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GENERALIZED CHARACTERISTICS OF SMOOTHNESS AND SOME EXTREME PROBLEMS OF THE APPROXIMATION THEORY OF FUNCTIONS IN THE SPACE $L_2(\mathbb{R})$. I

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We consider the generalized characteristics of smoothness of the functions $\omega^w(f,t)$ and $\Lambda^w(f,t)$, $t>0$, in the space $L_2(\mathbb{R})$ and on the classes $L_2^\alpha(\mathbb{R})$ defined with the help of fractional-order derivatives $\alpha\in(0,\infty)$ and obtain the exact Jackson-type inequalities for $\omega^w(f)$.

1. Introduction

In [1], Bernstein laid the foundations of the investigations in the field of approximation of functions given on the entire real axis with the help of the space of entire functions of finite exponential type. Later, various aspects of this field of investigations were studied by N. Akhiezer, A. Timan, M. Timan, Nikol'skii, Ibragimov, Nasibov, Popov, Ponomarenko, Gaimnazarov, Stepanets, Ligun, Doronin, Arestov, Babenko, Vasil'ev, Vakarchuk, Shabozov, Yanchenko, Artamonov, and other researchers (see, e.g., [2–32]).

In the present paper, we continue the investigations aimed at the solution of numerous extreme problems of approximation theory of functions in the space $L_2(\mathbb{R})$ by using generalized characteristics of smoothness and the generalization of the notion of derivative. In the case of 2π-periodic functions, a similar, in a certain sense, class of extreme problems in the space $L_2([0,2\pi])$ was considered in [33–35]. Note that, in [21], one can find an extension of a brief survey of final (in a certain sense) results on the best polynomial approximations of 2π-periodic functions in the space $L_2([0,2\pi])$ to the case of the best approximation by entire functions of exponential type in the space $L_2(\mathbb{R})$.

We now present necessary notions and definitions. By $L_2(\mathbb{R})$ we denote the space of all measurable functions f given on the entire real axis $\mathbb{R}$. The squared modulus of these functions is Lebesgue integrable on any finite interval and their norm is given by the formula

$$\|f\| := \left\{\int\limits_{-\infty}^{\infty} \left|f(x)\right|^2 dx\right\}^{1/2} < \infty.$$

We also present characteristics of smoothness of functions for which it is possible to obtain final solutions of a series of extreme problems of the approximation theory in the space $L_2(\mathbb{R})$.

1.1. For $\beta\in(0,\infty)$, we write the binomial coefficients

$$\binom{\beta}{0} := 1, \qquad \binom{\beta}{1} := \beta, \qquad \binom{\beta}{j} := \frac{\beta(\beta-1)\ldots(\beta-j+1)}{j!}, \tag{1.1}$$

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where $j \in \mathbb{N}\backslash\{1\}$. In the case where $\beta = m,\ m \in \mathbb{N}$, for (1.1), we set

$$\binom{m}{j} := \left\{ \frac{m!}{j!(m-j)!} \text{ for } j = 0, \ldots, m;\ 0 \text{ for } j = m+1, m+2, \ldots \right\}. \tag{1.2}$$

Since

$$\sum_{j=0}^{\infty} \left| \binom{\beta}{j} \right| < \infty,$$

the difference of fractional order β for a function $f \in L_2(\mathbb{R})$ with step $h \in \mathbb{R}$, i.e.,

$$\Delta_h^{\beta}(f, x) := \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} f(x - jh), \tag{1.3}$$

is defined almost everywhere on $\mathbb{R}$ and belongs to $L_2(\mathbb{R})$. Difference (1.3) is called left-hand sided for $h > 0$ and right-hand sided for $h < 0$. The modulus of continuity of fractional order $\beta \in (0, \infty)$ for a function $f \in L_2(\mathbb{R})$ is defined as follows:

$$\omega_{\beta}(f, t) := \sup \left\{ \left\| \Delta_h^{\beta}(f) \right\| :\ |h| \le t \right\}, \quad t \ge 0. \tag{1.4}$$

For $\beta = m,\ m \in \mathbb{N}$, we get the ordinary modulus of continuity of order m $\omega_m(f)$ from (1.1)–(1.4). In the case of approximation by entire functions of exponential type in $L_2(\mathbb{R})$, the characteristic of smoothness $\omega_m(f), m \in \mathbb{N}$, was used in [6–8, 15, 16, 21, 26, 27]. In a more general case of the space $L_p(\mathbb{R}),\ 1 \le p < \infty$, the characteristic of smoothness (1.4) was considered in [32] and the modulus of continuity of the fractional order was studied in [10, 11].

1.2. In [18, 19, 21–24], the following characteristic of smoothness was used for the solution of extremal problems in $L_2(\mathbb{R})$:

$$\Omega_m(f, t) := \left\{ \frac{1}{t^m} \int_0^t \ldots \int_0^t \left\| \Delta_{\overline{h}}^m(f) \right\|^2 dh_1 \ldots dh_m \right\}^{1/2}, \quad t > 0, \tag{1.5}$$

where

$$\overline{h} := (h_1, \ldots, h_m), \qquad \Delta_{\overline{h}}^m := \Delta_{h_1}^1 \circ \ldots \circ \Delta_{h_m}^1, \qquad \Delta_{h_j}^1(f, x) := f(x + h_j) - f(x), \quad j = \overline{1, m}.$$

1.3. For any function $f \in L_2(\mathbb{R})$, we write the Steklov function

$$S_h(f, x) := (1/(2h)) \int_{x-h}^{x+h} f(t)\, dt, \quad h > 0,$$

and denote

$$S_{h,j}(f) := S_h(S_{h,j-1}(f)), \quad j \in \mathbb{N}, \qquad \text{and} \qquad S_{h,0}(f) \equiv f.$$

Let $\mathbb{I}$ be the identity operator in the space $L_2(\mathbb{R})$. We define special finite differences of the first and higher orders at the point x with steps h as follows:

$$\widetilde{\Delta}_h^1(f,x) := S_h(f,x) - f(x) = (S_h - \mathbb{I})(f,x),$$

$$\widetilde{\Delta}_h^m(f,x) := \widetilde{\Delta}_h^1\big(\widetilde{\Delta}_h^{m-1}(f),x\big) = (S_h - \mathbb{I})^m(f,x)$$

$$= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}S_{h,j}(f,x), \quad m = 2,3,\ldots.$$

By using the introduced notation, we can write the special modulus of continuity of order $m,\ m \in \mathbb{N}$ as follows:

$$\widetilde{\Omega}_m(f,t) := \sup\big\{\big\|\widetilde{\Delta}_h^m(f)\big\| :\ 0 < h \le t\big\}, \quad t > 0. \tag{1.6}$$

The characteristic of smoothness (1.6) was used, e.g., in [25].

1.4. In [30], the following characteristic of smoothness was used for the solution of some extreme problems in the space $L_2(\mathbb{R})$,

$$\Lambda_m(f,t) := \left\{\frac{1}{t}\int_0^t \big\|\Delta_h^m(f)\big\|^2\,dh\right\}^{1/2}, \quad t > 0, \tag{1.7}$$

where $m \in \mathbb{N}$. We consider this quantity in more detail by using the results obtained by Ditzian and Totik in [36, p. 26].

We take the interval $D = (a,b)$ whose endpoints may take not only finite but also infinite values, e.g., $-\infty$ and $+\infty$, respectively. For a function $f \in L_p(D),\ 1 \le p < \infty$, the characteristic of smoothness

$$\overline{\omega}_\varphi^{*m}(f,t)_p := \left\{\frac{1}{t}\int_0^t\int_D \big|\overline{\Delta}_{h\varphi(x)}^m f(x)\big|^p\,dx\,dh\right\}^{1/p}, \quad t > 0, \tag{1.8}$$

was considered in [36]. The function φ defined on the interval D is positive and satisfies certain requirements formulated in [36], Sec. 1.2. By $\overline{\Delta}_{h\varphi(x)}^m f(x)$ we denote the direct or inverse finite difference of order m of a function f that exists almost everywhere on D. In this case,

$$\overline{\Delta}_{h\varphi(x)}^m f(x) := \overrightarrow{\Delta}_{h\varphi(x)}^m f(x) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}f\big(x + (m-j)h\varphi(x)\big)$$

or

$$\overline{\Delta}_{h\varphi(x)}^m f(x) := \overleftarrow{\Delta}_{h\varphi(x)}^m f(x) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}f(x - jh\varphi(x)).$$

We set

$$\overrightarrow{\Delta}_{h\varphi(x)}^m f(x) = 0 \quad \text{or} \quad \overleftarrow{\Delta}_{h\varphi(x)}^m f(x) = 0$$

if either the segment $\big[x, x + mh\varphi(x)\big]$ or the segment $\big[x - mh\varphi(x), x\big]$, respectively, does not belong to D.

Thus, if, in relation (1.8), we have $D = (-\infty, \infty)$ and $\varphi = \widetilde{\varphi}$, where $\widetilde{\varphi}(x) \equiv 1,\ p = 2$, and

$$\overline{\Delta}^m_{h\widetilde{\varphi}(x)} f(x) := \overrightarrow{\Delta}^m_{h\widetilde{\varphi}(x)} f(x) = \Delta^m_h(f, x),$$

then, by using relations (1.7) and (1.8), we obtain

$$\Lambda_m(f, t) = \overline{\omega}^{*m}_{\widetilde{\varphi}}(f, t)_2, \quad t > 0, \quad f \in L_2(\mathbb{R}),$$

i.e., (1.7) is definitely a natural characteristic of smoothness in the space $L_2(\mathbb{R})$.

1.5. In [31], Artamonov proposed the modulus of continuity $\omega_{\langle\prime\rangle}(f)$, where $f \in L_p(\mathbb{R}),\ 1 \le p \le \infty$. In terms of the notation accepted in [31], we now formulate its definition in the space $L_2(\mathbb{R})$, namely, with the help of the operators $\overline{\Delta}_h := \overline{T}_h - \mathbb{I}$ and

$$\overline{T}_h(f, x) := \frac{3}{\pi^2} \sum_{\substack{j \in \mathbb{Z} \\ (j \ne 0)}} \frac{f(x + jh)}{j^2},$$

where $h \in \mathbb{R}$, we write the characteristic of smoothness

$$\omega_{\langle\prime\rangle}(f, t) := \sup\big\{\|\overline{\Delta}_h(f)\|:\ 0 \le h \le t\big\}, \quad t \ge 0. \tag{1.9}$$

The modulus of continuity (1.9) is a modification of the modulus of continuity introduced by Runovski and Schmeisser in the space $L_p\big([0, 2\pi]\big),\ 1 \le p < \infty$, corresponding to the Riesz derivative [37]. In the space $L_2(\mathbb{R})$, the generalization of the modulus of continuity from [37] takes the form

$$\widehat{\omega}(f, t) := \sup\big\{\|\widehat{\Delta}_h(f)\|:\ 0 \le h \le t\big\}, \quad t \ge 0, \tag{1.10}$$

where $\widehat{\Delta}_h := \widehat{T}_h - \mathbb{I}$ and

$$\widehat{T}_h(f, x) := \frac{4}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{f(x + (2j+1)h)}{(2j+1)^2}, \quad h \in \mathbb{R}.$$

In this connection, in our opinion, it is reasonable to consider more general structures in the space $L_2(\mathbb{R})$ playing the role of characteristics of smoothness and containing, as special cases, the moduli of continuity considered in Secs. 1.1–1.5. Moreover, these structures should preserve the possibility of accumulation of new types of moduli of continuity that may appear in the future.

2. Fourier Transform and Generalized Characteristics of Smoothness of Functions in the Space $L_2(\mathbb{R})$

2.1. For the first time, the Fourier transform in the space $L_2(\mathbb{R})$ was constructed and studied by Plancherel. Therefore, this transformation is sometimes called the Fourier–Plancherel transform.

Plancherel Theorem ([3], Chap. III, Sec. 3.11.21)**.** *For any function $f \in L_2(\mathbb{R})$, the integral*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{e^{-itx} - 1}{-it}\, dt$$

possesses the derivative, which is almost everywhere finite

$$\mathcal{F}(f,x)=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} f(t)\frac{e^{-itx}-1}{-it}\,dt \tag{2.1}$$

and such that

$$\int\limits_{-\infty}^{\infty} \left|\mathcal{F}(f,x)\right|^2 dx=\int\limits_{-\infty}^{\infty} \left|f(x)\right|^2 dx \tag{2.2}$$

and

$$f(x)=\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)\frac{e^{itx}-1}{it}\,dt \tag{2.3}$$

almost everywhere. Moreover, as $k\to\infty$,

$$\int\limits_{-\infty}^{\infty}\left|\mathcal{F}(f,x)-\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k} f(t)e^{-itx}\,dt\right|^2 dx\to 0, \tag{2.4}$$

$$\int\limits_{-\infty}^{\infty}\left|f(x)-\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k} \mathcal{F}(f,t)e^{itx}\,dt\right|^2 dx\to 0. \tag{2.5}$$

Function (2.1) is called the Fourier transform of f in the space $L_2(\mathbb{R})$. Sometimes, relations (2.1) and (2.3) are called the inversion formulas.

Relations (2.1) and (2.4) show that the Fourier transform in $L_2(\mathbb{R})$ can be defined not only as a pointwise (almost everywhere) limit but also as a limit in the mean, which is denoted by $\mathrm{l.i.m}$. The same is also true for relations (2.3) and (2.5). Hence, we get

$$\mathcal{F}(f,x):=\mathrm{l.i.m}\left\{\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k} f(t)e^{-itx}\,dt:\ k\to\infty\right\},$$

$$f(x):=\mathrm{l.i.m}\left\{\frac{1}{\sqrt{2\pi}}\int\limits_{-k}^{k} \mathcal{F}(f,t)e^{itx}\,dt:\ k\to\infty\right\}.$$

In the inversion formulas

$$\mathcal{F}(f,x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)e^{-itx}\,dt,\qquad f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)e^{itx}\,dt$$

written for the function $f\in L_2(\mathbb{R})$, the integrals are understood in a sense of mean square convergence, i.e., the respective relations (2.4) and (2.5) are true.

2.2. By $\mathbb{B}_{\sigma,2}$, $\sigma \in (0, \infty)$, we denote the collection of all entire functions g of exponential type not greater than σ whose restrictions to the entire real axis $\mathbb{R}$ belong to the space $L_2(\mathbb{R})$.

Let $L_2(a, b)$, $-\infty < a < b < +\infty$, be the space of functions measurable on (a, b) whose squared moduli are Lebesgue integrable, i.e.,

$$\int_a^b |f(x)|^2 dx < \infty, \quad \sigma \in (0, \infty).$$

Thus, if the Fourier transform of the function g, i.e., $\mathcal{F}(g)$, belongs to $L_2(-\sigma, \sigma)$, then

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \mathcal{F}(g, t) e^{itx}\, dt \tag{2.6}$$

is an element of the space $L_2(\mathbb{R})$ and admits the analytic extension onto the entire complex plane to an entire function of exponential type not greater than σ. In other words, any function $g(z)$ admitting representation (2.6) on the real axis belongs to $\mathbb{B}_{\sigma,2}$. The converse statement is also true.

Wiener–Paley Theorem ([48], Chap. II, Sec. 2.5)**.** *In order that a function $g \in L_2(\mathbb{R})$ be representable in the form (2.6), where $\mathcal{F}(g) \in L_2(-\sigma, \sigma)$, i.e., in order that $g(x)$ be a function with finite and square integrable spectrum, it is necessary and sufficient that $g(x)$ can be defined in the plane of complex variable $z = x + iy$ as an entire function of finite exponential type $\leq \sigma$.*

2.3. As the subsequent development of the Shapiro–Boman ideas presented in [38, 39], the generalized moduli of continuity of 2π-periodic functions in the space $L_2([0, 2\pi])$ were studied by Vasil'ev, Babenko, Kozko, Rozhdestvenskii, and Vakarchuk (see, e.g., [40–42, 25–27]). The notion of generalized modulus of continuity was extended to the space of functions of n variables $L_2(\mathbb{R}^n)$ by Vasil'ev [17] and then used by Gorbachev [43].

2.3.1. By using the notation introduced in [17, 43], we give the definition of generalized modulus of continuity in the space $L_2(\mathbb{R})$. Let $\mathcal{M} := \{\mu_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers satisfying the conditions

$$0 < \sum_{j\in\mathbb{Z}} |\mu_j| < \infty \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \mu_j = 0.$$

Also let

$$\mu(z) := \sum_{j\in\mathbb{Z}} \mu_j z^j$$

and let H^h be an operator of shift, i.e., $H^h f(x) := f(x + h)$, $h \in \mathbb{R}$ and, in addition, $(H^h)^j := H^{hj}$. By $\Delta_h^{\mathcal{M}}$ we denote a generalized difference operator with constant coefficients acting from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$. Moreover, almost everywhere on $\mathbb{R}$, we have

$$\Delta_h^{\mathcal{M}}(f, x) := \sum_{j\in\mathbb{Z}} \mu_j f(x + jh) = \mu(H^h) f(x). \tag{2.7}$$

Thus, in the case $\widetilde{\mu}_1(z) := (z - 1)^m$, $m \in \mathbb{N}$, for the number sequence corresponding to this function, we get

$$\mathcal{M}_{1,m} := \left\{ \mu_j = (-1)^{m-j} \binom{m}{j} \ \text{ for } j = 0, \ldots, m; \ \ \mu_j = 0 \ \text{ for } j < 0 \ \text{ or } j > m \right\}_{j\in\mathbb{Z}}.$$

Note that, by virtue of (2.7), $\Delta_h^{\mathcal{M}_{1,m}}$ is a finite-difference operator Δ_h^m. For the function $f \in L_2(\mathbb{R})$, this operator (almost everywhere) takes the form

$$\Delta_h^m(f,x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jh).$$

For $\widetilde{\mu}_2(z) := (1-z)^\beta,\ |z| \le 1,\ \beta \in (0,\infty)\backslash\mathbb{N},$ we find

$$\mathcal{M}_{2,\beta} := \left\{ \mu_j = (-1)^j \binom{\beta}{j} \ \text{ for } \ j = 0, 1, \ldots; \ \ \mu_j = 0 \ \text{ for } \ j = -1, -2, \ldots \right\}_{j\in\mathbb{Z}}$$

and the corresponding operator $\Delta_h^{\mathcal{M}_{2,\beta}}$ is the difference $\Delta_{-h}^{\beta}(f,x)$ of fractional order β. Here,

$$\Delta_{-h}^{\beta}(f,x) := \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{\beta}{j} f(x-jh),$$

where $h \in \mathbb{R},\ \mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. This difference is called left-hand-sided for $h > 0$ and right-hand-sided for $h < 0$. Thus, by virtue of (2.7), we obtain

$$\Delta_{-h}^{\mathcal{M}_{2,\beta}}(f,x) = \Delta_{-h}^{\beta}(f,x).$$

For

$$\widetilde{\mu}_3(z) := (-1)^m \prod_{j=0}^{m-1} (1 - z^{a^j}),$$

where $m, a \in \mathbb{N},$ we arrive at the Thue–Morse difference operator [42]:

$$\widetilde{\Delta}_{ah}^m(f,x) = \prod_{j=0}^{m-1} \Delta_{a^j h}^1(f,x) = \prod_{j=0}^{m-1} \big(f(x+a^j h) - f(x)\big).$$

In the case of a numerical sequence

$$\mathcal{M}_3 := \big\{\mu_j = 3/(\pi j)^2 \text{ for } j \in \mathbb{Z}\backslash\{0\};\ \mu_j = -1 \text{ for } j = 0\big\}_{j\in\mathbb{Z}},$$

the operator $\Delta_h^{\mathcal{M}_3}$ coincides with the operator $\overline{\Delta}_h$ in the definition of the modulus of continuity (1.9) and, for the numerical sequence

$$\mathcal{M}_4 := \big\{\mu_j = 4/(\pi j)^2 \ \text{ for } \ j = 2\nu+1, \ \ \nu \in \mathbb{Z}; \ \ \mu_j = 0 \ \text{ for } \ j = 2\nu, \ \ \nu \in \mathbb{Z}\backslash\{0\}; \ \ \mu_j = -1 \ \text{ for } \ j = 0\big\}_{j\in\mathbb{Z}},$$

by virtue of (2.7), the corresponding operator $\Delta_h^{\mathcal{M}_4}$ turns into the difference operator $\widehat{\Delta}_h$ from the definition of the characteristic of smoothness (1.10).

By using relations (2.5) and (2.7) and the conditions imposed on terms of the numerical sequence $\mathcal{M} = \{\mu_j\}_{j\in\mathbb{Z}}$, we obtain (almost everywhere on $\mathbb{R}$)

$$\Delta_h^{\mathcal{M}}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)\left(\sum_{j\in\mathbb{Z}}\frac{e^{i(x+jh)t}-1}{it}\,\mu_j\right)dt$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)\left(\sum_{j\in\mathbb{Z}}\mu_j e^{ijht}\right)\frac{e^{ixt}}{it}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}(f,t)\left(\sum_{j\in\mathbb{Z}}\mu_j e^{ijht}\right)\frac{e^{ixt}-1}{it}\,dt. \tag{2.8}$$

Since, for any function f belonging to $L_2(\mathbb{R})$, its generalized difference $\Delta_h^{\mathcal{M}}(f)$ is also an element of $L_2(\mathbb{R})$, in view of (2.3), we can write

$$\Delta_h^{\mathcal{M}}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\mathcal{F}\Big(\Delta_h^{\mathcal{M}}(f),t\Big)\frac{e^{ixt}-1}{it}\,dt. \tag{2.9}$$

Setting

$$w_{\mathcal{M}}(x) := \mu(e^{ix}) = \sum_{j\in\mathbb{Z}}\mu_j e^{ijx}, \tag{2.10}$$

we derive the following relation from (2.8) and (2.9) almost everywhere on $\mathbb{R}$:

$$\mathcal{F}\big(\Delta_h^{\mathcal{M}}(f),x\big) = w_{\mathcal{M}}(hx)\mathcal{F}(f,x). \tag{2.11}$$

By using relations (2.2) and (2.11), we get

$$\big\|\Delta_h^{\mathcal{M}}(f)\big\|^2 = \int_{-\infty}^{\infty}\big|\mathcal{F}\big(\Delta_h^{\mathcal{M}}(f),x\big)\big|^2 dx = \int_{-\infty}^{\infty}\big|\mathcal{F}(f,x)\big|^2\big|w_{\mathcal{M}}(hx)\big|^2 dx. \tag{2.12}$$

It follows from relation (2.10) that the complex-valued function $w_{\mathcal{M}}$ is continuous, 2π-periodic, and such that $w_{\mathcal{M}}(0) = 0$. All arguments presented above are also true for the real-valued function $|w_{\mathcal{M}}|^2$, which, in addition, can be even if all elements of the numerical sequence $\mathcal{M} = \{\mu_j\}_{j\in\mathbb{Z}}$ are real numbers.

2.3.2. By virtue of (2.10), we get the following result for the numerical sequence $\mathcal{M}_{1,m}$:

$$w_{\mathcal{M}_{1,m}}(x) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}e^{ijx} = (e^{ix}-1)^m.$$

Therefore,

$$\big|w_{\mathcal{M}_{1,m}}(x)\big|^2 = 2^m(1-\cos x)^m. \tag{2.13}$$

Further, we consider a numerical sequence $\mathcal{M}_{2,\beta}$ for which $\beta \in (0,\infty)\backslash\mathbb{N}$. This yields

$$w_{\mathcal{M}_{2,\beta}}(x) = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{\beta}{j} e^{ijx} = (1-e^{ix})^\beta$$

and

$$\left|w_{\mathcal{M}_{2,\beta}}(x)\right|^2 = 2^\beta(1-\cos x)^\beta. \tag{2.14}$$

By virtue of (2.11), in the analyzed case, almost everywhere on $\mathbb{R}$, we find

$$\mathcal{F}\big(\Delta_h^\beta(f),x\big) = \mathcal{F}\big(\Delta_{-h}^{\mathcal{M}_{2,\beta}}(f),x\big) = w_{\mathcal{M}_{2,\beta}}(-hx)\mathcal{F}(f,x) = \big(1-e^{-ixh}\big)^\beta \mathcal{F}(f,x).$$

For the numerical sequence $\mathcal{M}_3$, by using the results obtained in [44, p. 776] (Sec. 5.4.2.7), for $0 \le x \le \pi$, we get

$$w_{\mathcal{M}_3}(x) = -1 + \frac{6}{\pi^2}\sum_{j\in\mathbb{N}} \cos\frac{jx}{j^2} = \frac{3x(x/(2\pi)-1)}{\pi},$$

i.e.,

$$\left|w_{\mathcal{M}_3}(x)\right|^2 = \frac{9}{\pi^2}x^2\Big(1-\frac{x}{2\pi}\Big)^2. \tag{2.15}$$

For the numerical sequence $\mathcal{M}_4$, by using the results obtained in [44, p. 771] (Sec. 5.4.6.5), for $0 \le x \le \pi$, we find

$$w_{\mathcal{M}_4}(x) = -1 + \frac{8}{\pi^2}\sum_{\nu\in\mathbb{Z}_+} \cos((2\nu+1)x)/(2\nu+1)^2 = -\frac{2x}{\pi},$$

i.e.,

$$\left|w_{\mathcal{M}_4}(x)\right|^2 = \frac{4x^2}{\pi^2}. \tag{2.16}$$

Note that functions (2.13)–(2.16) are 2π-periodic, continuous, and even (for this reason, they are considered on the segment $[0,\pi]$). Moreover, they are equal to 0 at the origin.

Following Vasil'ev [17], the generalized modulus of continuity of an arbitrary element $f \in L_2(\mathbb{R})$ generated by a numerical sequence $\mathcal{M} = \{\mu_j\}_{j\in\mathbb{Z}}$ is defined as a function

$$w_{\mathcal{M}}(f,t) := \sup\big\{\|\Delta_h^{\mathcal{M}}(f)\|:\ |h| \le t\big\}, \quad t \ge 0. \tag{2.17}$$

In the general case, including the numerical sequences $\mathcal{M}_{1,m}$, $\mathcal{M}_{2,\beta}$, $\mathcal{M}_3$, and $\mathcal{M}_4$ considered above, by using (2.12) and (2.17), we obtain

$$w_{\mathcal{M}}(f,t) = \sup\left\{\left(\int_{-\infty}^{\infty} \left|\mathcal{F}(f,\tau)\right|^2 \left|w_{\mathcal{M}}(h\tau)\right|^2 d\tau\right)^{1/2}:\ 0 \le h \le t\right\}, \quad t \ge 0. \tag{2.18}$$

2.3.3. The characteristics of smoothness (1.5)–(1.7) do not fit the general scheme of formation of the generalized modulus of continuity of the form (2.17) in the space $L_2(\mathbb{R})$. We consider relations (1.6) and (1.7). According to [25, 30], for any function $f \in L_2(\mathbb{R})$, we obtain

$$\widetilde{\Omega}_m(f,t) = \sup\left\{\left(\int\limits_{-\infty}^{\infty} \big|\mathcal{F}(f,\tau)\big|^2\big(1-\operatorname{sinc}(h\tau)\big)^{2m} d\tau\right)^{1/2} : 0 < h \le t\right\}, \quad t > 0, \tag{2.19}$$

and

$$\Lambda_m(f,t) = \left\{\int\limits_{-\infty}^{\infty} \big|\mathcal{F}(f,\tau)\big|^2 \eta_m(t\tau)\, d\tau\right\}^{1/2}, \quad t > 0, \tag{2.20}$$

where $m \in \mathbb{N},\ \operatorname{sinc}(x) := \big\{\sin(x)/x \text{ for } x \ne 0;\ 1 \text{ for } x = 0\big\}$, and

$$\eta_m(x) := (2^m/x)\int\limits_0^x (1-\cos v)^m\, dv, \quad x \ne 0.$$

We set $\eta_m(0) = 0$. The functions $\big(1-\operatorname{sinc}(x)\big)^{2m}$ and $\eta_m(x)$ in relations (2.19) and (2.20), respectively, are continuous, even and bounded. Moreover, they are not equal to zero almost everywhere and equal to zero at $x = 0$ on the set $\mathbb{R}$. However, none of these functions is 2π-periodic.

2.3.4. We continue the generalization of the characteristics of smoothness in the space $L_2(\mathbb{R})$. By $\mathbb{G}$ we denote the set of all continuous nonnegative even functions φ bounded on the entire real axis $\mathbb{R}$ that are not equal to zero almost everywhere on $\mathbb{R}$ and are such that $\varphi(0) = 0$. By $\mathfrak{M}$ we denote the class of all complex-valued functions $w\colon \mathbb{R} \to \mathbb{C}$ for which $|w|^2 \in \mathbb{G}$.

Let $f \in L_2(\mathbb{R})$ and let $\mathcal{F}(f)$ be the Fourier transform of the function $f,\ w \in \mathfrak{M},\ h \in \mathbb{R}$. Then

$$\big\|\mathcal{F}(f,\cdot)w(h\cdot)\big\| \le \|w\|_{C(\mathbb{R})}\|f\| < \infty,$$

i.e., $\mathcal{F}(f,x)w(hx) \in L_2(\mathbb{R})$. By using the generalized difference operator $\Delta_h^w\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$, where $h \in \mathbb{R}$ and $w \in \mathfrak{M}$, we define a function

$$\Delta_h^w(f,x) := \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,\tau)w(h\tau)\frac{e^{ix\tau}-1}{i\tau}\, d\tau \tag{2.21}$$

almost everywhere on $\mathbb{R}$. Thus, for $w := w_{\mathcal{M}}$, where $w_{\mathcal{M}}$ is given by relation (2.10), and $f \in L_2(\mathbb{R})$, from (2.8) and (2.21), we obtain

$$\Delta_h^{w_{\mathcal{M}}}(f) = \Delta_h^{\mathcal{M}}(f).$$

In this connection, relation (2.21) can be regarded as an extension of the generalized difference operator $\Delta_h^{\mathcal{M}}$ to a more general case Δ_h^w. According to (2.3) and (2.21), the equality $\mathcal{F}(\Delta_h^w(f),x) = \mathcal{F}(f,x)w(hx),\ h \in \mathbb{R}$, is true almost everywhere on $\mathbb{R}$. By using this equality and relation (2.2), we get

$$\big\|\Delta_h^w(f)\big\|^2 = \big\|\mathcal{F}(\Delta_h^w(f))\big\|^2 = \int\limits_{-\infty}^{\infty} \big|\mathcal{F}(f,\tau)\big|^2\big|w(h\tau)\big|^2 d\tau. \tag{2.22}$$

This yields the characteristic of smoothness of the functions $f \in L_2(\mathbb{R})$

$$\omega^w(f,t) := \sup\left\{\|\Delta_h^w(f)\|:\ |h| \le t\right\}, \quad t \ge 0, \tag{2.23}$$

which, in a certain sense, is more general than (2.17). Moreover, in view of (2.22) and (2.23), we get

$$\omega^w(f,t) = \sup\left\{\left(\int\limits_{-\infty}^{\infty} \left|\mathcal{F}(f,\tau)\right|^2 \left|w(h\tau)\right|^2 d\tau\right)^{1/2}:\ 0 \le h \le t\right\}, \quad t \ge 0. \tag{2.24}$$

Furthermore, $\lim\left\{\omega^w(f,t):\ t \to 0+\right\} = 0$ and $\omega^w(f,t)$ is a continuous function nondecreasing on the set $0 \le t < \infty$ and such that

$$\omega^w(f_1+f_2,t) \le \omega^w(f_1,t) + \omega^w(f_2,t),$$

where $f_1, f_2 \in L_2(\mathbb{R})$.

Comparing (2.18) with (2.24), we get $\omega^{w_{\mathcal{M}}}(f,t) = \omega_{\mathcal{M}}(f,t),\ t \ge 0$. Since the difference operator $\widetilde{\Delta}_h^m(f)$, $m \in \mathbb{N},\ h \in (0,\infty),\ f \in L_2(\mathbb{R})$, considered in Sec. 1.3 can be represented in the form [25]

$$\widetilde{\Delta}_h^m(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int\limits_{-\infty}^{\infty} \mathcal{F}(f,\tau)(\operatorname{sinc}(h\tau)-1)^m\,\frac{e^{ix\tau}-1}{i\tau}\,d\tau,$$

almost everywhere on $\mathbb{R}$, in view of (2.21), we conclude that the operator $\widetilde{\Delta}_h^m(f)$ is a special case of the generalized operator $\Delta_h^w(f)$ for $w = \widetilde{w}_m$, where $\widetilde{w}_m(x) := (\operatorname{sinc}(x)-1)^m$. By using (2.19) and (2.24), we obtain $\omega^{w_m}(f,t) = \widetilde{\Omega}_m(f,t),\ t > 0$.

2.3.5. Consider the second group of functions that can be used as characteristics of smoothness in the space $L_2(\mathbb{R})$. Let $f \in L_2(\mathbb{R})$ and $w \in \mathfrak{M}$. We set

$$\Lambda^w(f,t) := \left\{\frac{1}{t}\int\limits_0^t \left\|\Delta_h^w(f)\right\|^2 dh\right\}^{1/2}, \quad t > 0. \tag{2.25}$$

Moreover, $\lim\left\{\Lambda^w(f,t):\ t \to 0+\right\} = 0;$ $\Lambda^w(f,t)$ is a continuous function on the set $0 < t < \infty;$

$$\Lambda^w(f,t) \le \omega^w(f,t), \quad t > 0; \qquad \Lambda^w(f_1+f_2,t) \le \sqrt{2}(\Lambda^w(f_1,t) + \Lambda^w(f_2,t)), \quad t > 0,$$

where $f_1, f_2 \in L_2(\mathbb{R})$. Thus, in the case $w = w_{\mathcal{M}_{1,m}},\ m \in \mathbb{N}$, from (1.7) and (2.25), we obtain

$$\Lambda^{w_{\mathcal{M}_{1,m}}}(f,t) = \Lambda_m(f,t), \quad t > 0.$$

By using (2.22) and (2.25), for any element $f \in L_2(\mathbb{R})$, we can write

$$\Lambda^w(f,t) = \left\{\int\limits_{-\infty}^{\infty} \left|\mathcal{F}(f,\tau)\right|^2 \left(\frac{1}{t}\int\limits_0^t \left|w(h\tau)\right|^2 dh\right) d\tau\right\}^{1/2}, \quad t > 0. \tag{2.26}$$

It is clear that

$$\int_0^t |w(h\tau)|^2 dh = \frac{1}{\tau}\int_0^{t\tau} \big|w(h)\big|^2 dh, \qquad t>0, \quad \tau\neq 0. \tag{2.27}$$

Let

$$\mathcal{W}(x) := \begin{cases} 0 & \text{for} \quad x=0, \\ (1/x)\displaystyle\int_0^x \big|w(h)\big|^2 dh & \text{for} \quad x\in\mathbb{R} \quad \text{and} \quad x\neq 0, \end{cases} \tag{2.28}$$

where $w \in \mathfrak{M}$. Since the function $|w|^2$ is even, it follows from (2.28) that $\mathcal{W}(x) = \mathcal{W}(-x),\ x\in\mathbb{R}$. In view of (2.27) and (2.28), relation (2.26) takes the form

$$\Lambda^w(f,t) = \left\{ \int_{-\infty}^{\infty} |\mathcal{F}(f,\tau)|^2 \mathcal{W}(t\tau)\, d\tau \right\}^{1/2}, \quad t>0. \tag{2.29}$$

It is worth noting that, for the functions $w = w_{\mathcal{M}_{2,\beta}},\ \beta\in(0,\infty),\ w = w_{\mathcal{M}_3}$, and $w = w_{\mathcal{M}_4}$ as well as for the functions $w = \widetilde{w}_m,\ m\in\mathbb{N}$, the class of extreme problems of the approximation theory of functions in the space $L_2(\mathbb{R})$ considered in what follows was not investigated earlier with the use of the characteristic of smoothness Λ^w.

3. Fractional-Order Derivatives of Functions in the Space $L_2(\mathbb{R})$

We now recall the definition of the derivative of fractional order $\alpha\in(0,\infty)$ of an arbitrary function $f\in L_2(\mathbb{R})$ (see, e.g., [10, 45–47, 32]). Assume that q is a function from $L_2(\mathbb{R})$ such that

$$\lim\big\{\|\Delta^{\alpha}_{-h}(f)/h^{\alpha} - q\|:\ h\to 0+\big\} = 0, \tag{3.1}$$

where

$$\Delta^{\alpha}_{-h}(f,x) = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{\alpha}{j} f(x-jh)$$

almost everywhere on $\mathbb{R}$. Then q is called the strong Liouville–Grünwald–Letnikov derivative of fractional order α for a function $f\in L_2(\mathbb{R})$ and denoted by $\mathcal{D}^{\alpha} f$, i.e., $q = \mathcal{D}^{\alpha} f$. In particular, it follows from equality (3.1) that

$$\|\mathcal{D}^{\alpha} f\| = \lim\big\{\|\Delta^{\alpha}_{-h}(f)/h^{\alpha}\|:\ h\to 0+\big\}.$$

In [10], Gaimnazarov showed that the equality

$$\mathcal{F}(\mathcal{D}^{\alpha} f, x) = (ix)^{\alpha}\mathcal{F}(f,x) \tag{3.2}$$

is true for any function $f\in L_2(\mathbb{R})$ and $\alpha\in(0,\infty)$ almost everywhere on $\mathbb{R}$.

In [32], it was indicated that if there exists a strong Liouville–Grünwald–Letnikov derivative $D^\alpha f$ in the sense indicated above, then the equality

$$D^\alpha f(x) = \lim\left\{\Delta^\alpha_{-h}(f,x)/h^\alpha :\ h \to 0+\right\}$$

holds almost everywhere on $\mathbb{R}$,.

By $L^\alpha_2(\mathbb{R})$, $\alpha \in (0,\infty)$, we denote a class of functions $f \in L_2(\mathbb{R})$ with fractional-order derivatives $D^\alpha f$ that belong to the space $L_2(\mathbb{R})$. Note that $L^\alpha_2(\mathbb{R})$ is a Banach space with the norm $\|f\| + \|D^\alpha f\|$. For $\alpha = r,\ r \in \mathbb{N}$, by $L^r_2(\mathbb{R})$ we denote a class of functions $f \in L_2(\mathbb{R})$ whose derivatives of order $(r-1)$ are locally absolutely continuous and derivatives of order r belong to the space $L_2(\mathbb{R})$. In this case, it is clear that $D^r f = f^{(r)}$ almost everywhere on $\mathbb{R}$.

4. Best Mean-Square Approximations by Entire Functions of Exponential Type $\sigma \in (0,\infty)$ on the Classes $L_2(\mathbb{R})$ and $L^\alpha_2(\mathbb{R}),\ \alpha \in (0,\infty)$, Expressed in Terms of the Characteristic of Smoothness ω^w

4.1. Prior to presentation of the main results of this section, we introduce some necessary notions and definitions. Since the functions of the set $\mathbb{G}$ introduced in Sec. 2.3.4 are even, it suffices to consider them only on the semiaxis $\mathbb{R}_+$. For any element $\varphi \in \mathbb{G}$, by $t_* \in (0,\infty)$ we denote the value of the argument x for which

$$\varphi(t_*) = \sup\left\{\varphi(x) :\ 0 < x < \infty\right\}. \tag{4.1}$$

It is clear that t_* depends on φ. In the case where the upper bound in relation (4.1) is attained for several values of the argument, as t_*, we take the least of these values.

We say that a function $\varphi \in \mathbb{G}$ possesses *property A* if it is monotonically increasing on the segment $[0, t_*]$. For any element $\varphi \in \mathbb{G}$ with this property, we set

$$\varphi_*(x) := \left\{\varphi(x)\ \text{ for }\ 0 \le x \le t_*;\ \varphi(t_*)\ \text{ for }\ t_* \le x < \infty\right\}, \tag{4.2}$$

$$\varphi(\widetilde{t}_*) = \inf\left\{\varphi(x) :\ t_* < x < \infty\right\}, \tag{4.3}$$

where the value t_* is determined from relation (4.1) In the case where the lower bound in relation (4.3) is attained for several values of the argument, as $\widetilde{t}_*$, we choose the least of these values.

We say that a function $\varphi \in \mathbb{G}$ satisfies *property B* if $\varphi(\widetilde{t}_*) > 0$.

Note that the functions

$$|w_{\mathcal{M}_{1,m}}|^2,\quad m \in \mathbb{N},\qquad |w_{\mathcal{M}_{2,\beta}}|^2,\quad \beta \in (0,\infty),\qquad |w_{\mathcal{M}_3}|^2,\qquad \text{and}\qquad |w_{\mathcal{M}_4}|^2$$

belong to the space $\mathbb{G}$, satisfy *property A*, and, for each of them, $t_* = \pi$. As for the functions $|\widetilde{w}_m|^2,\ m \in \mathbb{N}$, they are also elements of the set $\mathbb{G}$, satisfy *properties A* and *B*, and take the same value $t_* \in (4.49; 4.51)$, which is the least positive root of the equation $\tan(x) = x$ (see, e.g., [21, 25]).

For any function $f \in L_2(\mathbb{R})$, by $\mathcal{A}_\sigma(f)$, $\sigma \in (0,\infty)$, we denote its best mean-square approximation by elements of the subspace $\mathbb{B}_{\sigma,2}$ formed by functions of exponential type $\le \sigma$ whose restrictions to $\mathbb{R}$ belong to the space $L_2(\mathbb{R})$, i.e.,

$$\mathcal{A}_\sigma(f) := \inf\left\{\|f-g\| :\ g \in \mathbb{B}_{\sigma,2}\right\}.$$

In what follows, we need the following statement established by Ibragimov and Nasibov [6]:

Lemma 1. *Assume that a function f belongs to the space $L_2(\mathbb{R})$ and that $\mathcal{F}(f)$ is its Fourier transform in a sense of $L_2(\mathbb{R})$, i.e.,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \mathcal{F}(f,t) \frac{e^{ixt}-1}{it}\, dt,$$

where $\mathcal{F}(f) \in L_2(\mathbb{R})$. Then

$$\mathcal{L}_\sigma(f,x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \mathcal{F}(f,t) e^{ixt}\, dt \tag{4.4}$$

is an entire function from the subspace $\mathbb{B}_{\sigma,2}$ that has the least deviation from f in a sense of metric of the space $L_2(\mathbb{R})$, i.e.,

$$\mathcal{A}_\sigma(f) = \big\|f - \mathcal{L}_\sigma(f)\big\| = \left\{ \int_{|t|\geq\sigma} |\mathcal{F}(f,t)|^2\, dt \right\}^{1/2}. \tag{4.5}$$

4.2. **Theorem 1.** *Suppose that α and σ belong to $(0,\infty)$, a complex-valued function $w\colon \mathbb{R} \to \mathbb{C}$ is such that $|w|^2$ belongs to the set $\mathbb{G}$ and satisfies properties A and B, and a point $\bar{t} \in (0, t_*)$ is defined as follows:*

$$\big|w(\bar{t})\big| = \big|w(\widetilde{t}_*)\big|, \tag{4.6}$$

where the quantity $\widetilde{t}_*$ is determined from relation (4.3) for the function $\varphi = |w|^2$. Then, for any $\tau \in (0,\bar{t}]$, the equality

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\omega^w(D^\alpha f, \tau/\sigma)} = \frac{1}{|w(\tau)|} \tag{4.7}$$

is true.

Proof. By using relations (2.24), (3.2), and (4.5) and taking into account the fact that the function $|w|^2$ satisfies property *B*, for $0 < t \leq \bar{t}/\sigma$, we conclude that

$$\omega^w(D^\alpha f, t) \geq \left\{ \int_{|\tau|\geq\sigma} \big|\mathcal{F}(D^\alpha f,\tau)\big|^2 \big|w(t\tau)\big|^2\, d\tau \right\}^{1/2}$$

$$= \left\{ \int_{|\tau|\geq\sigma} |\tau|^{2\alpha} \big|\mathcal{F}(f,\tau)\big|^2\, |w(t\tau)|^2\, d\tau \right\}^{1/2}$$

$$\geq \sigma^\alpha \left\{ \int_{|\tau|\geq\sigma} \big|\mathcal{F}(f,\tau)\big|^2 \big|w(t\tau)\big|^2\, d\tau \right\}^{1/2}$$

$$\geq \sigma^{\alpha}|w(t\sigma)|\left\{\int\limits_{|\tau|\geq\sigma}\left|\mathcal{F}(f,\tau)\right|^{2}d\tau\right\}^{1/2}$$

$$=\sigma^{\alpha}\big|w(t\sigma)\big|\mathcal{A}_{\sigma}(f).$$

Thus, setting $t=\tau/\sigma$, where $0<\tau\leq\bar{t}$, we get the following upper bound:

$$\sup_{f\in L_2^{\alpha}(\mathbb{R})}\frac{\sigma^{\alpha}\mathcal{A}_{\sigma}(f)}{\omega^{w}(D^{\alpha}f,\tau/\sigma)}\leq\frac{1}{|w(\tau)|}. \tag{4.8}$$

We now establish a lower bound for the extreme characteristic on the left-hand side of inequality (4.8). To this end, we consider a function $\lambda_a(x):=a\mathrm{sinc}\,(ax),\ a\in(0,\infty)$. Since

$$|\lambda_a(z)|\leq k\exp(a|z|),$$

where $k=\mathrm{const}(k>0),\ z\in\mathbb{C}$, the quantity λ_a is an entire function of exponential type $\leq a$. The function λ_a is not an element of the space $L_1(\mathbb{R})$ [48] (Chap. II, Sec. 2.3). However, $\lambda_a\in L_2(\mathbb{R})$ and, hence, it has the Fourier transform (2.1) in a sense of the space $L_2(\mathbb{R})$ equal to $\mathcal{F}(\lambda_a,x)=\sqrt{\pi/2}\,\big\{1$ for $|x|<a;\ 1/2$ for $|x|=a$; and 0 for $|x|>a\big\}$ [49] (Chap. 5). By using this result, we consider a function

$$q_{\sigma+\varepsilon}(x):=\sqrt{2/\pi}\big(\lambda_{\sigma+\varepsilon}(x)-\lambda_{\sigma}(x)\big),\quad\varepsilon>0, \tag{4.9}$$

which is an entire function of finite exponential type $\leq\sigma+\varepsilon$, belongs to the space $L_2(\mathbb{R})$, and has the Fourier transform

$$\mathcal{F}(q_{\sigma+\varepsilon},x)=\big\{1\text{ for }\sigma<|x|<\sigma+\varepsilon;$$
$$1/2\text{ for }|x|=\sigma\text{ or }|x|=\sigma+\varepsilon'\text{ and }0\text{ for }|x|<\sigma\text{ or }|x|>\sigma+\varepsilon\big\}. \tag{4.10}$$

It follows from (2.2), (3.2), and (4.10) that $q_{\sigma+\varepsilon}$ belongs to $L_2^{\alpha}(\mathbb{R})$. In view of (4.5) and (4.10), for $q_{\sigma+\varepsilon}$, we obtain

$$\mathcal{A}_{\sigma}(q_{\sigma+\varepsilon})=\sqrt{2\varepsilon}. \tag{4.11}$$

According to (2.22) and (3.2),

$$\big\|\Delta_h^{w}(\mathcal{D}^{\alpha}f)\big\|^2=\int\limits_{-\infty}^{\infty}\big|\mathcal{F}(f,\tau)\big|^2|\tau|^{2\alpha}\big|w(h\tau)\big|^2d\tau. \tag{4.12}$$

Thus, by using relation (4.2), where $\varphi=|w|^2$, and relations (4.10)–(4.12) and taking into account the fact that $|w|^2\in\mathbb{G}$, for the function $q_{\sigma+\varepsilon}$, we find

$$\big\|\Delta_h^{w}(\mathcal{D}^{\alpha}q_{\sigma+\varepsilon})\big\|^2=2\int\limits_{\sigma}^{\sigma+\varepsilon}\tau^{2\alpha}|w(h\tau)|^2d\tau$$

$$\leq 2(\sigma+\varepsilon)^{2\alpha} \int\limits_{\sigma}^{\sigma+\varepsilon} |w(h\tau)|^2 d\tau$$

$$\leq (\sigma+\varepsilon)^{2\alpha} \mathcal{A}_\sigma^2(q_{\sigma+\varepsilon}) \left|w(h(\sigma+\varepsilon))\right|_*^2. \tag{4.13}$$

By using definition (2.23) for the characteristic of smoothness ω^w and (4.13), for any $t \in (0, \bar{t}/\sigma]$, we obtain

$$\omega^w(\mathcal{D}^\alpha q_{\sigma+\varepsilon}, t) \leq \mathcal{A}_\sigma(q_{\sigma+\varepsilon})(\sigma+\varepsilon)^\alpha \left|w(t(\sigma+\varepsilon))\right|_*. \tag{4.14}$$

Setting $t = \tau/\sigma$, where $0 < \tau \leq \bar{t}$, and introducing the notation

$$\theta_\varepsilon(\sigma, \tau) := (1+\varepsilon/\sigma)^\alpha \left|w(\tau(1+\varepsilon/\sigma))\right|_*, \tag{4.15}$$

in view of (4.14), we get

$$\frac{\sigma^\alpha \mathcal{A}_\sigma(q_{\sigma+\varepsilon})}{\omega^w(\mathcal{D}^\alpha q_{\sigma+\varepsilon}, \tau/\sigma)} \geq \frac{1}{\theta_\varepsilon(\sigma,\tau)}.$$

Since $q_{\sigma+\varepsilon}$ belongs to $L_2^\alpha(\mathbb{R})$, this yields

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\omega^w(\mathcal{D}^\alpha f, \tau/\sigma)} \geq \frac{1}{\theta_\varepsilon(\sigma,\tau)}. \tag{4.16}$$

It follows from (4.15) that the quantity $\theta_\varepsilon(\sigma,\tau)$ monotonically decreases as $\varepsilon \to 0+$ for constant values of σ and τ. Therefore, as $\varepsilon \to 0+$, the quantity $1/\theta_\varepsilon(\sigma,\tau)$ monotonically increases and is bounded from above by $1/|w(\tau)|$. Thus, for an arbitrarily small number $\delta > 0$, there exists a value $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\delta) \in (0, \sigma_*)$, where $\sigma_* := \min(\sigma, 1/\sigma)$, for which $1/\theta_{\widetilde{\varepsilon}}(\sigma,\tau) > 1/|w(\tau)| - \delta$. By using this result and the definition of the upper bound for a number set, we get

$$\sup\left\{1/\theta_\varepsilon(\sigma,\tau) \colon\ 0 < \varepsilon < \sigma_*\right\} = 1/|w(\tau)|. \tag{4.17}$$

Finding the upper bound of the right-hand side of inequality (4.16) with respect to $\varepsilon \in (0, \sigma_*)$ and using (4.17), we obtain the lower bound as follows:

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\omega^w(D^\alpha f, \tau/\sigma)} \geq \frac{1}{|w(\tau)|}. \tag{4.18}$$

The required equality (4.7) follows from relations (4.8) and (4.18).
Theorem 1 is proved.

Remark 1. Reasoning as in the proof of Theorem 1, we get

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\omega^w(f, \tau/\sigma)} = \frac{1}{|w(\tau)|}, \tag{4.19}$$

where $0 < \tau \leq \bar{t}$. In this case, the upper bound is taken over all functions f from $L_2(\mathbb{R})$ not equivalent to zero.

Combining, e.g., (4.7) with $\alpha = r \in \mathbb{N}$ and $w = \widetilde{w}_m,\ m \in \mathbb{N},$ with relation (4.19) also with $w = \widetilde{w}_m,$ $m \in \mathbb{N},$ we arrive at one of our results from [25]

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\widetilde{\Omega}_m(f^{(r)}, \tau/\sigma)} = \frac{1}{(1-\operatorname{sinc}(\tau))^m},$$

where $r \in \mathbb{Z}_+,\ L_2^0(\mathbb{R}) \equiv L_2(\mathbb{R}),\ f^{(0)} \equiv f$, and $0 < \tau \le \bar{t}$.

4.3. Further, we set

$$\mathfrak{N}(f; u, \tau) := |\mathcal{F}(f,u)|^p\ |u|^{\alpha p}\ |w(\tau u)|^p\ \xi(\tau), \tag{4.20}$$

$$\Xi_{u,p,\alpha,w}(\xi, t) := |u|^\alpha \left\{ \int_0^t |w(\tau u)|^p\, \xi(\tau)\, d\tau \right\}^{1/p}. \tag{4.21}$$

Theorem 2. *Suppose that α and σ belong to $(0,\infty)$, a complex-valued function $w\colon \mathbb{R} \to \mathbb{C}$ is such that $|w|^2$ belongs to the set $\mathbb{G}$ and satisfies property A, $0 < p \le 2,\ t \in (0, t_*/\sigma]$, where t_* is given by (4.1) for $\varphi = |w|^2$, and ξ is a nonnegative function summable on the segment $[0,t]$ and not equivalent to zero. Then the two-sided inequality*

$$\frac{1}{\Xi_{\sigma,p,\alpha,w}(\xi,t)} \le \sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\int_0^t (\omega^w(\mathcal{D}^\alpha f, \tau))^p \xi(\tau)\, d\tau\right\}^{1/p}} \le \frac{1}{\inf\left\{\Xi_{u,p,\alpha,w}(\xi,t)\colon\ \sigma \le |u| < \infty\right\}} \tag{4.22}$$

is true.

Proof. By using relations (2.23), (4.12), (4.20), (4.21), and (4.5) and the generalized Minkowski inequality (see, e.g., [5], Chap. I, Sec. 1.3.2), for any $t \in (0, t_*/\sigma]$, we can write

$$\begin{aligned}
\left\{ \int_0^t \big(\omega^w(\mathcal{D}^\alpha f, \tau)\big)^p \xi(\tau)\, d\tau \right\}^{1/p} &\ge \left\{ \int_0^t \big\|\Delta_\tau^w(\mathcal{D}^\alpha f)\big\|^p \xi(\tau)\, d\tau \right\}^{1/p} \\
&= \left\{ \int_0^t \left[\int_{-\infty}^{\infty} \big|\mathcal{F}(f,u)\big|^2 |u|^{2\alpha} \big|w(\tau u)\big|^2 du \right]^{p/2} \xi(\tau)\, d\tau \right\}^{1/p} \\
&\ge \left\{ \int_0^t \left[\int_{|u|\ge\sigma} \big|\mathcal{F}(f,u)\big|^2 |u|^{2\alpha} \big|w(\tau u)\big|^2 du \right]^{p/2} \xi(\tau)\, d\tau \right\}^{1/p} \\
&= \left\{ \int_0^t \left[\int_{|u|\ge\sigma} \mathfrak{N}^{2/p}(f; u, \tau) du \right]^{p/2} d\tau \right\}^{1/p}
\end{aligned}$$

$$\geq \left\{ \int\limits_{|u|\geq\sigma} \left[\int\limits_0^t \mathfrak{N}(f;u,\tau)\,d\tau \right]^{2/p} du \right\}^{1/2}$$

$$= \left\{ \int\limits_{|u|\geq\sigma} |\mathcal{F}(f,u)|^2 \left[|u|^{\alpha p} \int\limits_0^t \left|w(\tau u)\right|^p \xi(\tau)\,d\tau \right]^{2/p} du \right\}^{1/2}$$

$$= \left\{ \int\limits_{|u|\geq\sigma} |\mathcal{F}(f,u)|^2\, \Xi^2_{u,p,\alpha,w}(\xi,t)du \right\}^{1/2}$$

$$\geq \mathcal{A}_\sigma(f) \inf\left\{\Xi_{u,p,\alpha,w} :\, \sigma \leq |u| < \infty\right\}.$$

This yields the upper bound

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\int_0^t (\omega^w(\mathcal{D}^\alpha f,\tau))^p \xi(\tau)\,d\tau\right\}^{1/p}} \leq \frac{1}{\inf\left\{\Xi_{u,p,\alpha,w}(\xi,t) :\, \sigma \leq |u| < \infty\right\}}. \tag{4.23}$$

We now deduce the lower bound for the extreme characteristic on the left-hand side of inequality (4.23). To this end, we use the entire function $q_{\sigma+\varepsilon} \in L_2^\alpha(\mathbb{R})$ of exponential type $\leq \sigma+\varepsilon$ given by relation (4.9) and introduced in the proof of Theorem 1. Further, we set

$$\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t) := (\sigma+\varepsilon)^\alpha \left\{ \int\limits_0^t \left|w(\tau(\sigma+\varepsilon))\right|_*^p \xi(\tau)\,d\tau \right\}^{1/p}, \quad \varepsilon > 0. \tag{4.24}$$

By using inequality (4.14) valid in a wider range $0 < t < \infty$, and relation (2.24), we obtain

$$\int\limits_0^t (\omega^w(\mathcal{D}^\alpha q_{\sigma+\varepsilon},\tau))^p \xi(\tau)\,d\tau \leq \mathcal{A}_\sigma^p(q_{\sigma+\varepsilon})(\sigma+\varepsilon)^{\alpha p} \int\limits_0^t |w(\tau(\sigma+\varepsilon))|_*^p \xi(\tau)\,d\tau$$

$$= \left(\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t)\right)^p \mathcal{A}_\sigma^p(q_{\sigma+\varepsilon}),$$

i.e.,

$$\left\{ \int\limits_0^t (\omega^w(\mathcal{D}^\alpha q_{\sigma+\varepsilon},\tau))^p \xi(\tau)\,d\tau \right\}^{1/p} \leq \mathcal{A}_\sigma(q_{\sigma+\varepsilon}) \widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t).$$

This yields

$$\frac{\mathcal{A}_\sigma(q_{\sigma+\varepsilon})}{\left\{\int_0^t (\omega^w(\mathcal{D}^\alpha q_{\sigma+\varepsilon},\tau))^p \xi(\tau)\,d\tau\right\}^{1/p}} \geq \frac{1}{\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t)}.$$

In view of the fact that, as indicated above, $q_{\sigma+\varepsilon}$ is an element of the class $L_2^\alpha(\mathbb{R})$ and the last inequality, for $0 < t \le t_*$, we find

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\displaystyle\int_0^t \big(\omega^w(\mathcal{D}^\alpha f,\tau)\big)^p \xi(\tau)\,d\tau\right\}^{1/p}} \ge \frac{1}{\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t)}. \tag{4.25}$$

It follows from relation (4.24) that, as $\varepsilon \to 0+$, the quantity $\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t)$ monotonically decreases as a function of ε for fixed values of the other parameters. By using (4.21), we get

$$\lim\big\{\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t) : \varepsilon \to 0+\big\} = \Xi_{\sigma,p,\alpha,w}(\xi,t).$$

Therefore, for an arbitrarily small $\delta > 0$, there exists a number $\widehat{\varepsilon} = \widehat{\varepsilon}(\delta) \in (0,\sigma_*)$ such that the inequality

$$\frac{1}{\widehat{\Xi}_{\sigma+\widehat{\varepsilon},p,\alpha,w}(\xi,t)} > \frac{1}{\Xi_{\sigma,p,\alpha,w}(\xi,t)} - \delta$$

holds. By using this relation and the definition of the upper bound for a number set, we get

$$\sup\left\{\frac{1}{\widehat{\Xi}_{\sigma+\varepsilon,p,\alpha,w}(\xi,t)} : 0<\varepsilon<\sigma_*\right\} = \frac{1}{\Xi_{\sigma,p,\alpha,w}(\xi,t)}. \tag{4.26}$$

Since the left-hand side of inequality (4.25) is independent of ε, we can find the upper bound of its right-hand side over $\varepsilon \in (0,\sigma_*)$ by using (4.26) and obtain

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\displaystyle\int_0^t \big(\omega^w(\mathcal{D}^\alpha f,\tau)\big)^p \xi(\tau)\,d\tau\right\}^{1/p}} \ge \frac{1}{\Xi_{\sigma,p,\alpha,w}(\xi,t)}. \tag{4.27}$$

The required relation (4.22) follows from inequalities (4.23) and (4.27). This completes the proof of Theorem 2.

Remark 2. Repeating almost exactly the proof of Theorem 2, we get a two-sided inequality for elements of the space $L_2(\mathbb{R})$:

$$\frac{1}{\widetilde{\Xi}_{\sigma,p,w}(\xi,t)} \le \sup_{f\in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\displaystyle\int_0^t (\omega^w(f,\tau))^p \xi(\tau)\,d\tau\right\}^{1/p}} \le \frac{1}{\inf\big\{\widetilde{\Xi}_{u,p,w}(\xi,t) : \sigma \le |u| < \infty\big\}}, \tag{4.28}$$

where $\widetilde{\Xi}_{\sigma,p,w}(\xi,t) := \Xi_{\sigma,p,0,w}(\xi,t)$. The upper bound in relation (4.28) is taken over all functions f from $L_2(\mathbb{R})$ not equivalent to zero.

If we now specify w in relation (4.22) by setting $w = w_{\mathcal{M}_{2,\beta}}$, $\beta \in (0,\infty)$, then we arrive at the main result of Theorem 1 in [32]:

$$\frac{1}{\gamma_{\sigma,\beta,\alpha,p}(\xi,t)} \le \sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\displaystyle\int_0^t \omega_\beta^p(\mathcal{D}^\alpha f,\tau)\xi(\tau)\,d\tau\right\}^{1/p}} \le \frac{1}{\inf\big\{\gamma_{u,\beta,\alpha,p}(\xi,t) : \sigma \le |u| < \infty\big\}}, \tag{4.29}$$

where $0 < t \le \pi/\sigma;\ \sigma, \alpha \in (0, \infty);\ 0 < p \le 2;$

$$\gamma_{u,\beta,\alpha,p}(\xi, t) := 2^{\beta/2}|u|^{\alpha} \left\{ \int\limits_0^t (1 - \cos(u\tau))^{\beta p/2} \xi(\tau)\, d\tau \right\}^{1/p}, \quad u \in \mathbb{R}.$$

Combining relations (4.29), where $\alpha \in \mathbb{N},\ \beta = m \in \mathbb{N}$, and $0 < p \le 2$, with (4.28), where $w = w_{\mathcal{M}_{1,m}},\ m \in \mathbb{N}$, and $0 < p \le 2$, we get the result of Theorem 1 in [26].

Now let $w = \widetilde{w}_m,\ m \in \mathbb{N};\ 0 < p \le 2;\ \sigma \in (0, \infty);\ 0 < t \le t_*/\sigma$ in relations (4.22) and (4.28) and let $\alpha = r \in \mathbb{N}$ in relation (4.22). Thus, combining these two relations under the indicated conditions, we get one of the main results of Theorem 2 in [25]:

$$\frac{1}{\widehat{\gamma}_{\sigma,m,r,p}(\xi, t)} \le \sup_{f \in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{ \int_0^t \widetilde{\Omega}_m^p(f^{(r)}, \tau)\xi(\tau)\, d\tau \right\}^{1/p}} \le \frac{1}{\inf\left\{ \widehat{\gamma}_{u,m,r,p}(\xi, t) \colon \sigma \le |u| < \infty \right\}},$$

where $r \in \mathbb{Z}_+$,

$$\widehat{\gamma}_{u,m,r,p}(\xi, t) := |u|^r \left\{ \int\limits_0^t (1 - \operatorname{sinc}(u\tau))^{mp} \xi(\tau)\, d\tau \right\}^{1/p}, \quad u \in \mathbb{R}.$$

It is worth noting that a relation of the form (4.22) for the moduli of continuity $\omega_{\mathcal{M}}$ given by relation (2.17) was earlier unknown, with the exception of two above-mentioned special cases

$$\mathcal{M} = \mathcal{M}_{1,m}, \quad m \in \mathbb{N}, \qquad \text{and} \qquad \mathcal{M} = \mathcal{M}_{2,\beta}, \quad \beta \in (0, \infty).$$

5. Some Corollaries of Theorem 2

In our opinion, the investigation of conditions under which it is possible to find the exact values of the extreme characteristic in relation (4.22) is of especial interest.

5.1. Corollary 1. *Suppose that α and σ belong to $(0, \infty)$, a complex-valued function $w \colon \mathbb{R} \to \mathbb{C}$ is such that $|w|^2$ belongs to the set $\mathbb{G}$ and satisfies properties A and B, $0 < p \le 2,\ 0 < t \le \bar{t}/\sigma$, where $\bar{t} \in (0, t_*)$ is the value of the argument of the function $|w|^2$ given by relation (4.6), and ξ is a nonnegative function summable on the segment $[0, t]$ and not equivalent to zero. Then the equality*

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{ \int_0^t (\omega^w(\mathcal{D}^\alpha f, \tau))^p \xi(\tau)\, d\tau \right\}^{1/p}} = \frac{1}{\Xi_{\sigma,p,\alpha,w}(\xi, t)} \tag{5.1}$$

is true.

Proof. To obtain relation (5.1), it is necessary to show that the equality

$$\inf\left\{ \Xi_{u,p,\alpha,w}(\xi, t) \colon \sigma \le |u| < \infty \right\} = \Xi_{\sigma,p,\alpha,w}(\xi, t) \tag{5.2}$$

holds and then apply Theorem 2. Let $0 < y \le \bar{t},\ x, z \in [1, \infty)$, and let $\nu, \mu \in (0, \infty)$ be arbitrary numbers. Since the function $|w|^2 \in \mathbb{G}$ has properties *A* and *B*, the inequality

$$x^{\nu/\mu}|w(zy)|^2 \ge |w(y)|^2$$

is true. Raising both sides of this inequality to the $(\mu/2)$th power, we obtain

$$x^{\nu/2}|w(zy)|^\mu \ge |w(y)|^\mu. \tag{5.3}$$

Since $|w|^2$ is an even function, it is clear that

$$|w(x)| = |w(|x|)|, \quad x \in \mathbb{R}.$$

Setting

$$z = x = |u|/\sigma, \quad \sigma \le |u| < \infty, \qquad \text{and} \qquad y = \sigma\tau, \quad 0 < \tau \le \bar{t}/\sigma,$$

in (5.3), we get

$$|u|^{\nu/2}|w(u\tau)|^\mu \ge \sigma^{\nu/2}|w(\sigma\tau)|^\mu.$$

For $\nu = 2\alpha p$ and $\mu = p$, we arrive at the inequality

$$|u|^{\alpha p}|w(u\tau)|^p \ge \sigma^{\alpha p}|w(\sigma\tau)|^p. \tag{5.4}$$

We multiply both sides of relation (5.4) by the function $\xi(\tau)$, integrate both sides of the obtained inequality with respect to the variable τ from 0 to t, $0 < t \le \bar{t}/\sigma$, and raise to the $(1/p)$th power. This gives

$$|u|^\alpha \left\{ \int_0^t |w(u\tau)|^p \xi(\tau)\, d\tau \right\}^{1/p} \ge \sigma^\alpha \left\{ \int_0^t |w(\sigma\tau)|^p \xi(\tau)\, d\tau \right\}^{1/p}$$

or, in view of (4.21),

$$\Xi_{u,p,\alpha,w}(\xi, t) \ge \Xi_{\sigma,p,\alpha,w}(\xi, t),$$

where $\sigma \le |u| < \infty$. Hence, equality (5.2) holds and Corollary 1 is proved.

Remark 3. Similarly, it is possible to show that the equality

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{ \int_0^t \left(\omega^w(f, \tau)\right)^p \xi(\tau)\, d\tau \right\}^{1/p}} = \frac{1}{\widetilde{\Xi}_{\sigma,p,w}(\xi, t)} \tag{5.5}$$

is true under the conditions of Corollary 1. In this case, $\widetilde{\Xi}_{\sigma,p,w}(\xi, t)$ is given in Remark 2 and the upper bound is taken over all functions f from $L_2(\mathbb{R})$ that are not equivalent to zero.

5.1.1. Setting, e.g., $w = \widetilde{w}_m,\ m \in \mathbb{N}$, and using (1.6), (2.19), and (4.21) for $0 < t \le \bar{t}/\sigma,\ \sigma \in (0,\infty)$, and $p \in (0,2]$, we obtain the following relation for $\alpha \in (0,\infty)$ from (5.1) and (5.5):

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^t \widetilde{\Omega}_m^p(\mathcal{D}^\alpha f,\tau)\xi(\tau)\,d\tau\right\}^{1/p}} = \sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\int_0^t \widetilde{\Omega}_m^p(f,\tau)\xi(\tau)\,d\tau\right\}^{1/p}}$$

$$= \frac{1}{\left\{\int_0^t (1-\operatorname{sinc}(\tau\sigma))^{mp}\xi(\tau)\,d\tau\right\}^{1/p}}. \tag{5.6}$$

For $p = 1/m,\ m \in \mathbb{N}$, and $\xi(\tau) \equiv \tau$, we arrive at the following relation from (5.6) for $\alpha \in (0,\infty)$ and $0 < t \le \bar{t}$:

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha-2m} \mathcal{A}_\sigma(f)}{\left\{\int_0^{t/\sigma} \widetilde{\Omega}_m^{1/m}(\mathcal{D}^\alpha f,\tau)\tau d\tau\right\}^{m}} = \sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\sigma^2\int_0^{t/\sigma} \widetilde{\Omega}_m^{1/m}(f,\tau)\tau d\tau\right\}^{m}}$$

$$= \frac{2^m}{t^{2m}(1-\operatorname{sinc}^2(t/2))^m}.$$

For $p = 1/m,\ m \in \mathbb{N}$, and $\xi(\tau) \equiv 1$, we derive the following relation from (5.6):

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha-m} \mathcal{A}_\sigma(f)}{\left\{\int_0^{t/\sigma} \widetilde{\Omega}_m^{1/m}(\mathcal{D}^\alpha f,\tau)\,d\tau\right\}^{m}} = \sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\left\{\sigma\int_0^{t/\sigma} \widetilde{\Omega}_m^{1/m}(f,\tau)\,d\tau\right\}^{m}} = \frac{1}{(1-\operatorname{Si}(t))^m},$$

where

$$\operatorname{Si}(x) := \int_0^x \operatorname{sinc}(t)\,dt$$

is the integral sine, $\alpha \in (0,\infty)$, and $0 < t \le \bar{t}$.

5.2. Corollary 2. *Assume that $\sigma \in (0,\infty),\ \alpha \in [1/2,\infty)$, a complex-valued function $w : \mathbb{R} \to \mathbb{C}$ is such that the function $|w|^2$ belongs to the set $\mathbb{G}$, is differentiable almost everywhere on $\mathbb{R}$, and has property A, $t \in (0, t_*/\sigma]$, and ξ is a nonnegative function measurable on the segment $[0,t]$, not equivalent to zero, and differentiable almost everywhere in the interval $(0,t)$. If, for some $p \in [1/\alpha, 2]$, the inequality*

$$(\alpha p - 1)\xi(\tau) - \tau\xi'(\tau) \ge 0 \tag{5.7}$$

holds almost everywhere on $[0,t]$, then relation (5.1) is true.

Proof. Suppose that, for some $p \in [1/\alpha, 2]$, inequality (5.7) is true for almost all τ from the segment $[0,t]$. To deduce relation (5.1) under the indicated conditions, it is necessary to show that equality (5.2) is true and then apply relation (4.22). To this end, we consider an auxiliary function $\lambda(u) := \big(\Xi_{u,p,\alpha,w}(\xi,t)\big)^p$, all parameters of which except u (including the variable t) are arbitrary but fixed. In view of (4.21), the function λ is even and nonnegative on $\mathbb{R}$. Hence, it suffices to consider its behavior on the semiaxis $\mathbb{R}_+$ and show that λ is a nondecreasing function. Since

$$\lambda(u) = u^{\alpha p}\int\limits_0^t \big|w(\tau u)\big|^p \xi(\tau)\,d\tau,$$

we conclude that

$$\lambda'(u) = \alpha p u^{\alpha p-1}\int\limits_0^t \big|w(\tau u)\big|^p \xi(\tau)\,d\tau + u^{\alpha p}\int\limits_0^t \xi(\tau)\frac{\partial}{\partial u}\big|w(\tau u)\big|^p d\tau. \tag{5.8}$$

Setting $z = \tau u$, almost everywhere on $\mathbb{R}_+\backslash\{0\}$, we obtain

$$\frac{\partial}{\partial u}|w(z)|^p = p|w(z)|^{p-1}\big(|w(z)|\big)'_z\tau \qquad \text{and} \qquad \frac{\partial}{\partial \tau}|w(z)|^p = p|w(z)|^{p-1}\big(|w(z)|\big)'_z u,$$

i.e.,

$$\frac{1}{\tau}\frac{\partial}{\partial u}\big|w(\tau u)\big|^p = \frac{1}{u}\frac{\partial}{\partial \tau}\big|w(\tau u)\big|^p. \tag{5.9}$$

In view of (5.9), equality (5.8) takes the form

$$\lambda'(u) = u^{\alpha p-1}\left\{\alpha p\int\limits_0^t \big|w(\tau u)\big|^p \xi(\tau)\,d\tau + \int\limits_0^t \tau\xi(\tau)\frac{\partial}{\partial \tau}\big|w(\tau u)\big|^p d\tau\right\}. \tag{5.10}$$

Integrating the second integral in (5.10) by parts, we obtain

$$\lambda'(u) = u^{\alpha p-1}\left\{t\xi(t)\big|w(tu)\big|^p + \int\limits_0^t \big(\alpha p\xi(\tau) - \xi(\tau) - \tau\xi'(\tau)\big)\big|w(\tau u)\big|^p d\tau\right\}. \tag{5.11}$$

By using inequality (5.7), we find $\lambda'(u) \ge 0$, where $0 < u < \infty$, from (5.11), i.e., λ is a nondecreasing function on the analyzed set.

Corollary 2 is proved.

5.2.1. Setting $w = w_{\mathcal{M}_{2,\beta}}$, $\beta \in (0,\infty)$, and using (2.23), (2.17), and (1.4), we get

$$\omega^{w_{\mathcal{M}_{2,\beta}}}(f,t) = \omega_{\mathcal{M}_{2,\beta}}(f,t) = \omega_\beta(f,t),$$

where $f \in L_2(\mathbb{R})$, $t \ge 0$. Note that, in this case, $t_* = \pi$. Thus, by virtue of Corollary 2 and relations (2.14), (4.21), and (5.1), for $0 < t \le \pi/\sigma$, we find

$$\sup_{f\in L_2^\alpha(\mathbb{R})}\frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^t \omega_\beta^p(\mathcal{D}^\alpha f,\tau)\xi(\tau)\,d\tau\right\}^{1/p}} = \frac{1}{2^{\beta/2}\left\{\int_0^t (1-\cos(\sigma\tau))^{p\beta/2}\xi(\tau)\,d\tau\right\}^{1/p}}. \tag{5.12}$$

If, e.g., $\beta = 2/p,\ p \in [1/\alpha, 2],\ \alpha \in [1/2, \infty)$, and $\xi(\tau) \equiv 1$ in (5.12), then we get

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha-1/p}\mathcal{A}_\sigma(f)}{\left\{\int_0^{t/\sigma} \omega_{2/p}^p(\mathcal{D}^\alpha f, \tau)\, d\tau\right\}^{1/p}} = \frac{1}{\left\{2t(1-\operatorname{sinc}(t)\right\}^{1/p}}, \tag{5.13}$$

where $0 < t \leq \pi$. For $\alpha = r \in \mathbb{N}$ and $p = 2/m$, where $m \in \mathbb{N}$ and $1 \leq m \leq 2r$, relation (5.13) yields one of the results of Corollary 2 in [26]:

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^{r-1/p}\mathcal{A}_\sigma(f)}{\left\{\int_0^{t/\sigma} \omega_m^{2/m}(f^{(r)}, \tau)\, d\tau\right\}^{m/2}} = \frac{1}{\left\{2t(1-\operatorname{sinc}(t)\right\}^{m/2}}.$$

Here, $0 < t \leq \pi$ and ω_m is the ordinary modulus of continuity of order m.

5.2.2. Let $\mathcal{M} = \mathcal{M}_4$ and let $w = w_{\mathcal{M}_4}$. Thus, according to (2.7), (2.17), (2.23), and (1.10), for any element $f \in L_2(\mathbb{R})$, we obtain

$$\omega^{w_{\mathcal{M}_4}}(f, t) = \omega_{\mathcal{M}_4}(f, t) = \widehat{\omega}(f, t),$$

where $\widehat{\omega}$ is the modulus of continuity introduced by Runovski and Schmeisser [37]. In this case, by using relations (2.16) and (4.21), we derive the following relation from Corollary 2:

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha+1}\mathcal{A}_\sigma(f)}{\left\{\int_0^t \widehat{\omega}^p(\mathcal{D}^\alpha f, \tau)\xi(\tau)\, d\tau\right\}^{1/p}} = \frac{\pi}{2\left\{\int_0^t \tau^p\, \xi(\tau)\, d\tau\right\}^{1/p}}, \tag{5.14}$$

where $0 < t \leq \pi/\sigma$. We set $\xi(\tau) := \tau^m,\ m \in [0, \infty)$. Then inequality (5.7) takes the form $(m+1)/\alpha \leq p \leq 2$, where $(m+1)/2 \leq \alpha < \infty$. Hence, in view of relation (5.14) with $0 < t \leq \pi/\sigma$, we get

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha+1}\mathcal{A}_\sigma(f)}{\left\{\int_0^t \widehat{\omega}^p(\mathcal{D}^\alpha f, \tau)\tau^m\, d\tau\right\}^{1/p}} = \frac{\pi(p+m+1)^{1/p}}{2t^{1+(m+1)/p}}.$$

Further, consider a function $\xi(\tau) := \sin(\tau)$. Since the inequality $\operatorname{sinc}(\tau) > \cos(\tau)$ is true for $0 < \tau \leq \pi$, we get

$$(\alpha p - 1)\sin(\tau) - \tau\cos(\tau) = \frac{1}{\tau}\left\{(\alpha p - 1)\operatorname{sinc}(\tau) - \cos(\tau)\right\} \geq \frac{1}{\tau}(\alpha p - 2)\operatorname{sinc}(\tau)$$

and the right-hand side of this relation is nonnegative for any $\tau \in (0, \pi]$ for $p \geq 2/\alpha$. Hence, for Corollary 2, condition (5.7) is satisfied in this specific case for $2/\alpha \leq p \leq 2$ and $\alpha \in [1, \infty)$. Thus, it follows from (5.14) with $0 < t \leq \pi/\sigma$ that

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha+1}\mathcal{A}_\sigma(f)}{\left\{\int_0^t \widehat{\omega}^p(\mathcal{D}^\alpha f, \tau)\sin(\tau)\, d\tau\right\}^{1/p}} = \frac{\pi}{2\left\{\int_0^t \tau^p\ \sin(\tau)\, d\tau\right\}^{1/p}}. \tag{5.15}$$

Setting, e.g., $p = 1$ in (5.15), for $\alpha \in [2, \infty)$ and $0 < t \le \pi/\sigma$, we find

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha+1} \mathcal{A}_\sigma(f)}{\displaystyle\int_0^t \widehat{\omega}(\mathcal{D}^\alpha f, \tau) \sin(\tau)\, d\tau} = \frac{\pi}{2t\big(\operatorname{sinc}(t) - \cos(t)\big)}.$$

5.2.3. Now let $\mathcal{M} = \mathcal{M}_3$ and $w = w_{\mathcal{M}_3}$. In view of (2.7), (2.23), and (1.9), for any function $f \in L_2(\mathbb{R})$, we obtain

$$\omega^{w_{\mathcal{M}_3}}(f, t) = \omega_{\mathcal{M}_3}(f, t) = \omega_{\langle\prime\rangle}(f, t), \quad t \ge 0.$$

In this case, by using relations (4.21) and (2.15) and Corollary 2, we get the following relation for $t \in (0, \pi/\sigma]$:

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha+1} \mathcal{A}_\sigma(f)}{\left\{\displaystyle\int_0^t \omega_{\langle\prime\rangle}^p(\mathcal{D}^\alpha f, \tau)\, \xi(\tau)\, d\tau\right\}^{1/p}} = \frac{\pi}{3\left\{\displaystyle\int_0^t \tau^p (1 - \sigma\tau/(2\pi))^p\, \xi(\tau)\, d\tau\right\}^{1/p}}. \tag{5.16}$$

As already mentioned, if $\xi(\tau) := \tau^m,\ m \in [0, \infty)$, then condition (5.7) is satisfied for $p \in [(m+1)/\alpha; 2]$ and $\alpha \in [(m+1)/2; \infty)$ and the indicated power function can be used in relation (5.16). For $p = 1$, by virtue of (5.7) with $\alpha \in [m+1; \infty)$, equality (5.16) takes the following form for $0 < t \le \pi$:

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^{\alpha-m-1} \mathcal{A}_\sigma(f)}{\displaystyle\int_0^{t/\sigma} \omega_{\langle\prime\rangle}(\mathcal{D}^\alpha f, \tau) \tau^m d\tau} = \frac{\pi}{3t^{m+2}} \left\{\frac{1}{m+2} - \frac{t}{2\pi(m+3)}\right\}^{-1}.$$

5.2.4. We now consider one more case where $w = \widetilde{w}_m,\ m \in \mathbb{N}$. For any function $f \in L_2(\mathbb{R})$, we get $\omega^{\widetilde{w}_m}(f, t) = \widetilde{\Omega}_m(f, t),\ t > 0$. By using relation (4.21) and Corollary 2, for $0 < t \le t_*/\sigma$, we arrive at equality (5.6), where $\sigma \in (0, \infty),\ \alpha \in [1/2, \infty)$, and $p \in [1/\alpha, 2]$ is a number for which inequality (5.7) is true for almost all $\tau \in [0, t]$. Recall that, in this case, t_* is the least positive root of the equation $\tan(x) = x$, $4.49 < t_* < 4.51$ [25].

5.3. We set $\xi := \widetilde{\xi}$, where $\widetilde{\xi}(\tau) = \eta(\sigma\tau),\ \sigma \in (0, \infty),\ \tau \in (0, y/\sigma]$, and $y \in (0, t_*]$. Denoting $t = y/\sigma$, we rewrite relation (4.21) in the form

$$\Xi_{u,p,\alpha,w}\left(\widetilde{\xi}, \frac{y}{\sigma}\right) = |u|^\alpha \left\{\int_0^{y/\sigma} \big|w(\tau u)\big|^p \eta(\sigma\tau)\, d\tau\right\}^{1/p}$$

$$= \sigma^{\alpha - 1/p} \left\{\frac{|u|^{\alpha p}}{\sigma} \int_0^y \left|w\left(\frac{|u|}{\sigma}\tau\right)\right|^p \eta(\tau)\, d\tau\right\}^{1/p}, \quad \sigma \le |u| < \infty. \tag{5.17}$$

Let $z = |u|/\sigma$, i.e., $1 \le z < \infty$. By using (5.17), we obtain

$$\inf_{\sigma \le |u| < \infty} \Xi_{u,p,\alpha,w}\left(\widetilde{\xi}, \frac{y}{\sigma}\right) \ge \sigma^{\alpha - 1/p} \inf_{1 \le z < \infty} \left\{z^{\alpha p} \int_0^y |w(z\tau)|^p \eta(\tau)\, d\tau\right\}^{1/p}. \tag{5.18}$$

Denote

$$\overline{\Xi}_{p,\alpha,w}(\eta; y, z) := z^{\alpha p} \int\limits_0^y \big|w(z\tau)\big|^p \eta(\tau)\, d\tau. \tag{5.19}$$

By virtue of Theorem 2 and relations (5.17)–(5.19), we arrive at the following statement:

Corollary 3. *Suppose that $\alpha, \sigma \in (0, \infty)$, $0 < p \le 2$, a complex-valued function $w\colon \mathbb{R} \to \mathbb{C}$ is such that $|w|^2 \in \mathbb{G}$ and $|w|^2$ has property A, $y \in [0, t_*]$, where the number t_* is given by relation (4.1) for the function $\varphi := |w|^2$, and η is a measurable function summable on the segment $[0, y]$, which is nonnegative and not equivalent to zero. Then the two-sided inequality*

$$\frac{1}{\big\{\overline{\Xi}_{p,\alpha,w}(\eta; y, 1)\big\}^{1/p}} \le \sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y (\omega^w(\mathcal{D}^\alpha f, \tau/\sigma))^p \eta(\tau)\, d\tau\right\}^{1/p}}$$

$$\le \frac{1}{\left\{\inf\limits_{1 \le z < \infty} \overline{\Xi}_{p,\alpha,w}(\eta; y, z)\right\}^{1/p}} \tag{5.20}$$

is true. If the function η is such that

$$\inf_{1 \le z < \infty} \overline{\Xi}_{p,\alpha,w}(\eta; y, z) = \overline{\Xi}_{p,\alpha,w}(\eta; y, 1), \tag{5.21}$$

then the equality

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y \big(\omega^w(\mathcal{D}^\alpha f, \tau/\sigma)\big)^p \eta(\tau)\, d\tau\right\}^{1/p}} = \frac{1}{\big\{\overline{\Xi}_{p,\alpha,w}(\eta; y, 1)\big\}^{1/p}} \tag{5.22}$$

is true.

Note that, in a special case where

$$\mathcal{M} = \mathcal{M}_{2,\beta}, \quad \beta \in (0, \infty), \qquad \text{and} \qquad w = w_{\mathcal{M}_{2,\beta}},$$

for the characteristic of smoothness of a function $f \in L_2(\mathbb{R})$ of the form

$$\omega^{w_{\mathcal{M}_{2,\beta}}}(f, t) = \omega_{\mathcal{M}_{2,\beta}}(f, t) = \omega_\beta(f, t), \quad t \ge 0,$$

this corollary was obtained in [32].

5.4. The next assertion establishes the conditions for the function η under which equality (5.21) is true:

Corollary 4. *Suppose that $\alpha, \sigma \in (0, \infty)$, $0 < p \le 2$, $y \in [0, t_*]$, a complex-valued function $w\colon \mathbb{R} \to \mathbb{C}$ is such that $|w|^2 \in \mathbb{G}$ and $|w|^2$ has property A, and*

$$\widehat{\eta}(\tau) := \tau^{\alpha p - 1} \widetilde{\eta}(\tau),$$

where $\widetilde{\eta}$ is a measurable nonincreasing function summable on the set $(0, y]$, which is nonnegative and not equivalent to zero. Then, for $\eta = \widehat{\eta}$, equality (5.21) is true and the following relation holds:

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y \big(\omega^w(\mathcal{D}^\alpha f, \tau/\sigma)\big)^p \tau^{\alpha p-1}\widetilde{\eta}(\tau)\,d\tau\right\}^{1/p}} = \frac{1}{\{\Xi_{p,\alpha,w}(\widehat{\eta}; y, 1)\}^{1/p}}. \tag{5.23}$$

.

Proof. Let y be an arbitrary fixed number from the set $(0, t_*]$. We extend the definition of the function $\widetilde{\eta}$ as follows: $\widetilde{\eta}_y(\tau) := \{\widetilde{\eta}(\tau)$ for $0 < \tau \le y$; $\widetilde{\eta}(y)$ for $y \le \tau < \infty\}$. Since $\widetilde{\eta}$ is a nonincreasing and nonnegative function on the set $(0, y]$, for any value $z \in [1, \infty)$ and $0 < \tau < zy$, we find $\widetilde{\eta}(\tau/z) \ge \widetilde{\eta}_y(\tau)$. By using relation (5.19) with $\eta = \widehat{\eta}$, we get

$$\overline{\Xi}_{p,\alpha,w}(\widehat{\eta}; y, z) = z^{\alpha p}\int_0^y |w(z\tau)|^p\tau^{\alpha p-1}\widetilde{\eta}(\tau)\,d\tau = \int_0^{zy} |w(\tau)|^p\tau^{\alpha p-1}\widetilde{\eta}(\tau/z)\,d\tau$$

$$\ge \int_0^{zy} |w(\tau)|^p\tau^{\alpha p-1}\widetilde{\eta}_y(\tau)\,d\tau \ge \int_0^{y} |w(\tau)|^p\tau^{\alpha p-1}\widetilde{\eta}(\tau)\,d\tau = \overline{\Xi}_{p,\alpha,w}(\widehat{\eta}; y, 1),$$

where $1 \le z < \infty$. Hence, equality (5.21) is true for $\eta = \widehat{\eta}$, and, therefore, relation (5.22) is also true. In the analyzed case, this relation takes the form (5.23).

Corollary 4 is proved.

In special cases of Corollary 4 with

$$\mathcal{M} = \mathcal{M}_{2,\beta}, \quad \beta \in (0,\infty), \qquad w = w_{\mathcal{M}_{2,\beta}}, \qquad \omega^{w_{\mathcal{M}_{2,\beta}}}(f,t) = \omega_{\mathcal{M}_{2,\beta}}(f,t) = \omega_\beta(f,t), \quad t \ge 0,$$

and

$$w = \widetilde{w}_m, \quad m \in \mathbb{N}, \qquad \omega^{\widetilde{w}_m}(f,t) = \widetilde{\Omega}_m(f,t), \quad t \ge 0,$$

we obtain the results presented in [32] and [25], respectively.

5.4.1. Let $\mathcal{M} = \mathcal{M}_4$ and let $w = w_{\mathcal{M}_4}$. Then, for any function $f \in L_2(\mathbb{R})$, we get the characteristic of smoothness

$$\omega^{w_{\mathcal{M}_4}}(f,t) = \omega_{\mathcal{M}_4}(f,t) = \widehat{\omega}(f,t), \quad t \ge 0,$$

considered in [37]. We set $\widetilde{\eta}(\tau) := \tau^{-\gamma}$, $0 < \tau \le y$, and assume that $\gamma \in (0,1)$, $\alpha \in (\gamma/2, \infty)$, and $p \in (\gamma/\alpha, 2]$. By using relations (2.16), (5.20), and (5.23), we conclude that

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y \widehat{\omega}^p(\mathcal{D}^\alpha f, \tau/\sigma)\tau^{\alpha p-1-\gamma}\,d\tau\right\}^{1/p}} = \frac{\pi(p(1+\alpha)-\gamma)^{1/p}}{2y^{1+\alpha-\gamma/p}}, \quad 0 < y \le \pi.$$

5.4.2. Now let $\mathcal{M} = \mathcal{M}_3$ and $w = w_{\mathcal{M}_3}$. In this case, for $f \in L_2(\mathbb{R})$, we get the characteristic of smoothness

$$\omega^{w_{\mathcal{M}_3}}(f,t) = \omega_{\mathcal{M}_3}(f,t) = \omega_{\langle\prime\rangle}(f,t), \quad t \geq 0,$$

studied in [31]. By using the function $\widetilde{\eta}(\tau)$, under the above-mentioned restrictions imposed on γ, α, and p, for $0 < y \leq \pi$, by virtue of (2.15), (5.19), and (5.23), we obtain

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y \omega_{\langle\prime\rangle}^p(\mathcal{D}^\alpha f, \tau/\sigma)\tau^{\alpha p - 1 - \gamma}\, d\tau\right\}^{1/p}}$$

$$= \frac{\pi}{3}\left\{\int_0^y \left(1 - \frac{\tau}{2\pi}\right)^p \tau^{(1+\alpha)p-1-\gamma}\, d\tau\right\}^{-1/p}. \tag{5.24}$$

For $p = 2$ and $0 < y \leq \pi$, relation (5.24) implies that

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\left\{\int_0^y \omega_{\langle\prime\rangle}^2(\mathcal{D}^\alpha f, \tau/\sigma)\tau^{2\alpha - 1 - \gamma}\, d\tau\right\}^{1/2}}$$

$$= \frac{\pi}{3} y^{\gamma - 2(\alpha+1)} \left\{\frac{1}{2(\alpha+1) - \gamma} - \frac{y}{\pi(2(\alpha+1) - \gamma + 1)} + \frac{y^2}{4\pi^2(2(\alpha+1) - \gamma + 2)}\right\}^{-1/2}.$$

Now let $\alpha \in (\gamma, \infty)$, where $\gamma \in (0, 1)$. If we set $p = 1$, then, it follows from (5.24) that

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{\int_0^y \omega_{\langle\prime\rangle}(\mathcal{D}^\alpha f, \tau/\sigma)\tau^{\alpha - 1 - \gamma}\, d\tau} = \frac{\pi}{3} y^{\gamma - \alpha - 1}\left\{\frac{1}{\alpha - \gamma + 1} - \frac{y}{2\pi(\alpha - \gamma + 2)}\right\}^{-1}$$

for $0 < y \leq \pi$.

In the second part of the present paper, we plan to consider the extreme problems for the characteristic of smoothness (2.25) and compute the exact values of mean ν-widths for the classes of functions defined with the help of ω^w and Λ^w.

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