(\alpha+\beta+3)}{(\alpha+1)}\left(\frac{1-x}{2}\right) \\
&\qquad + \alpha\sum_{p=2}^{\infty}\frac{(\alpha+\beta+2)_p}{p(p-1)(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\Biggr].
\end{aligned}
\tag{5.5}
$$

Differentiating the recurrence formula for the Jacobi polynomials (with respect to n)

$$
\begin{aligned}
P_{n+1}^{(\alpha,\beta)}(x) &= (a(n)x + b(n))P_n^{(\alpha,\beta)}(x) - c(n)P_{n-1}^{(\alpha,\beta)}(x), \\
a(n) &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\
b(n) &= \frac{\alpha^2-\beta^2}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\
c(n) &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}
\end{aligned}
\tag{5.6}
$$

and taking into account (5.4), we arrive at the recursion relation

$$
\begin{aligned}
u_{n+1}(x) = -\frac{1}{2n+\alpha+\beta+3}\Big[&-(2n+\alpha+\beta+1)\big(a(n)x+b(n)\big)u_n(x) \\
&+ (2n+\alpha+\beta-1)c(n)u_{n-1}(x) \\
&+ \big(a'(n)x+b'(n)\big)P_n^{(\alpha,\beta)}(x) \\
&- c'(n)P_{n-1}^{(\alpha,\beta)}(x)\Big], \qquad n = 1,2,\ldots.
\end{aligned}
\tag{5.7}
$$

Thus, by using the initial conditions (5.5) we can find $u_n(x)$ for any n.

It is quite difficult to get an explicit formula for the solution of the Jacobi resonant equation for any α and β. Therefore, we only consider an example.

Example 5.1. We consider the case of Jacobi resonant equation of the first kind with $\alpha = 1$ and $\beta = 2$. From (5.4) with $n = 0, 1,$ we get the following particular solutions;

$$
\begin{aligned}
\chi_0(x) = &-\frac{1}{64}\big[5\ln(x+1) + 11\ln(x-1)\big] \\
&+ \frac{5(x-1)+10(x^2-1)-11(x+1)^2}{96(x+1)^2(x-1)}, \\
\chi_1(x) = &-\frac{5x-1}{384}\big[10\ln(x+1)+17\ln(x-1)\big] \\
&+ \frac{1075x^4+1298x^3-1842x^2-1918x+487}{96(x+1)^2(x-1)}.
\end{aligned}
\tag{5.8}
$$

The initial values for the recursion relation (5.7) are chosen in the form

$$
\begin{aligned}
u_0(x) &= \chi_0(x) + d_0Q_0^{(1,2)}(x) + c_0P_0^{(1,2)}(x), \\
u_1(x) &= \chi_1(x) + d_1Q_1^{(1,2)}(x) + c_1P_1^{(1,2)}(x), \quad x > 1,
\end{aligned}
\tag{5.9}
$$

where the undetermined coefficients are found to guarantee that $u_2(x)$ satisfies the resonant differential equation. We substitute (5.9) in (5.7) with $n = 1$ and then in the resonant differential equation. This yields

$$\begin{aligned} c_0 &= 0, \qquad d_0 = -\frac{7}{24}, \\ c_1 &= -\frac{47}{2880}, \qquad d_1 = 0. \end{aligned} \tag{5.10}$$

Further, we proceed according to our Algorithm 3.1.

5.2. The Jacobi Resonance Equation of the Second Kind. In this section, we consider a resonant equation

$$(1-x^2)\frac{d^2u(x)}{dx^2} + \big[\beta - \alpha - (\alpha+\beta+2)x\big]\frac{du(x)}{dx} + n(n+\alpha+\beta+1)u(x) \;=\; Q_n^{(\alpha,\beta)}(x), \tag{5.11}$$

where $Q_n^{(\alpha,\beta)}(x)$ is the Jacobi function of the second kind [5] (§ 10.8) given by relation (5.3).

In view of Theorem 3.1, we get the following general formula for a particular solution:

$$\begin{aligned} u_n(x) &= -\frac{1}{2n+\alpha+\beta+1}\left[\frac{\partial}{\partial\nu}Q_\nu^{(\alpha,\beta)}(x)\right]_{\nu=n} \\ &= -\frac{1}{2n+\alpha+\beta+1}\Bigg\{\Big[\ln(2) + \Psi(n+\alpha+1) + \Psi(n+\beta+1) \\ &\qquad - \ln(x-1) - 2\Psi(2n+\alpha+\beta+1)\Big]Q_n^{(\alpha,\beta)}(x) \\ &\qquad + \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(x-1)^{n+\alpha+1}(x+1)^{\beta}\Gamma(2n+\alpha+\beta+2)} \\ &\qquad \times \frac{\partial}{\partial\nu}\,F\left(\nu+1, \nu+\alpha+1; 2\nu+\alpha+\beta+1; \frac{2}{1-x}\right)\Bigg|_{\nu=n}\Bigg\}. \end{aligned} \tag{5.12}$$

This formula is quite complicated for practical applications. Therefore, we use our recursive algorithm. By differentiation of the recurrence equation we obtain the following recurrence formula:

$$\begin{aligned} u_{n+1}(x) = -\frac{1}{2n+\alpha+\beta+3}\Big[&-(2n+\alpha+\beta+1)\big(a(n)x+b(n)\big)u_n(x) \\ &+ (2n+\alpha+\beta-1)c(n)u_{n-1}(x) \\ &+ \big(a'(n)x+b'(n)\big)Q_n^{(\alpha,\beta)}(x) \\ &- c'(n)Q_{n-1}^{(\alpha,\beta)}(x)\Big], \qquad n = 1, 2, \ldots. \end{aligned} \tag{5.13}$$

Together with $\chi_0(x) = u_0(x),\ \chi_1(x) = u_1(x)$ from (5.12) and the corresponding ansatz for the initial values, this yields an algorithm for $u_n(x)$ for any $n = 2, 3, \ldots$. Since the formulas in the general case are quite cumbersome, we restrict ourself to an example.

Example 5.2. Let $\alpha = 1$ and $\beta = 2$. Hence, for the Jacobi functions of the second kind, we find

$$Q_0^{(1,2)}(x) = -\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + \frac{3x^2+3x-2}{3(x+1)^2(x-1)},$$

$$Q_1^{(1,2)}(x) = -\frac{5x-1}{4}\ln\left(\frac{x+1}{x-1}\right) + \frac{15x^3+12x^2-13x-8}{6(x+1)^2(x-1)}, \tag{5.14}$$

$$Q_2^{(1,2)}(x) = -\frac{21x^2-6x-3}{8}\ln\left(\frac{x+1}{x-1}\right) + \frac{105x^4+75x^3-115x^2-65x+16}{20(x+1)^2(x-1)}, \ldots .$$

The general formula is as follows:

$$Q_n^{(1,2)}(x) = -P_n^{(1,2)}(x)\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + q_n^{(1,2)}(x), \tag{5.15}$$

where the functions $q_n^{(1,2)}(x)$ satisfy the recurrence equation for the Jacobi polynomials but with the initial conditions given by the second terms in $Q_0^{(1,2)}(x)$ and $Q_1^{(1,2)}(x)$. From (5.12), we get the following particular solutions of the resonant equation of the second kind with $n = 0, 1$:

$$\chi_0(x) = -\frac{1}{4}w(x) + \frac{15x-11}{48(x-1)}\ln(x+1) - \frac{15x^2+22x+3}{48(x+1)^2}\ln(x-1) - \frac{9x^2+3x-4}{72(x+1)^2\,(x-1)},$$

$$\begin{aligned}\chi_1(x) = &-\frac{1}{12}\,(5x-1)\,w(x) + \frac{-135x^3-159x^2+42x+56}{360(x+1)^2}\ln(x-1)\\ &+\frac{45x^2-32x-8}{120(x-1)}\ln(x+1) + \frac{22025x^4+18340x^3-25422x^2-18452x+3413}{8640\,(x+1)^2\,(x-1)},\end{aligned} \tag{5.16}$$

$$w(x) = \text{dilog}\left(\frac{x+1}{2}\right) + \frac{1}{2}\ln(x+1)\,\ln(x-1) - \ln(2)\ln(x-1).$$

For the initial values in our recursive algorithm, we use the ansatzes

$$\begin{aligned}u_0(x) &= \chi_0(x) + c_0P_0^{(1,2)}(x) + d_0Q_0^{(1,2)}(x),\\ u_1(x) &= \chi_1(x) + c_1P_1^{(1,2)}(x) + d_1Q_1^{(1,2)}(x), \quad x > 1,\end{aligned} \tag{5.17}$$

with undetermined coefficients $c_0,\ d_0,\ c_1$, and d_1. We now substitute these relations in (5.13) with $n = 1$ and demand that the result must satisfy the resonant differential equation. As a result, we obtain

$$d_0 = \frac{3}{2}c_0 + \frac{7}{80}, \qquad d_1 = \frac{3}{2}c_1 + \frac{881}{516} \tag{5.18}$$

and, hence, the particular solution of the resonant equation

$$u_n^{(1,2)}(x) = -\frac{1}{2n+4} P_n^{(1,2)}(x)w(x) + p_n^{(1,2)}(x)\ln(x+1) + r_n^{(1,2)}(x)\ln(x-1) + v_n^{(1,2)}(x). \tag{5.19}$$

Here, the functions $p_n^{(1,2)}(x)$, $r_n^{(1,2)}(x)$ satisfy the recurrence relation for the Jacobi polynomials with the initial conditions

$$\begin{gathered} p_0^{(1,2)}(x) = \frac{15x-11}{48(x-1)}, \qquad p_1^{(1,2)}(x) = \frac{45x^2-32x-8}{120(x-1)}, \\ r_0^{(1,2)}(x) = -\frac{15x^2+22x+3}{48(x+1)^2}, \qquad r_1^{(1,2)}(x) = \frac{-135x^3-159x^2+42x+56}{360(x+1)^2}. \end{gathered} \tag{5.20}$$

The function $v_n^{(1,2)}(x)$ satisfies the recurrence relation

$$\begin{aligned} v_{n+1}(x) = -\frac{1}{2n+\alpha+\beta+3}\Big[&-(2n+\alpha+\beta+1)\big(a(n)x+b(n)\big)v_n(x) \\ &+ (2n+\alpha+\beta-1)c(n)v_{n-1}(x) \\ &+ \big(a'(n)x+b'(n)\big)Q_n^{(\alpha,\beta)}(x) \\ &- c'(n)Q_{n-1}^{(\alpha,\beta)}(x)\Big], \qquad n = 1,2,\ldots, \end{aligned} \tag{5.21}$$

with $\alpha = 1,\ \beta = 2,$ and the initial conditions

$$\begin{aligned} v_0(x) &= -\frac{9x^2+3x-4}{72(x+1)^2\,(x-1)} + d_0 Q_0^{(1,2)}(x), \\ v_1(x) &= \frac{22025x^4+18340x^3-25422x^2-18452x+3413}{8640\,(x+1)^2\,(x-1)} + d_1 Q_1^{(1,2)}(x). \end{aligned} \tag{5.22}$$

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