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TWO-DIMENSIONAL SURFACES IN 3-DIMENSIONAL AND 4-DIMENSIONAL EUCLIDEAN SPACES. RESULTS AND UNSOLVED PROBLEMS

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We present a survey of the results obtained for 2-dimensional surfaces in E^3 and E^4 and connected with the Gaussian curvature and Gaussian torsion. In this connection, we consider the Monge–Ampére equations, obtain the generalizations of Bernstein's integral formula, and establish lower estimates for the outer diameter of the surfaces in E^3.

1. Introduction

In the last century, much attention was given to the investigations of 2-dimensional surfaces in 3-dimensional and 4-dimensional Euclidean spaces (see, e.g., the works by Efimov [1], Poznyak [4], and Rozendorn [3]). To a certain extent, this was a peak of development of the Moscow school of geometry of 2-dimensional surfaces "as a whole," which still remains unbeatable in the world scientific literature.

The works by Burago [27], Rozendorn [28], and Sabitov [46] contain great amounts of data on the geometry of surfaces.

In our survey, we present the results that were not included in the above-mentioned works. We also present relatively new results obtained by Toponogov (Sec. 3), Aminov (Sec. 4), and Sabitov (Sec. 15). We analyze the behavior of the surfaces "as a whole" depending on the restrictions imposed on the Gaussian curvature. We first consider the surfaces specified in the explicit form in E^4 and establish simple formulas for the Gaussian curvature K and Gaussian torsion κ_Γ. Despite the fact that a 2-dimensional surface has more "freedom" in E^4 or, in other words, the arbitrariness in its presentation is greater, it is unknown whether it is possible to construct a surface in E^4 with regular one-to-one projection onto the entire plane (x, y) and Gaussian curvature $K \leq -K_0^2 < 0$, where K_0 is a constant.

We now recall that, in 1961, Rozendorn constructed a closed regular surface with Gaussian curvature $K \leq -K_0^2 < 0$ in E^4 (for details, see [3]). In [5], Perel'man constructed a complete regular saddle surface with Gaussian curvature separated from zero and a unique projection onto the plane (x, y) (except a countable unbounded set of points in which the surface is not defined). It is also necessary to mention the Blanuša construction of an isometric embedding of the entire Lobachevskii plane into E^6 [6]. This surface is bijectively projected onto the plane (x, y). By using the functions introduced by Blanuša, Rozendorn constructed an isometric immersion of the Lobachevskii plane into E^5 [7]. In this case, the surface has self-intersections and cannot be bijectively projected onto the plane (x, y). In [8], Sabitov constructed a piecewise analytic immersion of the Lobachevskii plane into E^4 with discretely many singular lines where the surface belongs to the Lipschitz class $C^{0,1}$.

Thus, the following problem remains open: *Is it possible to isometrically embed or immerse a Lobachevskii plane into E^4 in the form of a regular surface.*

Note that the theorems on impossibility of some special isometric immersions of a Lobachevskii space into Euclidean spaces were proved by Kadomtsev [29], Aminov [30, 31], Xavier [32], Masal'tsev [33], Nikolaevskii [34],

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and Bolotov [35]. Here, we do not analyze these works. An extensive survey of the problem of isometric immersions of space forms into Riemannian and pseudo-Riemannian spaces was presented by Borisenko in [42]. The properties of 2-dimensional surfaces in Euclidean spaces were also discussed in the monographs [41, 43].

An important role in geometry is played by the Monge–Ampére equations. In the present survey, much attention is given to the Monge–Ampére operator on a Riemannian manifold. We also present a very useful generalization of the integral Bernstein formula. In a special case where the Riemannian manifold is a plane, Heinz used the Bernstein formula to get an analytic proof of the Efimov estimate for the sizes of a circle or a square over which the surface $z = z(x, y)$ with Gaussian curvature $K \leq -K_0^2 < 0$ is defined in the space E^3.

Another important application of the generalized Bernstein formula is the author's result on the estimation of the outer diameter of a surface in E^3 depending on the Gaussian curvature. For the first time, estimates of this kind were established by Burago who used more complicated methods based on the approximation of surfaces by polyhedra. The generalized Bernstein formula and related formulas presented in our survey strongly simplify the analysis of the mentioned problems.

Earlier, general theorems on the existence of solutions of the Dirichlet problem for the elliptic Monge–Ampére equation were proved in numerous works. However, the corresponding solutions were not presented in the explicit form. In our survey, we present recently established theorems on the construction of polynomial solutions of the simplest Monge–Ampére equation

$$z_{xx}z_{yy} - z_{xy}^2 = f(x, y)$$

in the case where $f(x, y)$ is a polynomial. The first investigation was performed for the case of a quadratic polynomial $f(x, y)$ by the author together with a group of Turkish geometricians and published in [20].

The Sabitov theorem [24] gives an unusual structure of the surfaces $z(x, y)$ defined in the entire plane (x, y) with singularities at certain points in the case where $f(x, y) \equiv 0$. This theorem can be regarded, in a certain sense, as a result of discussions carried out within the framework of the Ukrainian–Russian joint investigation "Isometric Immersions of Metrics and External Geometric Properties of the Surfaces in the Spaces with Constant Curvature" in 2012–2013. In the case where the singularities are vertices of a convex polygon, the theorem is proved by the author.

In [17], with an aim to construct surfaces F^2 in E^4 given over closed surfaces M^2 in E^3 of complex topological form, the author and Gor'kavyi found simple algebraic surfaces in E^3 with symmetries called "symmetrons." A closed surface in E^4 with $K < 0$ constructed by Rozendorn is a surface of kind 7. Therefore, it is of interest to construct a surface of smaller kind with $K < 0$ in E^4 by using "symmetrons." Thus, we deduce the formulas expressing the Gaussian curvature of the surface F^2 in E^4 via the curvature M^2 and the value of the Monge–Ampére operator for a function on M^2 specifying the surface F^2. We analyze the behavior of the Gaussian curvature for specific "symmetrons" and determining functions by using computer methods.

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2. Gaussian Curvature of a 2-Dimensional Surface in E^4 Given in the Explicit Form

Consider two regular functions $u(x_1, x_2)$ and $v(x_1, x_2)$ given on a plane with coordinates x_1 and x_2. Then we can assume that a 2-dimensional surface with radius vector

$$r(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ u(x_1, x_2) \\ v(x_1, x_2) \end{pmatrix}$$

is given in E^4. Since the space is Euclidean, the metric of this surface has the form

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (du)^2 + (dv)^2$$

$$= E(dx_1)^2 + 2F dx_1 dx_2 + G(dx_2)^2,$$

where

$$E = 1 + u_1^2 + v_1^2, \quad F = u_1 u_2 + v_1 v_2, \quad \text{and} \quad G = 1 + u_2^2 + v_2^2.$$

Here and in what follows, the subscripts denote derivatives with respect to arguments.

By using the well-known formula for the Gaussian curvature K and the metric coefficients, one can represent K via the second derivatives of the functions $u(x_1, x_2)$ and $v(x_1, x_2)$. The famous Frobenius formula (for the first time, the expression for the Gaussian curvature via the coefficients of the first quadratic form was obtained by Gauss but in a more complicated form) has the form

$$K = -\frac{1}{4W^4} \begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2W} \left\{ \frac{\partial}{\partial v} \frac{E_v - F_u}{W} + \frac{\partial}{\partial u} \frac{G_u - F_v}{W} \right\}, \tag{1}$$

where u and v are coordinates of the surfaces and $W = \sqrt{EG - F^2}$. Since F is not equal to zero in the general case, the calculations are fairly cumbersome but elementary. We use another approach and rewrite the formula for the curvature via the second quadratic forms

$$K = \frac{\sum_{\alpha=1}^{2} \left[L_{11}^{\alpha} L_{22}^{\alpha} - (L_{12}^{\alpha})^2 \right]}{EG - F^2}, \tag{2}$$

where $L_{ij}^{\alpha} = (r_{ij} n_{\alpha})$ are the coefficients of the second quadratic form with respect to the unit normal n_{α}.

Further, we recall the formula from the Riemannian geometry. The intrinsic curvature of a surface in the Riemannian space M, i.e., the Gaussian curvature K_i, is connected with the extrinsic curvature K_e and the curvature of the space K_M over an area touching the surface by the formula $K_i = K_e + K_M$. In the analyzed case, the space M is Euclidean and, hence, $K_M = 0$ and $K_i = K_e$.

Assume that the normals have the following coordinates in E^4:

$$n_1 = (\xi_1, \xi_2, \xi_3, \xi_4),$$

$$n_2 = (\eta_1, \eta_2, \eta_3, \eta_4).$$

The derivatives of the radius vector take the form

$$r_1 = \begin{pmatrix} 1 \\ 0 \\ u_1 \\ v_1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 0 \\ 1 \\ u_2 \\ v_2 \end{pmatrix}, \qquad r_{ij} = \begin{pmatrix} 0 \\ 0 \\ u_{ij} \\ v_{ij} \end{pmatrix}.$$

Thus, the conditions of orthogonality of the normals to r_i give the following equations:

$$\xi_1 + \xi_3 u_1 + \xi_4 v_1 = 0,$$

$$\xi_2 + \xi_3 u_2 + \xi_4 v_2 = 0.$$

The equations for η_i are similar. On the other hand, we can write the condition of orthonormality of the basis n_1, n_2 with the help of the expressions for ξ_1 and ξ_2 in terms of ξ_3 and ξ_4 and, similarly, the expressions for η_1 and η_2 in terms of η_3 and η_4. As a result, we arrive at the system

$$\xi_3^2 A + 2\xi_3\xi_4 B + \xi_4^2 C = 1,$$

$$\xi_3\eta_3 A + (\xi_3\eta_4 + \xi_4\eta_3)B + \xi_4\eta_4 C = 0,$$

$$\eta_3^2 A + 2\eta_3\eta_4 B + \eta_4^2 C = 1,$$

where

$$A = 1 + u_1^2 + u_2^2, \qquad B = u_1 v_1 + u_2 v_2, \qquad \text{and} \qquad C = 1 + v_1^2 + v_2^2.$$

We also use the notation

$$D = \sqrt{AC - B^2} = \sqrt{1 + |\operatorname{grad} u|^2 + |\operatorname{grad} v|^2 + (u_1 v_2 - u_2 v_1)^2}.$$

In what follows, without loss of generality, we can take the vector n_1 such that $\xi_4 = 0$. Thus, it follows from the system that

$$\xi_3^2 = \frac{1}{A}, \qquad \eta_3 A + \eta_4 B = 0, \qquad \eta_4^2 = \frac{A}{D^2}.$$

Returning to formula (2), we get

$$L_{ij}^1 = (r_{ij} n_1) = u_{ij}\xi_3 + v_{ij}\xi_4,$$

$$L_{ij}^2 = (r_{ij} n_2) = u_{ij}\eta_3 + v_{ij}\eta_4.$$

Therefore,

$$\sum_{\alpha=1}^{2} \left(L_{11}^\alpha L_{22}^\alpha - (L_{12}^\alpha)^2\right) = (u_{11}u_{22} - u_{12}^2)(\xi_3^2 + \eta_3^2)$$
$$+ \ (u_{11}v_{22} - 2u_{12}v_{12} + u_{22}v_{11})(\xi_3\xi_4 + \eta_3\eta_4)$$
$$+ \ \left(v_{11}v_{22} - v_{12}^2\right)(\xi_4^2 + \eta_4^2). \tag{3}$$

By using the condition $\xi_4 = 0$, we obtain the coefficients of the second derivatives of the functions u and v:

$$\xi_3^2 + \eta_3^2 = \frac{C}{D^2}, \qquad \eta_3\eta_4 = -\frac{B}{D^2}, \qquad \eta_4^2 = \frac{A}{D^2}.$$

Substituting these coefficients in (3), we arrive at the final expression for the Gaussian curvature of the surface in E^4:

$$K = \frac{\left(u_{11}u_{22} - u_{12}^2\right)\left(1 + v_1^2 + v_2^2\right)}{D^4} - \frac{\left(u_{11}v_{22} - 2u_{12}v_{12} + u_{22}v_{11}\right)\left(u_1 v_1 + u_2 v_2\right)}{D^4} + \frac{\left(v_{11}v_{22} - v_{12}^2\right)\left(1 + u_1^2 + u_2^2\right)}{D^4}, \tag{4}$$

where

$$D^4 = \left[1 + u_1^2 + u_2^2 + v_1^2 + v_2^2 + (u_1 v_2 - u_2 v_1)^2\right]^2.$$

This relation was presented in [16].

In a special case, the expression for the surface $u = u(x_1, x_2)$ in E^3 has the form

$$K = \frac{u_{11}u_{22} - u_{12}^2}{\left(1 + u_1^2 + u_2^2\right)^2}.$$

In what follows, we use relation (4) for some special functions $u(x_1, x_2)$, and $v(x_1, x_2)$.

3. History of the Problem of Surfaces with Negative Curvature in E^3 Given in the Explicit Form

In 1953, Efimov proved the following theorem [9]:

Theorem A. *If the surface $z = f(x, y)$ is regular for all values of x and y, then its Gaussian curvature cannot be smaller than a certain negative number.*

In other words, a regular surface with strictly negative curvature $K \le -K_0 < 0$, where K_0 is an arbitrary positive constant, cannot exist over the entire plane (x, y).

Is it possible to generalize this theorem to the surfaces in E^4 ? In particular, is it possible to construct a surface of constant negative curvature $K = -1$ in E^4 projected onto the entire plane (x_1, x_2)?

Later, in [10], Efimov considerably generalized the result, namely, he proved that if a surface is defined over a square with side a, then a is bounded above. The following theorem is true:

Theorem B. *If a piece of a regular surface with Gaussian curvature $K \le -1$ is uniquely projected onto a square, then there exists a constant that cannot be exceeded by the side of this square.*

As this constant, we can take 18.9.

Theorem B immediately attracted much attention of the geometricians. As early as in 1955, Heinz [11] proposed another proof of this theorem based of the integral Bernstein formula for a surface given over a disk of radius r. Moreover, by using his method, Heinz proved that the radius r is bounded above.

Bernstein's formula is generalized by the author of the present paper for functions given on 2-dimensional and multidimensional Riemannian spaces (see [12–14]).

We now present the result obtained by the Heinz method in [13].

Consider a metric

$$ds^2 = d\sigma^2 + (du)^2,$$

where $(d\sigma)^2$ is a metric of constant negative curvature $-a^2$ given in a geodesic disk of radius R and u is a regular function of a point of this disk from the class C^2. The following theorem is true:

Theorem C. *Suppose that $d\sigma^2$ is a metric with constant negative curvature $-a^2$, the Gaussian curvature K of the metric ds^2 satisfies the inequality $K \leq -b^2$, and moreover, $b > 2a$. Then*

$$R \leq \frac{e\sqrt{3}}{b - 2a}.$$

In [13], we also considered a total metric $d\sigma^2$ with variable curvature and geodesic disks with radius r and area $S(r)$. It was proved that if

$$\int\limits_{r_1}^{\infty} \frac{dr}{\sqrt{S}(r)} = \infty,$$

then the total metric $ds^2 = d\sigma^2 + (du)^2$ with Gaussian curvature $K \leq -K_0^2$ cannot exist for any $K_0 > 0$.

This theorem can be used for metrics of the form

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (du)^2 + (dv)^2$$

in the case where $v(x_1, x_2)$ is a polynomial of x_1 and x_2 of a certain degree.

Another approach is based on the use of the relation for K and Bernstein-type formulas.

At the end of this section, in order to complete our historical overview, we present the well-known Efimov's theorem proved in 1963.

Efimov Theorem. *In E^3, the upper bound of the Gaussian curvature is not smaller than zero on any complete regular surface.*

In other words, a complete regular surface with Gaussian curvature $K \leq \text{const} < 0$ is impossible in E^3. As a condition of regularity, it is sufficient to take C^2 at every point of the surface.

The complete proof of this theorem was given in [36].

In [37], Klotz-Milnor presented a new version of the proof of this theorem with a dedication to "many Soviet geometricians who were very kind to me at the International Congress of Mathematicians in Moscow (1966)."

In the same paper, he advanced the following conjecture:

Conjecture (Milnor). *Let S be a complete surface that does not contain umbilical points and is C^2-immersed in E^3 so that the sum of squares of the principal curvatures on S is strictly separated from zero. Then K either changes sign or $K \equiv 0$.*

The proof of Efimov's theorem is based on the analysis of the mappings of E^2 onto E^2. This aspect, which is of interest in its own right, was considered in detail by Aleksandrov [38] who advanced some hypotheses concerning multidimensional mappings. We also mention the paper by Rozendorn and Shikin [39], which contains a presentation of Efimov's works on the surfaces of negative curvature in their historical relationship with the previous development of geometry.

Milnor's conjecture was analyzed by Toponogov. In [48], he proved the following assertion:

If, on any surface Φ of the class C^3, the principal curvatures k_1 and k_2 are connected by the formula

$$(1 - k_1 d)(1 - k_2 d) = -1,$$

then the surface Φ is a direct circular cylinder of radius $\dfrac{d}{2}$.

Here, $d = \mathrm{const}$. It is clear that, on the surface satisfying the equation presented above, the umbilical points are absent and

$$k_1^2 + k_2^2 \geq \frac{3}{d^2},$$

i.e., the Milnor condition is satisfied. Later, in [50], Toponogov proved a more general theorem for the case where the principal curvatures are connected by the formula $f(k_1, k_2) = 0$ and the surface does not contain umbilical points.

As far as we know, in the general statement, the Milnor conjecture is neither proved nor rejected.

4. Complex-Analytic Curve in E^4 and Other Surfaces

Assume that two regular functions $u(x_1, x_2)$ and $v(x_1, x_2)$ are given in a certain domain D of the plane (x_1, x_2). These functions specify a certain 2-dimensional surface F^2 in E^4. We now present several examples of simple surfaces in E^4.

1. We introduce a complex variable $z = x_1 + ix_2$. Assume that the complex function $f(z) = u + iv$ is complex-analytic. *In this case, the surface F^2 is called a complex-analytic curve.*

The functions u and v are conjugate harmonic functions. The subscripts denote the derivatives of these functions with respect to their arguments. We write the system of equations

$$u_1 = v_2, \qquad u_2 = -v_1,$$

$$u_{11} + u_{22} = 0, \qquad v_{11} + v_{22} = 0,$$

and deduce the expression for the Gaussian curvature of the surface F^2. We find

$$u_{11} = v_{12}, \qquad u_{12} = v_{22}, \qquad u_{22} = -v_{12}.$$

By using these relations, we get

$$u_{11}u_{22} - u_{12}^2 = v_{11}v_{22} - v_{12}^2 = -v_{11}^2 - v_{12}^2,$$

$$u_{11}v_{22} - 2u_{12}v_{12} + u_{22}v_{11} = 0.$$

Then the formula for the Gaussian curvature of the surface F^2 takes the form

$$K = \frac{2\big(v_{11}v_{22} - v_{12}^2\big)}{\big(1 + v_1^2 + v_2^2\big)^3} = \frac{-2\big(v_{11}^2 + v_{12}^2\big)}{\big(1 + v_1^2 + v_2^2\big)^3}.$$

This implies that *the Gaussian curvature of a complex-analytic curve is always nonpositive, i.e.,* $K \leq 0$.

Assume that a complex-analytic curve is given in a disk D of radius R and that its Gaussian curvature satisfies the inequality $K \leq -K_0 < 0$. Note that the Gaussian curvature $\bar{K}$ of the auxiliary surface

$$\big\{x_1, x_2, v(x_1, x_2)\big\}$$

satisfies the inequalities

$$\bar{K} = \frac{v_{11}v_{22} - v_{12}^2}{\big(1 + v_1^2 + v_2^2\big)^2} \leq \frac{v_{11}v_{22} - v_{12}^2}{\big(1 + v_1^2 + v_2^2\big)^3} \leq -\frac{K_0}{2}.$$

By using the Heinz estimate, we obtain

$$R \leq \frac{\sqrt{3}e}{\sqrt{2K_0}}.$$

Thus, in a complex-analytic curve given on the entire plane (x_1, x_2), *the Gaussian curvature is not separated from zero by a constant number.*

2. Consider the expression for the curvature K in the case where both functions u and v are harmonic and not necessarily conjugate. Note that the second derivatives in the numerator of K have the form $u_{22} = -u_{11}$ and $v_{22} = -v_{11}$. Then the numerator can be represented as follows:

$$\begin{aligned}
-(u_{11}^2 + u_{12}^2)&\big(1 + v_1^2 + v_2^2\big) + 2(u_{11}v_{11} + u_{12}v_{12})(u_1v_1 + u_2v_2) \\
&= -(v_{11}^2 + v_{12}^2)(1 + u_1^2 + u_2^2) - u_{11}^2 - u_{12}^2 - v_{11}^2 - v_{12}^2 \\
&\quad - (u_{11}v_1 - v_{11}u_1)^2 - (u_{11}v_2 - v_{11}u_2)^2 \\
&\quad - (u_{12}v_1 - v_{12}u_1)^2 - (u_{12}v_2 - v_{12}u_2)^2.
\end{aligned}$$

Hence, **if the components u and v of the surface $F^2 \subset E^4$ are harmonic functions, then the Gaussian curvature $K \leq 0$.** For the first time, this was indicated by Perel'man [5].

3. Let u and v be polynomials of x_1 and x_2 of a certain degree.

Thus, on surface (2a), we take

$$u = \frac{1}{2}(x_1^2 + x_2^2), \qquad v = 3(x_1^2 - x_2^2).$$

It is easy to see that the Gaussian curvature of surface (2a) has the form

$$K = \frac{-35}{\left(1 + 37(x_1^2 + x_2^2) + (12x_1x_2)^2\right)^2}.$$

Further, on surface (2b), we set

$$u = 3(x_1^2 + x_2^2), \qquad v = \frac{1}{2}(x_1^2 - x_2^2).$$

The Gaussian curvature of surface (2b) has the form

$$K = \frac{35}{\left(1 + 37(x_2^2 + x_2^2) + (12x_1x_2)^2\right)^2}.$$

Thus, the Gaussian curvature of surface (2a) $K < 0$, whereas the Gaussian curvature of surface (2b) $K > 0$. Both surfaces have projections onto three-dimensional spaces with both positive and negative curvatures.

4. Consider a surface

$$u = f(x_1)\cos x_2, \qquad v = f(x_1)\sin x_2.$$

This surface has the following remarkable property: Numerous geometric characteristics of this surface depend only on x_1. We denote derivatives of the function f with respect to x_1 by primes. We have

$$u_1 = f'\cos x_2, \qquad u_2 = -f\sin x_2, \qquad v_1 = f'\sin x_2, \qquad v_2 = f\cos x_2,$$

$$u_{11} = f''\cos x_2, \qquad u_{12} = -f'\sin x_2, \qquad u_{22} = -f\cos x_2,$$

$$v_{11} = f''\sin x_2, \qquad v_{12} = f'\cos x_2, \qquad v_{22} = -f\sin x_2.$$

This surface has the following metric:

$$ds^2 = (1 + f'^2)(dx_1)^2 + (1 + f^2)(dx_2)^2.$$

Since the coordinate lines x_1 and x_2 are orthogonal, by using this metric, we can easily determine the Gaussian curvature of this surface:

$$K = -\frac{f''f(1 + f^2) + f'^2(1 + f'^2)}{(1 + f^2)^2(1 + f'^2)^2}.$$

5. Gaussian Torsion of a Surface in E^4 Given in the Explicit Form

It is known that the Gaussian torsion κ_Γ is defined as follows:

$$\kappa_\Gamma = \frac{1}{\sqrt{g}}\left[L_{1i}^1 L_{2j}^2 - L_{2i}^1 L_{1j}^2\right] g^{ij},$$

where g^{ij} are the coefficients of the inverse metric tensor and g is the determinant of the matrix of metric tensor.

If a and b are the semiaxes of the ellipse of normal curvature, then $\kappa_\Gamma = \pm 2ab$ and the sign depends on the direction of traversing the ellipse of normal curvature by the end of the vector of normal curvature. Thus, we take the sign $+$ in the case of positive direction relative to the positive orientation in the normal plane or the sign $-$ for the negative direction. This quantity is the sole invariant of normal connectedness of the surface analogous to the curvature of tangential connectedness, i.e., to the Gaussian curvature. The integral of κ_Γ over the closed surface F^2 is equal to zero if the surface contains a regular normal unit vector field. In the general case, it is equal to $2\pi\nu$, where ν is the Whitney invariant, i.e., the sum of indices of the singularities of a unit normal vector field. This fact (and even a more general case) were established by Chern [40]. For details, see [41] (Chap. 6).

We have

$$g_{11} = 1 + u_1^2 + v_1^2, \qquad g_{12} = u_1u_2 + v_1v_2, \qquad g_{22} = 1 + u_2^2 + v_2^2,$$

$$g = 1 + u_1^2 + u_2^2 + v_1^2 + v_2^2 + (u_1v_2 - u_2v_1)^2.$$

The expressions for the coefficients of the inverse metric tensor take the form

$$g^{11} = \frac{1 + u_2^2 + v_2^2}{g}, \qquad g^{12} = -\frac{u_1u_2 + v_1v_2}{g}, \qquad g^{22} = \frac{1 + u_1^2 + v_1^2}{g}.$$

We now recall the expressions for the coefficients of the second quadratic forms obtained earlier:

$$L^1_{ij} = \xi_3 u_{ij} + \xi_4 v_{ij},$$

$$L^2_{ij} = \eta_3 u_{ij} + \eta_4 v_{ij}.$$

In view of the fact that $\xi_4 = 0$, we find

$$\begin{aligned}\big[L^1_{1i}L^2_{2j} - L^1_{2i}L^2_{1j}\big]g^{ij} &= \xi_3\eta_4(u_{1i}v_{2j} - u_{2i}v_{1j})g^{ij} \\ &= \xi_3\eta_4\big[(u_{11}v_{21} - u_{12}v_{11})g^{11} + (u_{11}v_{22} - u_{22}v_{11})g^{12} + (u_{12}v_{22} - u_{22}v_{12})g^{22}\big].\end{aligned}$$

Earlier, we have obtained

$$\xi_3^2 = \frac{1}{A}, \qquad \eta_4^2 = \frac{A}{D^2}.$$

In order to correctly choose the signs of the root, we consider a basis in E^4 formed by tangential and normal vectors. Assume that the orientation of this basis is positive. Then the determinant Δ formed by the components of these vectors is also positive. Hence,

$$\Delta = \begin{vmatrix} 1 & 0 & u_1 & v_1 \\ 0 & 1 & u_2 & v_2 \\ \xi_1 & \xi_2 & \xi_3 & 0 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{vmatrix} = \xi_3\eta_4 - \xi_2\eta_4u_2 + (\xi_2\eta_3 - \eta_2\xi_3)v_2 - u_1\xi_1\eta_4 + v_1(\xi_1\eta_3 - \xi_3\eta_1) + (u_1v_2 - u_2v_1)(\xi_1\eta_2 - \xi_2\eta_1).$$

We use the relations

$$\xi_1 = -\xi_3 u_1, \qquad \eta_1 = -\eta_3 u_1 - \eta_4 v_1,$$

$$\xi_2 = -\xi_3 u_2, \qquad \eta_2 = -\eta_3 u_2 - \eta_4 v_2.$$

Substituting these relations in the determinant, we obtain

$$\Delta = \xi_3\eta_4\big(1 + u_1^2 + u_2^2 + v_1^2 + v_2^2 + (u_1v_2 - u_2v_1)^2\big).$$

Therefore, $\xi_3\eta_4 > 0$. By using the above-mentioned relations, we get

$$\xi_3\eta_4 = \frac{1}{D}.$$

Note that $D^2 = g$. Hence, we can write the following expression for the Gaussian torsion:

$$\kappa_\Gamma = \frac{1}{g^2}\big[(u_{11}v_{12} - u_{12}v_{11})(1 + u_2^2 + v_2^2)$$

$$- (u_{11}v_{22} - u_{22}v_{11})(u_1u_2 + v_1v_2) + (u_{12}v_{22} - u_{22}v_{12})(1 + u_1^2 + v_1^2)\big].$$

As an example, we consider a complex-analytic curve. By using the deduced formula and the expressions for the first and second derivatives of the functions u and v, we get

$$\kappa_\Gamma = -2\frac{v_{11}v_{22} - v_{12}^2}{\big(1 + v_1^2 + v_2^2\big)^3}.$$

This expression for the Gaussian torsion differs from the expression for the Gaussian curvature K obtained above only by sign. Hence, for the complex-analytic curve, we can write

$$K + \kappa_\Gamma = 0.$$

The other interesting class of surfaces is specified by the equations

$$u = \Phi_x, \qquad v = \Phi_y,$$

where $\Phi(x, y)$ is a regular function. For this surface, the Gaussian curvature is expressed in terms of the third derivatives of the function Φ:

$$K = \frac{1}{g^2}\big[(\Phi_{xxx}\Phi_{xyy} - \Phi_{xxy}^2)(1 + \Phi_{xy}^2 + \Phi_{yy}^2)$$

$$- (\Phi_{xxx}\Phi_{yyy} - \Phi_{xxy}\Phi_{xyy})(\Phi_{xx} + \Phi_{yy})\Phi_{xy} + (\Phi_{xxy}\Phi_{yyy} - \Phi_{xyy}^2)(1 + \Phi_{xx}^2 + \Phi_{xy}^2)\big],$$

where

$$g = 1 + \Phi_{xx}^2 + 2\Phi_{xy}^2 + \Phi_{yy}^2 + (\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2)^2.$$

For the Gaussian torsion κ_Γ, we get the same expression, i.e., for this surface,

$$K = \kappa_\Gamma.$$

More general relations in E^4 with $K \pm \kappa_\Gamma = 0$ were obtained in [15].

For the analyzed surface $u = f(x)\cos y,\ v = f(x)\sin y$, the following expression for the Gaussian torsion was established in [16]:

$$\kappa_\Gamma = \frac{f''f'(1+f^2) + f'f(1+f'^2)}{(1+f^2)^2(1+f'^2)^2}.$$

By using this result, we found the surface with constant nonzero Gaussian torsion. It was proved that the width of the strip $t_1 \le x \le t_2$ in a regular part of the surface is bounded above.

6. Bernstein Formula and Its Application by Heinz

Assume that a regular function $z = z(x, y)$ from the class C^2 is given in a disk D of radius R in the plane (x, y). This function specifies a surface F^2 in E^3. We assume that the center of the disk coincides with the origin of coordinates and introduce polar coordinates $r,\ \phi$ in this disk. By $D(r)$ we denote the disk of radius r centered at the origin and by $\Gamma(r)$ we denote its boundary circle.

To prove Efimov's theorem, Heinz used the Bernstein formula

$$\frac{d}{dr}\int\limits_{\Gamma(r)} \frac{z_\phi^2}{r}\,d\phi = 2\int\limits_{D(r)} (z_{xy}^2 - z_{xx}z_{yy})\,dx\,dy + \int\limits_{\Gamma(r)} z_r^2\,d\phi. \tag{5}$$

This formula is proved in Sec. 7.

Assume that the Gaussian curvature K of the surface F^2 satisfies the inequality

$$K \le -K_0 < 0,$$

where K_0 is a positive constant. Since

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2},$$

the derivatives of the function z satisfy the inequality

$$z_{xy}^2 - z_{xx}z_{yy} \ge K_0(1+z_x^2+z_y^2)^2.$$

We introduce a function of single variable t as follows:

$$f(t) = \int\limits_0^t \left(\int\limits_{\Gamma(r)} \frac{z_\phi^2}{r}\,d\phi\right) dr + S(t),$$

where $S(t)$ is the area of the disk of radius t, i.e., πt^2.

This function possesses the derivative

$$f'(t) = \int\limits_{\Gamma(t)} \frac{z_\phi^2}{t}\, d\phi + 2\pi t.$$

Note that $f(0) = 0$ and $f'(0) = 0.$ Indeed, the integral in the expression for f' can be rewritten in the form

$$\int\limits_{\Gamma(t)} z_s^2\, ds,$$

where s is the length of arc of the circle $\Gamma(t)$. Since the function $z(x, y)$ is regular, the modulus of the derivative z_s is bounded above. Since the length of the circle $\Gamma(t)$ approaches zero as $t \to 0,$ this integral also tends to zero. In addition, we note that $f(t) \geq \pi t^2$.

It follows from the Bernstein formula (5) that

$$f''(t) = 2 \int\limits_{D(t)} (z_{xy}^2 - z_{xx} z_{yy})\, dx\, dy + \int\limits_{\Gamma(t)} z_t^2\, d\phi + 2\pi$$

$$\geq 2K_0 \int\limits_{D(t)} (1 + z_x^2 + z_y^2)^2\, dx\, dy + 2\pi.$$

On the other hand, we can estimate the function $f(t)$ from above as follows:

$$f(t) = \int\limits_0^t \left(\int\limits_{\Gamma(r)} z_s^2 ds \right) dr + \int\limits_{D(t)} dx\, dy \leq \int\limits_{D(t)} (1 + z_x^2 + z_y^2)\, dx\, dy.$$

By using the Cauchy–Bunyakovskii inequality

$$\left(\int\limits_{D(t)} (1 + z_x^2 + z_y^2)\, dx\, dy \right)^2 \leq \left[\int\limits_{D(t)} \big((1 + z_x^2 + z_y^2)^2\, dx\, dy \right] \pi t^2,$$

we get

$$f^2(t) \leq \left[\int\limits_{D(t)} (1 + z_x^2 + z_y^2)\, dx\, dy \right] \pi t^2.$$

Comparing this inequality with the inequality for the second derivative, we find

$$f''(t) \geq \frac{2K_0 f^2(t)}{\pi t^2}.$$

Note that $f'(t) > 0$ for $t > 0$. We multiply both sides of the inequality by $f'(t)$ and integrate it from 0 to t. In view of $f(0) = f'(0) = 0$, this yields

$$\frac{f'(t)}{f^{\frac{3}{2}}(t)} \geq \frac{2\sqrt{K_0}}{\sqrt{3\pi}t}.$$

Let $0 < t_1 < t_2$. Integrating from t_1 to t_2, we get

$$\frac{1}{\sqrt{f(t_1)}} - \frac{1}{\sqrt{f(t_2)}} \geq \frac{2\sqrt{K_0}}{\sqrt{3\pi}} \ln \frac{t_2}{t_1}.$$

Since $f(t) \geq \pi t^2$, we find

$$\frac{1}{t_1} \geq \frac{2\sqrt{K_0}}{\sqrt{3}} \ln \frac{t_2}{t_1}.$$

The estimate for t_2 is expressed via t_1 as follows:

$$\frac{1}{t_1} + \frac{2\sqrt{K_0}}{\sqrt{3}} \ln t_1 \geq \frac{2\sqrt{K_0}}{\sqrt{3}} \ln t_2.$$

Consider the function on the left-hand side of this inequality

$$\theta(t_1) = \frac{1}{t_1} + \frac{2\sqrt{K_0}}{\sqrt{3}} \ln t_1.$$

We now find the point of minimum of this function. We get

$$\theta'(t_1) = -\frac{1}{t^2} + \frac{2\sqrt{K_0}}{\sqrt{3}t_1} = 0.$$

Therefore, the function $\theta(t_1)$ takes the minimum value for

$$\frac{1}{t_1} = \frac{2\sqrt{K_0}}{\sqrt{3}}.$$

Further, we get the following estimate for t_2:

$$1 + \ln \frac{\sqrt{3}}{2\sqrt{K_0}} \geq \ln t_2.$$

Setting $t_2 = R$, we conclude that the *radius R of the disk D over which a regular surface of the form $z = z(x, y)$ with Gaussian curvature $K \leq -K_0$ may exist is bounded above, i.e.,*

$$R \leq \frac{\sqrt{3}e}{2\sqrt{K_0}}.$$

The Heinz estimate somewhat improves the Efimov estimate.

7. Proof of the Bernstein Formula (5)

We now write the transformation of coordinates

$$x = r\cos\phi, \qquad r = \sqrt{x^2+y^2},$$
$$y = r\sin\phi, \qquad \phi = \arctan\frac{y}{x},$$

and determine the relations between the derivatives:

$$z_x = z_r r_x + z_\phi \phi_x = z_r \cos\phi - z_\phi \frac{\sin\phi}{r},$$
$$z_y = u_r r_y + z_\phi \phi_y = z_r \sin\phi + z_\phi \frac{\cos\phi}{r}.$$

We represent the Hessian of the function z in the form

$$z_{xx}z_{yy} - z_{xy}^2 = \frac{1}{2}\left[\frac{\partial}{\partial x}(z_x z_{yy} - z_y z_{xy}) + \frac{\partial}{\partial y}(z_y z_{xx} - z_x z_{xy})\right].$$

Differentiating the right-hand side, we conclude that the third derivatives are mutually cancelled. Integrating the Hessian of the function z over the disk D and using the Green formula, we obtain

$$J = \int\limits_D (z_{xx}z_{yy} - z_{xy}^2)\,dx\,dy = \frac{1}{2}\left[\int\limits_{\Gamma(r)} (z_x z_{xy} - z_y z_{xx})dx + (z_x z_{yy} - z_y z_{xy})dy\right]$$
$$= \int\limits_{\Gamma(r)} z_x(z_{yx}dx + z_{yy}dy) - z_y(z_{xx}dx + z_{xy}dy) = \int\limits_{\Gamma(r)} (z_x dz_y - z_y dz_x).$$

Substituting the expressions for the derivatives of the function z and taking into account the fact that the differentials of these derivatives are taken along the boundary circle for fixed r, we get

$$J = \frac{1}{2}\left[\int\limits_{\Gamma(r)} \left\{\left(z_r\cos\phi - z_\phi\frac{\sin\phi}{r}\right)\left(z_{r\phi}\sin\phi + z_{\phi\phi}\frac{\cos\phi}{r} + z_r\cos\phi - z_\phi\frac{\sin\phi}{r}\right)\right.\right.$$
$$\left.\left. - \left(z_r\sin\phi + z_\phi\frac{\cos\phi}{r}\right)\left(z_{r\phi}\cos\phi - z_{\phi\phi}\frac{\sin\phi}{r} - z_r\sin\phi - z_\phi\frac{\cos\phi}{r}\right)\right\} d\phi\right]$$
$$= \frac{1}{2}\left[\int\limits_{\Gamma(r)} \left(-\frac{z_\phi z_{r\phi}}{r} + \frac{z_r z_{\phi\phi}}{r} + z_r^2 + \frac{z_\phi^2}{r^2}\right) d\phi\right].$$

Note that

$$\int\limits_{\Gamma(r)} \frac{z_r z_{\phi\phi}}{r} d\phi = \int\limits_{\Gamma(r)} \left(\frac{\partial z_r z_\phi}{r \partial \phi} - \frac{z_{r\phi} z_\phi}{r} \right) d\phi = - \int\limits_{\Gamma(r)} \frac{z_\phi z_{r\phi}}{r} d\phi.$$

In addition, we can write

$$-2 \frac{z_\phi z_{r\phi}}{r} + \frac{z_\phi^2}{r^2} = -\frac{\partial}{\partial r} \left(\frac{z_\phi^2}{r} \right).$$

Since the integral is taken over the circle $r = \text{const}$, we can factor out the derivative with respect to r from the integrand. As a result, we get the Bernstein formula (5). For the first time, this formula was obtained in the hard-to-reach work [45].

8. Generalization of the Bernstein Formula (5)

We generalize the Bernstein formula (5) by assuming that the function z is given in a certain domain with general Riemannian metric $ds^2 = g_{ij} dx^i dx^j$.

1. We first consider the integral of divergence of a certain vector over the surface. If the vector field a is given by its contravariant components a^i, then the quantity $a^i_{,i}$ is called the divergence of the field a, i.e.,

$$\operatorname{div} a = a^i_{,i}.$$

Here and in what follows, the comma in the subscript denotes the corresponding covariant derivative.

The divergence of the field a can be also represented in the form

$$\operatorname{div} a = \frac{1}{\sqrt{g}} \frac{\partial a^i \sqrt{g}}{\partial x^i},$$

where $g = g_{11} g_{22} - g_{12}^2$. Indeed, the expression on the right-hand side admits a representation

$$\frac{\partial a^i}{\partial x^i} + \frac{a^i}{2g} \frac{\partial g}{\partial x^i} = \frac{\partial a^i}{\partial x^i} + \Gamma^k_{ki} a^i = a^i_{,i}$$

because it is known that

$$\Gamma^k_{ki} = \frac{\partial g}{2g \partial x^i}$$

in the Riemannian geometry. Integrating $\operatorname{div} a$ over a domain D with boundary Γ and using the Stokes formula, we obtain

$$\int\limits_D \operatorname{div} a \, dS = \int\limits_D \frac{1}{\sqrt{g}} \frac{\partial a^i \sqrt{g}}{\partial x^i} \sqrt{g} \, dx^1 dx^2$$

$$= \int\limits_\Gamma \sqrt{g} \big(-a^2 dx^1 + a^1 dx^2 \big).$$

The vector tangential to the curve Γ has the contravariant components dx^1 and dx^2. We introduce the unit normal vector τ to the curve Γ with the help of its covariant components by setting

$$\tau_1 = \lambda\, dx^2, \qquad \tau_2 = -\lambda\, dx^1.$$

Then the condition of orthogonality is satisfied: $dx^i \nu_i = 0$. We determine λ from the condition guaranteeing that this is a unit vector, namely,

$$1 = \tau_i \tau_j g^{ij} = \lambda^2 \big[(dx^2)^2 g^{11} - 2dx^1 dx^2 g^{12} + (dx^1)^2 g^{22}\big].$$

However,

$$g^{11} = g_{22}/g, \qquad g^{12} = -g_{12}/g, \qquad g^{22} = g_{11}/g.$$

Therefore,

$$\lambda = \frac{\sqrt{g}}{ds},$$

where ds is an element of arc length for the curve Γ. In view of the relations established above, we can write

$$\int\limits_D \operatorname{div} a\, dS = \int\limits_\Gamma (a\tau)\, ds, \tag{6}$$

i.e., the integral of divergence of the field a over the surface is expressed via the integral of scalar product of this field by the unit vector normal to the boundary over the arc length of the boundary.

2. In 1901, in their work "Méthodes de calcul différentiel absolu et leurs applications," Ricci and Levi–Civita defined differential invariants of a function $\phi(x^1, \dots, x^n)$ given on an n-dimensional Riemannian space as coefficients of $\lambda^{n-1}, \lambda^{n-2}, \dots, \lambda$ of the equation

$$\frac{1}{g} |\phi_{,ij} - \lambda g_{ij}| = 0.$$

We now write this equation for $n = 2$:

$$\frac{1}{g}\begin{vmatrix} \phi_{,11} - \lambda g_{11} & \phi_{,12} - \lambda g_{12} \\ \phi_{,21} - \lambda g_{21} & \phi_{,22} - \lambda g_{22} \end{vmatrix}$$

$$= \frac{1}{g}\big[\phi_{,11}\phi_{,22} - (\phi_{,12})^2 - \lambda(\phi_{,11} g_{22} - 2\phi_{,12} g_{12} + \phi_{,22} g_{11}) + \lambda^2 g\big] = 0.$$

Note that the second covariant derivatives of the function satisfy the equality $\phi_{,ij} = \phi_{,ji}$. The coefficient of $-\lambda$ is the Laplace–Beltrami operator

$$\nabla_2 \phi = \phi_{,ij} g^{ij}.$$

The following expression is a generalization of the Monge–Ampére operator (or the generalized Hessian):

$$\nabla_{22}\phi = \frac{\phi_{,11}\phi_{,22} - (\phi_{,12})^2}{g_{11}g_{22} - (g_{12})^2}.$$

We transform this operator by separating its divergence part

$$\nabla_{22}\phi = \nu^1_{,1} + \nu^2_{,2} + \frac{1}{2g}\big[-\phi_{,1}(\phi_{,221} - \phi_{,122}) + \phi_{,2}(\phi_{,121} - \phi_{,112})\big],$$

where

$$\nu^1 = \frac{\phi_{,1}\phi_{,22} - \phi_{,2}\phi_{,12}}{2g},$$

$$\nu^2 = \frac{\phi_{,2}\phi_{,11} - \phi_{1}\phi_{,12}}{2g}.$$

By ν we denote the vector field with the components ν^i. In what follows, we show that the quantities ν^i form a tensor. By using the formula from the Riemannian geometry, for the difference of the third covariant derivatives, we find

$$\phi_{,221} - \phi_{,122} = R^i_{.221}\phi_{,i} = R_{1221}\phi^{,1},$$

$$\phi_{,121} - \phi_{,112} = R^i_{.121}\phi_{,i} = R_{2121}\phi^{,2}.$$

Here, R_{ijkl} is the Riemannian tensor of the metric ds^2. However, the Gaussian curvature K of the metric ds^2 is given by the formula

$$K = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}.$$

Thus, we can write

$$\nabla_{22}\phi = \operatorname{div}\nu + \frac{1}{2}K\nabla_1\phi, \tag{7}$$

where $\nabla_1\phi = |\operatorname{grad}\phi|^2$ is the first differential Beltrami parameter.

Consider the integral of $\nabla_{22}z$ over a certain domain D. By using relation (6), we get

$$\int\limits_D \nabla_{22}z\, dS = \int\limits_\Gamma (\nu\tau)\, ds + \int\limits_D \frac{1}{2}K\nabla_1 z\, dS. \tag{8}$$

3. We now prove that a collection of ν^i forms a tensor. To do this, we introduce new coordinates

$$u^i = u^i(x^1, x^2), \quad i = 1, 2.$$

The overbars denote the quantities in new coordinates. If ν^i form a tensor, then, in the new coordinates, we have

$$\bar{\nu}^\alpha = \nu^i \frac{\partial u^\alpha}{\partial x^i}.$$

In particular, we check that

$$\bar{\nu}^1 = \nu^i \frac{\partial u^1}{\partial x^i}.$$

By definition, we find

$$\bar{\nu}^1 = \frac{\bar{\phi}_{,1}\bar{\phi}_{,22} - \bar{\phi}_{,2}\bar{\phi}_{,12}}{\bar{g}}.$$

According to the definition of a tensor, we get

$$\bar{\phi}_{,\alpha} = \phi_i \frac{\partial x^i}{\partial u^\alpha}, \qquad \bar{\phi}_{,\beta\gamma} = \phi_{,ij}\frac{\partial x^i}{\partial u^\beta}\frac{\partial x^j}{\partial u^\gamma}, \qquad \bar{g} = gJ^2\left(\frac{x^1, x^2}{u^1, u^2}\right),$$

where $J\left(\frac{x^1, x^2}{u^1, u^2}\right)$ is the Jacobian of transformation from the old coordinates to the new coordinates. Substituting these expressions, we get

$$\bar{\nu}^1 = \left[\nu^1 \frac{\partial x^2}{\partial u^2} - \nu^2 \frac{\partial x^1}{\partial u^2}\right] \frac{1}{J\left(\frac{x^1, x^2}{u^1, u^2}\right)}.$$

However, the relations

$$\frac{\partial u^1}{\partial x^1} = \frac{\frac{\partial x^2}{\partial u^2}}{J\left(\frac{x^1, x^2}{u^1, u^2}\right)}, \qquad \frac{\partial u^1}{\partial x^2} = -\frac{\frac{\partial x^1}{\partial u^2}}{J\left(\frac{x^1, x^2}{u^1, u^2}\right)}$$

are true. Therefore,

$$\bar{\nu}^1 = \nu^i \frac{\partial u^1}{\partial x^i},$$

Q.E.D.

4. We now prove the generalized Bernstein formula. In a neighborhood of the curve Γ, we introduce a semi-geodesic coordinate system $x^1,\ x^2$ (or, in a different notation, $r,\ \phi$). In this system, the metric takes the form

$$ds^2 = (dr)^2 + G(d\phi)^2.$$

To do this, we draw geodesic lines orthogonal to the curve Γ (the lines $\phi = \text{const}$) and take their orthogonal trajectories, i.e., the lines $r = \text{const}$ denoted by $\Gamma(r)$. The unit tangential vector τ orthogonal to $\Gamma(r)$ has the components $\tau^1 = 1$ and $\tau^2 = 0$. Hence, the scalar product $(\nu\tau) = \nu^1$. In this case, $\nu^1 = \nu_1$ because $g_{11} = 1$ and $g_{12} = 0$. We now return to relation (8) and consider the contour integral denoted by A:

$$A = \int\limits_{\Gamma(r)} (\nu\tau)\, ds = \int\limits_{\Gamma(r)} \frac{z_1 z_{,22} - z_2 z_{,12}}{2g}\sqrt{G}\, du^2.$$

Further, we write the expressions for the covariant derivatives of the function $z.$ The first derivatives coincide with the ordinary derivatives. For the second derivatives, we obtain

$$z_{,22} = z_{u^2u^2} - \Gamma^1_{22} z_{u^1} - \Gamma^2_{22} z_{u^2},$$

$$z_{,12} = z_{u^1u^2} - \Gamma^1_{12} z_{u^1} - \Gamma^2_{12} z_{u^2}.$$

By using the expressions for the Christoffel symbols

$$\Gamma^1_{22} = -\frac{1}{2} G_{u^1}, \qquad \Gamma^2_{22} = \frac{1}{2G} G_{u^2}, \qquad \Gamma^1_{12} = 0, \qquad \Gamma^2_{12} = \frac{1}{2G} G_{u^1}$$

and the formulas $g = G$ and $ds = \sqrt{G} du^2$, we obtain

$$A = \int\limits_{\Gamma(r)} \left[\frac{z_{u^1} z_{u^2u^2}}{2\sqrt{G}} - \frac{z_{u^1} z_{u^2} G_{u^2}}{4G^{3/2}} - \frac{z_{u^2} z_{u^1u^2}}{2\sqrt{G}} + \frac{G_{u^1} z^2_{u^2}}{4G^{3/2}} + \frac{G_{u^1} z^2_{u^1}}{4\sqrt{G}} \right] du^2.$$

We now use the relations

$$\frac{z_{u^1} z_{u^2u^2}}{2\sqrt{g}} - \frac{G_{u^2} z_{u^1} z_{u^2}}{4G^{3/2}} = \frac{\partial}{\partial u^2} \left(\frac{z_{u^1} z_{u^2}}{2\sqrt{G}} \right) - \frac{z_{u^2} z_{u^1u^2}}{2\sqrt{G}}.$$

Integrating the first term on the right-hand side along the closed curve $\Gamma(r)$, we obtain zero. Combining the second term on the right-hand side with the third term in the expression for A, we can write

$$-\frac{z_{u^2} z_{u^1u^2}}{\sqrt{G}} + \frac{G_{u^1} z^2_{u^2}}{4G^{3/2}} = -\frac{\partial}{\partial u^1} \left(\frac{z^2_{u^2}}{2\sqrt{G}} \right), \qquad \frac{1}{\rho_g} = \frac{G_{u^1}}{2G}, \qquad \frac{z_{u^2}}{\sqrt{G}} = z_s,$$

where $\dfrac{1}{\rho_g}$ is the geodesic curvature of the curve $\Gamma(r)$. We substitute these expressions in the expression for A. Applying relation (8), we obtain **the generalization of the Bernstein formula (5) for arbitrary 2-dimensional surfaces:**

$$2 \int\limits_{D(r)} \nabla_{22} z \, dS = -\frac{d}{dr} \int\limits_{\Gamma(r)} (z_s)^2 \, ds + \int\limits_{\Gamma(r)} \frac{(z_r)^2}{\rho_g} \, ds + \int\limits_{D(r)} K \nabla_1 z \, dS. \tag{9}$$

Here, K is the Gaussian curvature of the metric ds^2, $\dfrac{1}{\rho_g}$ is the geodesic curvature of the curve $\Gamma(r)$,

$$\nabla_1 z = |\operatorname{grad} z|^2$$

is the first differential Beltrami parameter of the function z or, in other words, the squared modulus of the gradient of this function. The first two integrals on the right-hand side of this formula are taken over the length of the arc s.

This formula was obtained by the author in [12].

9. Relationship Between the Curvatures of the Metrics $d\sigma^2$ and $ds^2 = d\sigma^2 + du^2$

Assume that a regular function $u(x^1, \ldots, x^n)$ is given on an n-dimensional Riemannian manifold with the metric

$$d\sigma^2 = a_{ij} dx^i dx^j.$$

We now find the relationship between the curvatures of the metrics $d\sigma^2$ and $ds^2 = d\sigma^2 + du^2$. By g_{ij} we denote the coefficients of the metric ds^2. Thus, we get

$$g_{ij} = a_{ij} + u_i u_j,$$

where u_i are the derivatives of the function u with respect to the coordinates x^i. We can always choose coordinates to guarantee that the Christoffel symbols of the metric $d\sigma^2$ are equal to zero at a fixed point P_0. At this point, the coordinates are orthogonal and $a_{ij} = \delta_{ij}$. Let R_{hijk} be the Riemannian tensor of the metric ds^2 and let $\bar{R}_{hijk}$ be the Riemannian tensor of the metric $d\sigma^2$. We now apply the following formula for the Riemannian tensor from the monograph "Riemannian Geometry" by Eisenhart:

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h}\right) + g^{lm}(\Gamma_{ij,m}\Gamma_{hk,l} - \Gamma_{ik,m}\Gamma_{hj,l}).$$

Here, $\Gamma_{pl.m}$ are the Christoffel symbols of the metric ds^2. However, at the point P_0, we have $\Gamma_{ij,k} = u_{ij}u_k$. Thus, setting $h = j$ and $i = k$, we get

$$R_{hihi} = \frac{1}{2}\left(2\frac{\partial^2 a_{hi}}{\partial x^h \partial x^i} - \frac{\partial^2 a_{hh}}{\partial x^i \partial x^i} - \frac{\partial^2 a_{ii}}{\partial x^h \partial x^h}\right)$$
$$+ \frac{1}{2}\left(2\frac{\partial^2 u_h u_i}{\partial x^h \partial x^i} - \frac{\partial^2 u_h^2}{\partial x^i \partial x^i} - \frac{\partial^2 u_i^2}{\partial x^h \partial x^h}\right) + g^{lm}(u_{ih}^2 - u_{ii}u_{hh})u_l u_m.$$

The first term on the right-hand side that contains solely the derivatives a_{ij} is the Riemannian tensor of the metric $d\sigma^2$ at the point P_0. The third mixed derivatives of the function u are mutually cancelled. As a result, we obtain

$$R_{hihi} = \bar{R}_{hihi} + (u_{,hh}u_{,ii} - u_{,ih}^2)(1 - u_l u_m g^{lm}).$$

We replace the ordinary derivatives of the function u with covariant derivatives because the Christoffel symbols of the metric $d\sigma^2$ are equal to zero at the point P_0. The curvature of a surface element touching the coordinate lines $x^h,\ x^i$ is given by the formula

$$K_{hi} = \frac{R_{hihi}}{g_{hh}g_{ii} - g_{hi}^2}.$$

Since $g_{hh}g_{ii} - g_{hi}^2 = 1 + u_h^2 + u_i^2$, we can write the following relationship between the curvatures of the surface elements for the metrics $d\sigma^2$ and ds^2:

$$K_{hi} = \frac{\bar{K}_{hi}}{1 + u_h^2 + u_i^2} + \frac{(u_{,hh}u_{,ii} - u_{,hi}^2)(1 - u_l u_m g^{lm})}{1 + u_h^2 + u_i^2}. \tag{10}$$

Note that $u_l u_m g^{lm}$ is the squared modulus of the gradient of the function u in the metric ds^2. We now determine the expression for this quantity via $\nabla_1 u$. By the definition of the metric ds^2 at a chosen point,

$$g_{ij} = \delta_{ij} + u_i u_j, \qquad g = \det|g_{ij}| = 1 + \sum_{j=1}^{n} u_j^2 = 1 + \nabla_1 u.$$

At the same time, the coefficients of the inverse metric tensor g^{jk} are given by the formula

$$g^{jk} = \frac{1}{g}(\delta_{jk} g - u_j u_k).$$

Indeed, we now check the validity of relations for the inverse metric tensor:

$$\begin{aligned} g_{ij} g^{jk} &= \frac{1}{g}(\delta_{ij} + u_i u_j)(\delta_{jk} g - u_j u_k) \\ &= \frac{1}{g}\left(\delta_{ij}\delta_{jk} g + u_i u_j \delta_{jk} g - u_j u_k \delta_{ij} - u_i u_k \sum_{j=1}^{n} u_j^2\right) = \delta_{ik}. \end{aligned}$$

By using the expressions for g^{ij} presented above, we obtain

$$\begin{aligned} u_j u_k g^{jk} &= \frac{1}{g} u_j u_k (\delta_{jk} g - u_j u_k) \\ &= \frac{1}{g}\left(g \sum_j u_j^2 - \sum_j u_j^2 \sum_k u_k^2\right) = \frac{\sum_j u_j^2}{g} = \frac{\nabla_1 u}{g}. \end{aligned}$$

Hence,

$$1 - u_l u_m g^{lm} = \frac{1}{1 + \nabla_1 u}.$$

These relations enable us to rewrite relation (10) as follows:

$$K_{hi} = \frac{\bar{K}_{hi}}{1 + u_h^2 + u_i^2} + \frac{u_{,hh} u_{,ii} - u_{,hi}^2}{(1 + u_h^2 + u_i^2)(1 + \nabla_1 u)}.$$

We now consider the case $n = 2$. In this case, $u_1^2 + u_2^2 = \nabla_1 u$ is the first differential Beltrami parameter. **The relationship between the Gaussian curvatures of 2-dimensional metrics $d\sigma^2$ and $ds^2 = d\sigma^2 + du^2$ has the form**

$$K(1 + \nabla_1 u)^2 = \bar{K}(1 + \nabla_1 u) + \nabla_{22} u. \tag{11}$$

Here, $\bar{K}$ is the curvature of the metric $d\sigma^2$ and K is the curvature of the metric ds^2. We use this relation in the next section.

10. Estimation of the Sizes of a Domain with 2-Dimensional Metric $d\sigma^2$ with Negative Curvature and Large Modulus and a Given Metric $ds^2 = d\sigma^2 + (du)^2$

Assume that there are two metrics given in a domain D, namely, $d\sigma^2$ and $ds^2 = d\sigma^2 + (du)^2$, where u is a regular function. Suppose that the Gaussian curvature K of the metric ds^2 is negative and its modulus is larger than the modulus of curvature $\bar{K}$ of the metric $d\sigma^2$. We now show that it is possible to get an upper bound for the sizes of the domain D similar to Efimov's estimate for the surfaces $z = z(x, y)$.

By $C(r)$ we denote a geodesic disk of radius r in the metric $d\sigma^2$. Assume that, for any $r \in [0, R]$, the boundary of the disk $C(r)$ has a geodesic curvature $\dfrac{1}{\rho_g} \geq 0$. This condition is satisfied if the Gaussian curvature of the metric $d\sigma^2$ is nonpositive or R is small. Let $S(r)$ be the area of the disk $C(r)$ and let $L(r)$ be the length of the circle $\Gamma(r)$ in the metric $d\sigma^2$. We define the quantity $M(R)$ that depends on the metric $d\sigma^2$:

$$M(R) = \inf_{0 \leq r \leq R} \frac{S(r)}{L(r)^2}.$$

The number $M(R)$ is not equal to zero because the ratio of $S(r)$ to $L^2(r)$ tends to $\dfrac{1}{4\pi}$ as $r \to 0$.

Note that, for $K = \bar{K}$, the estimate cannot be obtained. Indeed, in this case, we can set $u = \text{const}$ in the entire domain of definition of the metric $d\sigma^2$. Hence, it is necessary to "separate" K from $\bar{K}$. We do this with the help of a constant coefficient λ of K.

Theorem 1. *Suppose that there exists a constant $K_0 > 0$ such that the inequality $K \leq -K_0^2$ holds for the Gaussian curvatures K and $\bar{K}$ and there exists a number λ $(0 \leq \lambda < 1)$ such that $\lambda K \leq \bar{K}$.*

Then

$$S(R)M(R) \leq \frac{3e^2}{4(1-\lambda)K_0^2}.$$

To prove the theorem, we use the Heinz method somewhat modified as applied to the analyzed case. We introduce a function

$$f(r) = \int_0^r \int_{\Gamma(r)} u_\sigma^2 \, d\sigma \, dr + S(r).$$

Here, σ is the arc length of $\Gamma(r)$ in the metric $d\sigma^2$, u_σ is the derivative of u with respect to the arc length of this curve, and r is the arc length along the geodesic radius. This function can be also rewritten in the form

$$f(r) = \int_{C(r)} (1 + u_\sigma^2) \, dS.$$

We use the generalized Bernstein formula (9) with u instead of z and $\bar{K}$ instead of K. In view of relation (11) connecting the curvatures K and $\bar{K}$ and the Monge–Ampére operator, we get

$$\frac{d}{dr} \int_{\Gamma(r)} \frac{u_\sigma^2}{2} \, d\sigma = \int_{\Gamma(r)} \frac{u_r^2}{2} \frac{1}{\rho_g} \, d\sigma + \int_{C(r)} \left[-K(1 + \nabla_1 u)^2 + \bar{K}\left(1 + \frac{3}{2}\nabla_1 u\right)\right] dS. \tag{12}$$

By using the Cauchy–Bunyakovskii inequality, we find

$$f(r) \leq \left(\int\limits_{C(r)} (1 + \nabla_1 u)^2 \, dS \right)^{1/2} (S(r))^{1/2}. \tag{13}$$

The primes denote the derivatives with respect to r. We get

$$f'(r) = \int\limits_{\Gamma(r)} u_\sigma^2 \, d\sigma + L(r).$$

To find the second derivative $f''(r)$, we use Eq. (12). We also note that the derivative of $L(r)$ is equal to the integral of the geodesic curvature of the curve $\Gamma(r)$. Indeed, if we write the metric $d\sigma^2$ in the semigeodesic coordinate system

$$d\sigma^2 = dr^2 + G d\phi^2,$$

then we get

$$L(r) = \int\limits_{\Gamma(r)} \sqrt{G} \, d\phi, \qquad L' = \int\limits_{\Gamma(r)} \frac{G_r}{2G} \sqrt{G} \, d\phi = \int\limits_{\Gamma(r)} \frac{1}{\rho_g} \, d\sigma.$$

By assumption, the geodesic curvature is positive. Hence, $L' \geq 0$. We estimate the integrand of the last integral on the right-hand side of (12) and denote it by A:

$$A = -K(1 + \nabla_1 u)^2 + \bar{K}\left(1 + \frac{3}{2}\nabla_1 u\right).$$

If $\bar{K} \geq 0$ at the analyzed point, then, at this point,

$$A \geq K_0^2 (1 + \nabla_1 u)^2.$$

Now let $\bar{K} < 0$ at the same point. Then we get

$$\begin{aligned} A = &-(1-\lambda)K(1 + \nabla_1 u)^2 - \lambda K + \bar{K} \\ &+ 2(-\lambda K + \bar{K})\nabla_1 u - \lambda K (\nabla_1 u)^2 - \frac{1}{2}\bar{K}\nabla_1 u \\ \geq &\ (1-\lambda) K_0^2 (1 + \nabla_1 u)^2, \end{aligned}$$

where we have used the condition of the theorem $-\lambda K + \bar{K} \geq 0$ and $\bar{K} < 0$.

Therefore,

$$f'' = \frac{d}{dr} \int\limits_{\Gamma(r)} u_\sigma^2 \, d\sigma + L' \geq 2(1-\lambda) K_0^2 \int\limits_{C(r)} (1 + \nabla_1 u)^2 \, dS.$$

By using (13), we obtain

$$f''(r) \geq \frac{2(1-\lambda)K_0^2 f^2(r)}{S(r)}. \tag{14}$$

It is clear that $f'(r) > 0$ for $r > 0$ and $f(0) = f'(0) = 0$. Multiplying the right-hand and left-hand sides of inequality (14) by $f'(r)$ and integrating from 0 to r, we find

$$f'^2(r) \geq \frac{4(1-\lambda)K_0^2 f^3(r)}{3S(r)}.$$

We rewrite this inequality in the form

$$\frac{f'(r)}{f^{3/2}(r)} \geq \frac{2\sqrt{1-\lambda}K_0}{\sqrt{3S(r)}}.$$

Integrating it from $r_1 > 0$ to r_2, we get

$$\frac{1}{\sqrt{f(r_1)}} - \frac{1}{\sqrt{f(r_2)}} \geq \sqrt{1-\lambda}K_0 \int\limits_{r_1}^{r_2} \frac{dr}{\sqrt{3S(r)}}.$$

By using the inequality $f(r_1) \geq S(r_1)$, we obtain

$$\frac{1}{\sqrt{S(r_1)}} \geq \sqrt{1-\lambda}K_0 \int\limits_{r_1}^{r_2} \frac{dr}{\sqrt{3S(r)}}. \tag{15}$$

Note that

$$0 < \left(\sqrt{S(r)}\,\right)' = \frac{L(r)}{2\sqrt{S(r)}} \leq \frac{1}{2\sqrt{M(R)}}.$$

By virtue of this inequality and estimate (15) with $r_2 = R$, we find

$$\frac{1}{\sqrt{S(r_1)}} \geq \sqrt{1-\lambda}K_0 \int\limits_{r_1}^{R} \frac{\left(\sqrt{S(r)}\,\right)' dr}{\left(\sqrt{S(r)}\,\right)' \sqrt{3S(r)}} \geq \frac{\sqrt{1-\lambda}K_0 M^{1/2}(R)}{\sqrt{3}} \ln \frac{\sqrt{S(R)}}{\sqrt{S(r_1)}}.$$

Hence,

$$\frac{1}{\sqrt{S(r_1)}} + \frac{2\sqrt{1-\lambda}K_0 M^{1/2}(R)}{\sqrt{3}} \ln \sqrt{S(r_1)} \geq \frac{2\sqrt{1-\lambda}K_0 M^{1/2}(R)}{\sqrt{3}} \ln \sqrt{S(R)}.$$

The left-hand side has a minimum for

$$\sqrt{S(r_1)} = \frac{\sqrt{3}}{2\sqrt{M(R)(1-\lambda)}K_0}.$$

Substituting this value in the left-hand side, we arrive at the estimate for $S(R),$ which yields the required inequality

$$S(R)M(R) \leq \frac{3e^2}{4(1-\lambda)K_0^2}.$$

The validity of this estimate proves that, for a given metric $d\sigma^2$ and a given geodesic disk $C(R),$ the quantity K_0 cannot be arbitrarily large in this metric. Indeed, the right-hand side of this inequality tends to zero as $K_0 \to \infty,$ whereas the left-hand side takes a fixed nonzero value. The estimate becomes worse as λ approaches 1, i.e., in the case where the curvature K is close to $\bar{K}$.

11. Projection onto Complete Unbounded Manifolds

We now consider the problem of existence of the metric ds^2 on the complete manifold M^2 unbounded in the metric $d\sigma^2$. The following theorem is true:

Theorem 2. *If, on the manifold M^2 with metric $d\sigma^2$ complete and bounded in this metric with Gaussian curvature $\bar{K} \geq 0$, there exists a point P_0 without conjugate points and K is the Gaussian curvature of the metric $ds^2 = d\sigma^2 + (du)^2$, then*

$$\sup_{M^2} K \geq 0.$$

To prove the theorem, we construct a semigeodesic coordinate system centered at the point P_0 for which

$$d\sigma^2 = dr^2 + Gd\phi^2.$$

The absence of conjugate points for the point P_0 guarantees the convexity of geodesic circles. Since

$$\bar{K} = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$$

and $\bar{K} \geq 0,$ the quantity $\sqrt{G}$ increases not faster than a linear function: $\sqrt{G} \leq cr$ for sufficiently large $r.$ Here, $c = \mathrm{const}.$ Hence, the length of the circle $L(r)$ increases not faster than a linear function

$$L(r) = \int \sqrt{G}\, d\phi \leq 2\pi cr.$$

For sufficiently large $r,$ the area of the geodesic disks $S(r)$ satisfies the inequality

$$S(r) = \int \sqrt{G}\, d\phi\, dr \leq \pi cr^2.$$

Let

$$\sup_{M_2} K \leq -K_0^2 < 0.$$

We can use the arguments from the proof of Theorem 1 under the conditions of which we can set $\lambda = 0.$ We use inequality (11) and let r_2 tend to infinity. Thus, for large r, we get

$$\int_{r_1}^{r_2} \frac{1}{\sqrt{S}(r)}\,dr \geq \int_{r_1}^{r_2} \frac{1}{\sqrt{c\pi}r}\,dr \to \infty.$$

This contradicts inequality (15) for any $K_0 \neq 0$. Therefore, $\sup_{M^2} K = 0$.

Theorem 2 is proved.

12. Darboux Equation for the Squared Length of the Radius Vector of a Surface in E^3

Let $r = r(u^1, u^2)$ be the radius vector of a surface $F^2 \subset E^3$. Denote $\rho = \dfrac{1}{2}r^2$. The function ρ satisfies the **Darboux equation**

$$\nabla_{22}\rho - \nabla_2\rho + 1 = (2\rho - \nabla_1\rho)K. \tag{16}$$

Here, ∇_{22} is the Monge–Ampére operator in the metric of the surface, ∇_2 is the Laplace–Beltrami operator, and $\nabla_1\rho$ is the first differential Beltrami parameter for the function ρ or $|\operatorname{grad}\rho|^2$.

We now present a brief derivation of this equation. By using covariant derivatives, we represent the Gauss decompositions in the following simple form:

$$r_{,ij} = L_{ij}n,$$

where L_{ij} are the coefficients of the second quadratic form of the surface and n is the unit vector of normal. We now compute the covariant derivatives of the function ρ. Thus, we get

$$\rho_i = (rr_i), \qquad \rho_{,ij} = (r_ir_j) + (rr_{,ij}) = g_{ij} + (rn)L_{ij}.$$

Therefore, $\rho_{,ij} - g_{ij} = (rn)L_{ij}$. Hence,

$$(\rho_{,11} - g_{11})(\rho_{,22} - g_{22}) - (\rho_{,12} - g_{12})^2 = (rn)^2(L_{11}L_{22} - L_{12}^2).$$

In the expanded form, we can write

$$\rho_{,11}\rho_{,22} - \rho_{,12}^2 - \rho_{,11}g_{22} + 2\rho_{,12}g_{12} - \rho_{,22}g_{11} + g_{11}g_{22} - g_{12}^2 = (rn)^2(L_{11}L_{22} - L_{12}^2).$$

We divide the right-hand and left-hand sides of the last equation by $g_{11}g_{22} - g_{12}^2$ and use the relation

$$K = (L_{11}L_{22} - L_{12}^2)/(g_{11}g_{22} - g_{12}^2).$$

Note that

$$\frac{\rho_{,11}g_{22} - 2\rho_{,12}g_{12} + \rho_{,22}g_{11}}{g_{11}g_{22} - g_{12}^2} = \rho_{,ij}g^{ij} = \nabla_2\rho.$$

We now show that the support function (rn) can be expressed via the function ρ and its derivatives. We write the decomposition of r in the basis vectors r_1, r_2, and n as follows:

$$r = a^ir_i + n(rn).$$

Thus, we find

$$\rho_k = (rr_k) = a^i(r_ir_k) = a^ig_{ik}, \qquad \rho_kg^{kl} = a^l.$$

We now return to the decomposition of r:

$$r = \rho_kg^{kl}r_l + (rn)n.$$

Further, we scalarly multiply the right- and left-hand sides by r. This gives

$$2\rho = \rho_k\rho_lg^{kl} + (rn)^2,$$

i.e.,

$$(rn)^2 = 2\rho - \nabla_1\rho,$$

which completes the derivation of the Darboux equation (16). The presented relation implies that $\nabla_1\rho \leq 2\rho$.

13. Estimation of the Outer Diameter of a Surface in E^3

In 1968, Burago established estimates for the outer diameter of a surface in E^3 according to which the surface cannot be infinitely contractible in a class of regular surfaces. His method of proving presented in [18] was based on laborious and complicated analyses of polyhedral metrics. We now present the estimates obtained by Burago. Assume that a surface is located in a ball of radius R, S is its area, L is the length of the boundary, χ is the Euler characteristic, and ω^+ is the integral of the Gaussian curvature over the domain $K \geq 0$.

Burago Theorem. *There exists an absolute constant C such that if $\chi = 1$, then*

$$S \leq C(R^2\omega^+ + RL),$$

and if $\chi \leq 0$, then

$$S \leq C\big(R^2[\omega^+ - 2\pi\chi] + RL\big).$$

The proof of these inequalities proposed by Gromov and based on the use of the Lobachevskii space was presented in the monograph “Geometric Inequalities” by Burago and Zalgaller.

The class of regularity of the immersion C^2 is important in this theorem because, as shown by Kuiper [47], any 2-dimensional metric can be isometrically immersed in the class C^1 into the interior of a sphere of arbitrarily small diameter in a 3-dimensional Euclidean space. Moreover, it is also important that the dimension of the enveloping space is equal to 3. One can easily construct an isometric and regular immersion of the entire Euclidean plane into a 3-sphere of an arbitrarily small radius from E^4.

In 1973–1975, we applied the Darboux equation and the generalized Monge–Ampére operator to deduce estimates for the outer diameter of a surface in E^3.

We first consider a closed regular surface F^2 located in a ball of radius R. Denote

$$\omega^+ = \int\limits_{K\geq 0} K\,dS, \qquad \omega^- = -\int\limits_{K\leq 0} K\,dS.$$

Theorem 3. *If a closed regular oriented surface F^2 lies in a ball of radius R, then the inequality*

$$S \le R^2\left(\omega^+ + \frac{\omega^-}{2}\right) \tag{17}$$

is true.

We use the Darboux equation, integrate the right- and left-hand sides of this equation over the surface, and take into account relation (8). Since the surface is closed, this formula does not contain contour integrals. In addition, the integral of $\nabla_2\rho$ over the closed surface is equal to zero. As a result, we get

$$S = \int\limits_{F^2} K\left(2\rho - \frac{3}{2}\nabla_1\rho\right) dS.$$

In the domain where $K \ge 0$, we find

$$K\left(2\rho - \frac{3}{2}\nabla_1\rho\right) \le K2\rho \le KR^2.$$

At the same time, in the domain where $K \le 0$, we get

$$K\left(2\rho - \frac{3}{2}\nabla_1\rho\right) = K\left((rn)^2 - \frac{1}{2}\nabla_1\rho\right) \le |K|R^2/2.$$

This yields estimate (17).

We now consider a surface with edge. In this case, there are several possible boundary curves.

Theorem 4. *Suppose that the oriented surface F^2 with boundary Γ lies in a ball of radius R and its boundary lies in a ball of radius R_1 with the same center. Then*

$$S \le R^2\left(\omega^+ + \frac{\omega^-}{2}\right) + R_1^2\int\limits_{\Gamma} |k|\, ds. \tag{18}$$

Here, $k = r_{ss}$ is the vector of curvature of the curve Γ. The integral is taken over the arc length s of the curve Γ. Since $R_1 \le R$, we obviously get the lower bound for R.

To prove the theorem, we consider boundary integrals. We introduce a semigeodesic coordinate system r, ϕ in a neighborhood of the boundary curve Γ such that the curve Γ is given by the equation $r = 0$. Assume that the first quadratic form has the form $ds^2 = dr^2 + Gd\phi^2$. The coordinates r and ϕ are called the first and second coordinates, respectively. Then the unique vector τ normal to Γ has the coordinates $\tau_1 = 1$ and $\tau_2 = 0$. Integrating $-\nabla_2\rho$ over the surface, we obtain the following contour integral:

$$-\int\limits_{\Gamma} (\operatorname{grad}\rho\tau)\, ds = -\int\limits_{\Gamma} (rr_1)\, ds. \tag{19}$$

Moreover, integrating $\nabla_{22}\rho$ over the surface, we get the contour integral

$$\int\limits_{\Gamma} (\nu\tau)\, ds = \int\limits_{\Gamma} \nu^1\tau_1\, ds = \int\limits_{\Gamma} \frac{\rho_1\rho_{,22} - \rho_2\rho_{,12}}{2G}\, ds. \tag{20}$$

We write

$$-\frac{\rho_2\rho_{,12}}{2G} = -\left(\frac{\rho_2\rho_1}{2G}\right)_{,2} + \frac{\rho_{,22}\rho_1}{2G}.$$

Since the integral of the first term over the curve Γ is equal to zero, integral (20) takes the form

$$\int\limits_\Gamma \frac{\rho_1\rho_{,22}}{G}\,ds = \int\limits_\Gamma \frac{(rr_1)(G + (rr_{,22}))}{G}\,ds$$

$$= \int\limits_\Gamma (rr_1)ds + \int\limits_\Gamma (rr_1)(rk)\,ds.$$

Here, we have used the equality

$$r_{,22}/G = r_{ss} = k.$$

Note that, in the general sum, the first integral on the right-hand side and the integral on the right-hand side of (19) cancel each other. Thus, only the second integral is preserved. It can be estimated from above by the expression

$$R_1^2 \int\limits_\Gamma |k|\,ds,$$

Q.E.D.

If the curve Γ lies on a sphere of radius R_1 centered at the origin, then, in view of the fact that, in this case, $(rr_s) \equiv 0$, at points of the curve Γ, we can write

$$(rr_{ss}) = (rr_s)_s - r_s^2 = -1.$$

Therefore, the contour integral on the right-hand side of inequality (18) can be estimated as

$$\left|\int\limits_\Gamma (rr_1)(rr_{ss})\,ds\right| \le \int\limits_\Gamma |(rr_1)|ds \le R_1 L,$$

where L is the length of the spherical curve Γ.

14. Functions on a Surface in E^3 and the Monge–Ampére Operator

Assume that the surface $F^2 \subset E^3$ has the form $x^\alpha = x^\alpha(u^1, u^2)$ in Cartesian coordinates. We define a function Φ on the surface by specifying its dependence on the Cartesian coordinates x^α (which, in turn, depend on the curvilinear coordinates u^1 and u^2). Thus, it is supposed that the function Φ is first given in the space E^3 and then induced onto the surface F^2. If the function Φ is a polynomial of x^1, x^2, and x^3, then, on the surface, it can be regarded as an analog of a polynomial function in the plane.

Each Cartesian coordinate x^α is also regarded as a function on F^2. We now find the value of the Monge–Ampére operator for this function.

The derivatives of the function Φ with respect to x^α are denoted by Greek letter in the subscripts. At the same time, the covariant derivative with respect to u^i is denoted by a Latin letter. Moreover, the second covariant derivatives are, in addition, denoted by commas. Thus, we have

$$\Phi_i = \Phi_\alpha x_i^\alpha, \qquad \Phi_{,ij} = \Phi_{\alpha\beta} x_i^\alpha x_j^\beta + \Phi_\alpha x_{,ij}^\alpha.$$

We determine the numerator of the expression for the Monge–Ampére operator as follows:

$$\begin{aligned}
\Phi_{,11}\Phi_{,22} - \Phi_{,12}^2 &= (\Phi_{\alpha\beta} x_1^\alpha x_1^\beta + \Phi_\alpha x_{,11}^\alpha)(\Phi_{\gamma\sigma} x_2^\gamma x_2^\sigma + \Phi_\gamma x_{,22}^\gamma) \\
&\quad - (\Phi_{\alpha\beta} x_1^\alpha x_2^\beta + \Phi_\alpha x_{,12}^\alpha)(\Phi_{\gamma\sigma} x_1^\gamma x_2^\sigma + \Phi_\gamma x_{,12}^\gamma) \\
&= \frac{1}{2}\Phi_{\alpha\beta} x_1^\alpha x_2^\sigma (x_1^\beta x_2^\gamma - x_2^\beta x_1^\gamma) + \Phi_\alpha \Phi_{\gamma\sigma} x_{,11}^\alpha x_2^\gamma x_2^\sigma + \Phi_\gamma \Phi_{\alpha\beta} x_1^\alpha x_1^\beta x_{,22}^\gamma \\
&\quad - \Phi_\alpha \phi_{\gamma\sigma} x_{,12}^\alpha x_1^\gamma x_2^\sigma - \Phi_{\alpha\beta}\Phi_\gamma x_1^\alpha x_2^\beta x_{,12}^\gamma + \Phi_\alpha \Phi_\gamma (x_{,11}^\alpha x_{,22}^\gamma - x_{,12}^\alpha x_{,12}^\gamma).
\end{aligned} \tag{21}$$

Further, we apply the Gauss decompositions $r_{,ij} = L_{ij} n$ and introduce the following notation:

$$p^{\alpha\sigma} = \frac{x_1^\alpha x_2^\sigma - x_2^\alpha x_1^\sigma}{\sqrt{g}}, \qquad b_{ij} = \Phi_{\alpha\beta} x_i^\alpha x_j^\beta.$$

Here, g is the determinant of the metric tensor of the surface. The components $p^{\alpha\sigma}$ form a unique vector n normal to the surface:

$$n = \left(p^{23}, p^{31}, p^{12}\right) = (n^1, n^2, n^3).$$

We divide the right- and left-hand sides of Eq. (21) by g and use the Gauss equation. This yields

$$\begin{aligned}
\nabla_{22}\Phi = &- \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & n^1 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & n^2 \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & n^3 \\ n^1 & n^2 & n^3 & 0 \end{vmatrix} \\
&+ \frac{(L_{11}b_{22} - 2L_{12}b_{12} + L_{22}b_{11})(n, \operatorname{grad}\Phi)}{g} + K(n, \operatorname{grad}\Phi)^2.
\end{aligned} \tag{22}$$

Here, $\operatorname{grad}\Phi$ is the gradient of the function in E^3.

Remark 1. In a special case where

$$\Phi = \frac{1}{2}\left((x^1)^2 + (x^2)^2 + (x^3)^2\right),$$

we get the Darboux equation. In this case, $\operatorname{grad}\Phi = r$.

Remark 2. If the surfaces F^2 and $\Phi(x^1, x^2, x^3) = \mathrm{const}$ are orthogonal, i.e., $(\mathrm{grad}\,\Phi, n) = 0$, then

$$\nabla_{22}\Phi = -\begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & n^1 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & n^2 \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & n^3 \\ n^1 & n^2 & n^3 & 0 \end{vmatrix}. \tag{23}$$

Remark 3. For the simplest function on the surface (a component of the radius vector x^α), the expression for the Monge–Ampére operator takes a very simple form

$$\nabla_{22}x^\alpha = K(n^\alpha)^2, \quad \alpha = 1, 2, 3. \tag{24}$$

Since n is a unit vector, we get

$$\sum_{\alpha=1}^{3} \nabla_{22}x^\alpha = K.$$

Indeed, in view of $x^\alpha_{,ij} = L_{ij}n^\alpha$, we conclude that

$$x^\alpha_{,11}x^\alpha_{,22} - (x^\alpha_{,12})^2 = \big(L_{11}L_{22} - (L_{12})^2\big)(n^\alpha)^2.$$

Dividing both sides of the equation by $g_{11}g_{22} - g_{12}^2$, we obtain Eq. (24).

As an example, we apply relation (23) to a function Φ defined on an ordinary torus. In the implicit form, it is given by the equation

$$F(x, y, z) = z^2 + \big(\sqrt{x^2 + y^2} - a\big)\big(\sqrt{x^2 + y^2} - b\big) = 0,$$

where a and b are positive numbers and, moreover, $a \neq b$. We find

$$F_x = \frac{x}{\sqrt{x^2 + y^2}}\big(2\sqrt{x^2 + y^2} - a - b\big), \qquad F_y = \frac{y}{\sqrt{x^2 + y^2}}\big(2\sqrt{x^2 + y^2} - a - b\big), \quad F_z = 2z.$$

This yields

$$|\,\mathrm{grad}\, F|^2 = \big(2\sqrt{x^2 + y^2} - a - b\big)^2 + 4z^2.$$

Denote $\sqrt{x^2 + y^2} = \lambda$. Then the equation of torus can be rewritten in the form

$$z^2 = -\lambda^2 + \lambda(a + b) - ab.$$

Hence, on the surface of the torus, we find

$$|\,\mathrm{grad}\, F|^2 = (2\lambda - a - b)^2 + 4(-\lambda^2 + \lambda(a + b) - ab) = (a - b)^2.$$

We set

$$\Phi = \arctan \frac{y}{x}.$$

Thus, we get

$$\Phi_x = -\frac{y}{x^2+y^2}, \qquad \Phi_y = \frac{x}{x^2+y^2}, \qquad \Phi_z = 0.$$

Hence,

$$(\operatorname{grad} F, \operatorname{grad} \Phi) = 0$$

and we can use relation (23). We have

$$\Phi_{xx} = \frac{2xy}{(x^2+y^2)^2}, \qquad \Phi_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \Phi_{yy} = \frac{-2xy}{(x^2+y^2)^2}.$$

We substitute these expressions in relation (23) and note that the component of normal $n^3 = 2z/|\operatorname{grad} F|$. Therefore,

$$\nabla_{22}\Phi = -\frac{4z^2}{(a-b)^2(x^2+y^2)^2}.$$

Since, on the surface of the torus, z^2 is expressed in terms of x and y, we get the expression for $\nabla_{22}\Phi$ in terms of the parameters x and y.

15. Simplest Monge–Ampére Equation in the Plane

We now present some recent results obtained for the simplest Monge–Ampére equation

$$z_{xx}z_{yy} - z_{xy}^2 = f(x,y).$$

In the well-known Jörgens work [19], it was proved that a solution of this equation for $f(x,y) = 1$ defined on the entire plane (x,y) can be nothing but a polynomial of x and y of the second degree. Calabi and Pogorelov generalized this result to the many-dimensional case.

At the same time, there exist solutions of the equation $z_{xx}z_{yy} - z_{xy}^2 = -1$ defined on the entire plane that are not polynomials. Thus, Goursat determined the general parametric solution of this equation, and a specific example

$$z(x,y) = xy + x\ln\left(x + \sqrt{x^2+e^{-2y}}\right) - \sqrt{x^2+e^{-2y}}$$

was presented by Kantor [44].

The Monge–Ampére equation with right-hand side in the form of a polynomial was considered in [20, 22, 23]. It is quite natural to seek its solution also in the form of a polynomial of certain degree. Recall that every continuous function $f(x,y)$ on the right-hand side can be approximated in a bounded domain by a polynomial with unboundedly high accuracy. The best possible result in this case is to determine the coefficients of the polynomial $z(x,y)$ via the coefficients of the approximating polynomial in the explicit form (if this is possible). However, this is not always possible. *Thus, we arrive at a general problem of construction of a polynomial solution to the Monge–Ampére equation with polynomial right-hand side.*

We first consider the equation

$$z_{xx}z_{yy} - z_{xy}^2 = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00}, \tag{25}$$

where the constants b_{ij} satisfy the inequalities

$$b_{20} > 0, \qquad b_{02} > 0, \qquad 4b_{20}b_{02} - b_{11}^2 > 0, \qquad b_{00} > 0. \tag{26}$$

In [20], it was shown that the solution in the form of a polynomial of odd degree does not exist, while the solution in the form of a polynomial of even degree exists provided that $4b_{20}b_{02} - b_{11}^2 = 0$.

Moreover, in [22], we have proved the following theorem:

Theorem 5. *If the strict inequalities (26) are true, then Eq. (25) does not have solutions in the form of polynomials of any degree.*

At the same time, the analytic solution defined in the entire plane exists. If $f(x, y) = x^2 + y^2 + 1$, then the analytic solution takes the form

$$z(x, y) = \frac{1}{3\sqrt{2}}(x^2 + y^2 + 2)^{3/2}.$$

Another solution analytic in the entire plane except the origin has the form

$$z(x, y) = \frac{1}{\sqrt{2}}\Big((x^2 + y^2)^{3/2} + \sqrt{x^2 + y^2}\Big).$$

At the origin of coordinates, this solution is continuous but not differentiable.

Note that there exist polynomial strictly positive functions $f(x, y)$ for which the equation $z_{xx}z_{yy} - z_{xy}^2 = f(x, y)$ possesses a polynomial solution. Thus, setting

$$z(x, y) = \alpha(x^4 + y^4) + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$

where α is a positive constant, $a_{20} > 0$, $a_{02} > 0$, and $4a_{20}a_{02} - a_{11}^2 > 0$, we obtain

$$z_{xx}z_{yy} - z_{xy}^2 = (12\alpha xy)^2 + 24\alpha(a_{02}x^2 + a_{20}y^2) + 4a_{20}a_{02} - a_{11}^2 = f(x, y).$$

Here, the right-hand side $f(x, y)$ is positive for all values of x and y.

At the same time, Eq. (25) can be approximated by a polynomial in any bounded domain (despite the fact that, as already indicated, the polynomial solution does not exist). The following theorem is true:

Theorem 6. *For any given $\epsilon > 0$ and in any given bounded domain in the plane (x, y), under the strict conditions (26), one can find a polynomial of the fourth degree Q_ϵ such that*

$$\big|\nabla_{22}Q_\epsilon - (b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00})\big| \le \epsilon.$$

In what follows, we consider the Monge–Ampére equations with right-hand sides in the form of more general polynomial of degrees 2–4. We indicated the cases of existence and absence of a solution $z(x, y)$ in the form of a fourth-degree polynomial. The solutions are specified in the explicit form according to the coefficients b_{ij}. Thus, if

$$f(x,y) = 2x^4 + x^2y^2 + 2y^4 + x^2 + xy + y^2 + b_{00},$$

then the function

$$z(x,y) = \frac{1}{6}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{6}y^4 + \frac{1}{6}x^2 - \frac{1}{4}xy + \frac{1}{6}y^2$$

if and only if

$$b_{00} = \frac{1}{3^2} - \frac{1}{4^2}.$$

However, for $f(x,y)$ in the form of a polynomial of fourth degree, the solution was not found. Moreover, the problem of finding solutions in this case proves to be quite complicated.

The Monge–Ampére operator maps the space of fourth-degree polynomials of x and y into itself.

We can now indicate fixed points of the Monge–Ampére operator. Indeed, if

$$U = \frac{1}{4^2 \cdot 3}(x^2 + y^2)^2, \qquad W = -\frac{1}{4^2 \cdot 3}(x^2 - y^2)^2,$$

then the equalities

$$\nabla_{22}U = U, \qquad \nabla_{22}W = W$$

are true, i.e., these polynomials are fixed points of the Monge–Ampére operator.

In [24], Sabitov obtained an unexpected result for the simplest Monge–Ampére equation for $f(x,y) = 0$. Note that any regular cylindrical surface with unique projection onto the plane is a regular solution on the complete plane. Sabitov posed a problem: To describe the characteristics of a surface defined over the entire plane but with possible singularities and zero Gaussian curvature at points where this surface is regular. The simplest example is a cone $z = \sqrt{x^2 + y^2}$ with singular point at the vertex. Are there any surfaces with larger numbers of isolated singularities? The following theorem answers this question:

Sabitov Theorem. *Suppose that an arbitrary finite set of points M is given in the plane (x,y). Then the equation $z_{xx}z_{yy} - z_{xy}^2 = 0$ has infinitely many solutions $z(x,y)$ defined in the entire plane and C^∞-smooth at all points except the points of a given set M at which these solutions are continuous but not differentiable. Moreover, it is possible to state that there exist piecewise analytic solutions with violations of analyticity only at finitely many points of rectilinear rays.*

The first information about this theorem appeared in the abstract of Sabitov's report delivered at the "Geometry in Odessa-2015" conference [25]. The work by Gálvez and Nelli [26] was also devoted to the same problem and published in 2016.

16. Symmetrons

It is useful to have closed surfaces of any topological type analytically defined in E^3. In [17], these surfaces were constructed and the behavior of their Gaussian curvature was analyzed. Since these algebraic surfaces have certain symmetries, they are called **symmetrons**.

In E^3, we introduce the Cartesian coordinates $x,\ y,\ z$. In the plane $z = 0$, we take p mutually disjoint closed regular curves γ_i none of which lies inside the other. Assume that each curve γ_i is given by the equation $f_i(x, y) = 0$ and, inside this curve, we have $f_i(x, y) < 0$. Suppose that a closed curve γ envelopes all curves γ_i and is given by the equation $f(x, y) = 0$ and that, in addition, $f(x, y) < 0$ inside this curve. Then the surface M^2 is given in the implicit form by the equation

$$a(x, y, z) = z^2 + f(x, y) \prod_{i=1}^{p} f_i(x, y) = 0.$$

Moreover, it is closed oriented, and homeomorphic to a sphere with p handles. We consider specific functions f and f_i. Thus, if we set

$$x_i = \cos \frac{2\pi i}{p}, \qquad y_i = \sin \frac{2\pi i}{p},$$

then the surface

$$z^2 + (x^2 + y^2 - R^2) \prod_{i=1}^{p} \left[(x - x_i)^2 + (y - y_i)^2 - r^2\right] = 0$$

has symmetries for

$$r < \sin \frac{\pi}{p}, \quad r + 1 < R.$$

This surface is called a p-symmetron.

An ordinary circular torus of revolution in E^3 is specified by the equation

$$z^2 + \left(\sqrt{x^2 + y^2} - R\right)\left(\sqrt{x^2 + y^2} - r\right) = 0.$$

Therefore, it is also necessary to consider the surfaces specified by

$$\sqrt{(x - x_i)^2 + (y - y_i)^2} - r.$$

With the help of computer methods, by using relation (22), we studied the behavior of the Gaussian curvature of 2-symmetron and determined the domains on M^2 in which this curvature is positive or negative.

By adding the fourth coordinate as a function of $\Phi(x, y, z)$ bounded on M^2, we construct a closed surface $F^2 \subset E^4$. With the help of computer simulations, we analyzed the behavior of the Gaussian curvature K for the surface F^2. Our aim was to construct a surface F^2 over M^2 with Gaussian curvature $K < 0$. However, the results of numerical analyses show that, in all studied examples, one can find domains in F^2 for which $K \geq 0$. Therefore, in conclusion, we pose the following question: **Is it possible to find closed regular surfaces in E^4 with negative Gaussian curvature of the metric given on an n-symmetron for $n \geq 2$?**

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