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UPPER AND LOWER LEBESGUE CLASSES OF MULTIVALUED FUNCTIONS OF TWO VARIABLES

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We introduce a functional Lebesgue classification of multivalued mappings and obtain results on the upper and lower Lebesgue classifications of multivalued mappings $F\colon X\times Y \multimap Z$ for broad classes of spaces X, Y and Z.

1. Introduction

The investigations of the Lebesgue classification of separately continuous single-valued functions (i.e., functions of several variables continuous in each variable) and their analogs were started by Lebesgue [9] and Kuratowski [6]. Later, these investigations were continued by numerous mathematicians (see, e.g., [1, 2, 4, 7, 10, 12] and the references therein).

Some analogs of the Lebesgue classification are also known for multivalued mappings and connected with their upper and lower semicontinuity. Namely, a multivalued mapping $F\colon X \multimap [0,1]$ defined on the topological space X is called *upper (lower) semicontinuous at a point* $x_0 \in X$ if, for any open set $U \subseteq [0,1]$ with the property $F(x_0)\subseteq U$ $(F(x_0)\cap U \neq \varnothing)$, the set

$$F^+(U) = \{x\in X\colon F(x)\subseteq U\}$$

$$\big(F^-(U) = \{x\in X\colon F(x)\cap U \neq \varnothing\}\big)$$

is a neighborhood of the point x_0 in X. A multivalued mapping $F\colon X \multimap [0,1]$ is *continuous at a point* $x_0 \in X$ if it is simultaneously upper and lower semicontinuous at this point. It is known that the multivalued mapping $F\colon X \multimap [0,1]$ is continuous at a point $x_0\in X$ if and only if it is continuous at the point x_0 as a single-valued mapping with values in the space of all nonempty subsets of the segment $[0,1]$ with the Vietoris topology.

For topological spaces X and Y, by $\mathrm{U}(X,Y)$ and $(\mathrm{L}(X,Y))$ we denote the collections of all upper (lower) semicontinuous multivalued mappings $F\colon X \multimap Y$.

Let X and Y be topological spaces and let $\alpha < \omega_1$. A multivalued mapping $F\colon X \multimap Y$ belongs to

the *upper Lebesgue class* α if, for any open set $A\subseteq Y$, the set $F^+(A)$ belongs to the additive class α in X;

the *lower Lebesgue class* α if, for any open set $A\subseteq Y$, the set $F^-(A)$ belongs to the additive class α in X.

Note that the Lebesgue classes are also called Borel classes.

For the topological spaces X and Y, we denote the collection of all multivalued mappings $F\colon X \multimap Y$ of the upper (lower) Lebesgue class α by $\mathrm{U}_\alpha(X,Y)$ $\big(\mathrm{L}_\alpha(X,Y)\big)$.

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For the multivalued mapping $F\colon X \times Y \multimap Z$ of points $x \in X$ and $y \in Y,$ we denote

$$F^x(y) = F_y(x) = F(x, y).$$

Recall that a topological space is called *perfect* if every its closed set is a set of the type G_δ.

In [8], Kwiecińska obtained the following result for the Lebesgue classification of multivalued mappings of two variables:

Theorem 1.1 [8]. *Suppose that (X, d) is a metric space, $\mathcal{T}$ is a topology in the space $X,$ $D \subseteq X$ is an at most countable set, $(U(x)\colon x \in X)$ is a family of $\mathcal{T}$-open sets $U(x) \subseteq X,$ Y and Z are perfectly normal spaces, $\alpha < \omega_1,$ and $F\colon X \times Y \multimap Z$ is a compact-valued (multivalued) mapping with the following properties:*

(a) the set D is dense in $(X, \mathcal{T})$;

(b) for any $x \in D,$ the set $A(x) = \{u \in X\colon x \in U(u)\}$ belongs to the additive class α in (X, d);

(c) for any $x \in X,$ the sequence $(B_n(x))_{n\in\omega}$ of the sets

$$B_n(x) = U(x) \cap \left\{u \in X\colon d(x, u) < \frac{1}{n}\right\}$$

forms a base of the space $(X, \mathcal{T})$ at the point x;

(d) for any $y \in Y,$ the multivalued mapping $F_y\colon (X, \mathcal{T}) \multimap Z$ is continuous;

(e) for any $x \in D,$ the multivalued mapping $F^x\colon Y \multimap Z$ belongs to the lower (upper) class α.

Then F is a mapping of the lower (upper) Lebesgue class $\alpha + 1$ on the product $(X, d) \times Y.$

The other versions of Lebesgue classification of multivalued mappings of two variables were obtained in [5]. Theorem 1.1 yields the following result for a perfectly normal space Y:

Theorem 1.2 [5]. *Suppose that X is a metrizable space, D is a dense subset of the space $X,$ Y is a perfect space, Z is a perfectly normal space, $\alpha < \omega_1,$ and $F\colon X \times Y \multimap Z$ is a compact-valued (multivalued) mapping with the following properties:*

(a) for any $y \in Y,$ the multivalued mapping $F_y\colon X \multimap Z$ is continuous;

(b) for any $x \in D,$ the multivalued mapping $F^x\colon Y \multimap Z$ belongs to the lower (upper) Lebesgue class α.

Then F belongs to the upper (lower) Lebesgue class $\alpha + 1$ on the product $X \times Y.$

In the present paper, we introduce functional Lebesgue classes for multivalued mappings and generalize Theorems 1.1 and 1.2 to a broad class of topological spaces X.

2. Multivalued Mappings from the Upper and Lower Functional Lebesgue Classes α

Definition 2.1. *Let X and Y be topological spaces. The multivalued mapping $F\colon X \multimap Y$ is called*

functionally upper (lower) semicontinuous if the set $F^+(A)$ $(F^-(A))$ is functionally open in X for any functionally open set $A \subseteq Y$;

strongly functionally upper (lower) semicontinuous if the set $F^+(A)$ $(F^-(A))$ *is functionally open in* X *for any functionally open set* $A \subseteq Y$;

weakly functionally upper (lower) semicontinuous if the set $F^+(A)$ $(F^-(A))$ *is open in* X *for any functionally open set* $A \subseteq Y$.

By $\mathrm{U}^f(X)$ $\big(\mathrm{L}^f(X),\ \mathrm{U}^f_s(X),\ \mathrm{L}^f_s(X),\ \mathrm{U}^f_w(X),$ and $\mathrm{L}^f_w(X)\big)$ we denote the collection of all functionally upper (lower, strongly upper, strongly lower, weakly upper, and weakly lower) semicontinuous multivalued mappings $F\colon X \multimap Y$.

Let X be a topological space and let $\mathcal{A}_0(X)$ and $\mathcal{M}_0(X)$ be the systems of all functionally open and functionally closed sets in X, respectively. For any ordinal $\alpha \in [1,\omega_1)$, by $\mathcal{A}_\alpha(X)$ we denote the system of all unions $\bigcup_{n\in\omega} A_n$ of the sets A_n from $\bigcup_{\xi<\alpha}\mathcal{M}_\alpha(X)$ and by $\mathcal{M}_\alpha(X)$ we denote the system of all intersections $\bigcap_{n\in\omega} M_n$ of the sets M_n from $\bigcup_{\xi<\alpha}\mathcal{A}_\alpha(X)$. It is clear that

$$\mathcal{A}_\alpha(X) = \big\{X\setminus M\colon M \in \mathcal{M}_\alpha(X)\big\}.$$

Definition 2.2. *Let* X *and* Y *be topological spaces and let* $\alpha \in [0,\omega_1)$. *The multivalued mapping* $F\colon X \multimap Y$ *belongs to*

the upper functional Lebesgue class α *if* $F^+(A) \in \mathcal{A}_\alpha(X)$ *for any functionally open set* $A \subseteq Y$;

the lower functional Lebesgue class α *if* $F^-(A) \in \mathcal{A}_\alpha(X)$ *for any functionally open set* $A \subseteq Y$.

It is easy to see that the multivalued mapping F belongs to the upper (lower) functional Lebesgue class α if and only if $F^-(B) \in \mathcal{A}_\alpha(X)$ $\big(F^+(B) \in \mathcal{A}_\alpha(X)\big)$ for any functionally closed set $B \subseteq Y$.

For the topological spaces X and Y, by $\mathrm{U}^f_\alpha(X)$ $\big(\mathrm{L}^f_\alpha(X)\big)$ we denote the collection of all multivalued mappings $F\colon X \multimap Y$ of the upper (lower) functional Lebesgue class α. Note that

$$\mathrm{U}^f_0(X) = \mathrm{U}^f(X) \qquad \text{and} \qquad \mathrm{L}^f_0(X) = \mathrm{L}^f(X).$$

The properties presented in what follows readily follow from the definitions. For this reason, we do not present their proofs.

Proposition 2.1. *Let* X *and* Y *be topological spaces, let* $F\colon X \multimap Y$ *be a multivalued mapping, and let* $\alpha \in [0,\omega_1)$. *Then:*

(1) $\mathrm{U}(X,Y)\cup \mathrm{U}^f(X,Y) \subseteq \mathrm{U}^f_w(X,Y)$ *and* $\mathrm{L}(X,Y)\cup \mathrm{L}^f(X,Y) \subseteq \mathrm{L}^f_w(X,Y)$;

(2) $\mathrm{U}^f_s(X,Y) \subseteq \mathrm{U}(X,Y)\cap \mathrm{U}^f(X,Y)$ *and* $\mathrm{L}^f_s(X,Y) \subseteq \mathrm{L}(X,Y)\cap \mathrm{L}^f(X,Y)$;

(3) if the space X *is perfectly normal, then*

$$\mathrm{U}^f_s(X,Y) = \mathrm{U}(X,Y) \subseteq \mathrm{U}^f(X,Y) = \mathrm{U}^f_w(X,Y)$$

and

$$\mathrm{L}^f_s(X,Y) = \mathrm{L}(X,Y) \subseteq \mathrm{L}^f(X,Y) = \mathrm{L}^f_w(X,Y);$$

(4) if the space Y *is completely regular, then* $\mathrm{U}^f(X,Y) \subseteq \mathrm{U}(X,Y)$ *and* $\mathrm{L}^f(X,Y) \subseteq \mathrm{L}(X,Y)$;

(5) if the space Y is perfectly normal, then

$$\mathrm{U}_s^f(X,Y) = \mathrm{U}^f(X,Y) \subseteq \mathrm{U}(X,Y) = \mathrm{U}_w^f(X,Y)$$

and

$$\mathrm{L}_s^f(X,Y) = \mathrm{L}^f(X,Y) \subseteq \mathrm{L}(X,Y) = \mathrm{L}_w^f(X,Y);$$

(6) $\mathrm{U}_\alpha^f(X,Y) \subseteq \mathrm{L}_{\alpha+1}^f(X,Y)$*;*

(7) if the mapping F is compact-valued, then $\mathrm{L}_\alpha^f(X,Y) \subseteq \mathrm{U}_{\alpha+1}^f(X,Y)$.

Proposition 2.2. *Let Y be a topological space such that $\{\varnothing, Y\}$ is the collection of all functionally open sets in Y {see, e.g., [3] (2.7.18)}. Then:*

(1) for any topological space X, every multivalued mapping $F\colon X \multimap Y$ is functionally upper and lower semicontinuous;

(2) for any T_1-space Z, every strongly functionally upper semicontinuous mapping $F\colon Y \multimap Z$ is constant;

(3) for any (completely) regular space Z, every strongly functionally upper (lower) closed-valued mapping $F\colon Y \multimap Z$ is constant.

Proof. 1. Since $F^+(\varnothing) = F^-(\varnothing) = \varnothing$ and $F^+(Y) = F^-(Y) = X$, the mapping $F\colon X \multimap Y$ is functionally upper and lower semicontinuous.

2. Let Z be a T_1-space and let $F\colon Y \multimap Z$ be a nonconstant mapping. We choose points $y_1, y_2 \in Y$ such that $F(y_1) \not\subseteq F(y_2)$. Since Y is a T_1-space, there exists an open set $G \subseteq Z$ such that $F(y_1) \not\subseteq G \supseteq F(y_2)$. Then $y_1 \notin F^+(G) \ni y_2$. Therefore,

$$F^+(G) \notin \{\varnothing, Y\}$$

and $F^+(G)$ is not a functionally open set. Hence, the mapping F is not strongly functionally upper semicontinuous.

3. Let Z be a regular space and let $F\colon Y \multimap Z$ be a nonconstant mapping. We choose points $y_1, y_2 \in Y$ such that $F(y_1) \not\subseteq F(y_2)$. Since the space Y is regular and the set $F(y_2)$ is closed, there exists an open set $G \subseteq Z$ such that $G \cap F(y_1) \neq \varnothing$ and $G \cap F(y_2) = \varnothing$. Then $y_1 \in F^-(G) \not\ni y_2$. Hence,

$$F^-(G) \notin \{\varnothing, Y\}$$

and the set $F^-(G)$ is not functionally open. Thus, F is not strongly functionally lower semicontinuous. If the space Z is completely regular, then we can choose a functionally open set G and show that the mapping F is not functionally lower semicontinuous.

Example 2.1. Let $A \subseteq [0,1]$ be a set, which is not Borel measurable. A multivalued mapping $F\colon [0,1] \multimap [0,1]$ defined by the rule

$$F(x) = \begin{cases} [0,1], & x \in A, \\ [0,1), & x \in [0,1] \setminus A, \end{cases}$$

is (functionally) lower semicontinuous but not (functionally) measurable, i.e.,

$$F \notin \bigcup_{\alpha<\omega_1} \mathrm{U}_\alpha^f([0,1],[0,1]).$$

3. Functional Lebesgue Classification of Multivalued Mappings of Two Variables

Lemma 3.1 ([4], Proposition 1.4). *Suppose that X is a topological space, $\alpha \in [0,\omega_1)$, $(U_i\colon i \in I)$ is a locally finite family of functionally open sets in X, and $(A_i\colon i \in I)$ is a family of sets $A_i \in \mathcal{A}_\alpha(X)$ $(A_i \in \mathcal{M}_\alpha(X))$ such that $A_i \subseteq U_i$ for each $i \in I$. Then $\bigcup_{i\in I} A_i \in \mathcal{A}_\alpha(X)$ $\big(\bigcup_{i\in I} A_i \in \mathcal{M}_\alpha(X)\big)$.*

It is worth noting that the union of a locally finite family of sets of a functionally multiplicative class α is not necessarily a set of the same class even for $\alpha = 0$.

Indeed, consider a Nemyts'kyi plane $X = \mathbb{R} \times [0,+\infty)$ in which the base of neighborhoods of points $(x,y) \in X$ for $y > 0$ is formed by open balls centered at the point (x,y) and the base of neighborhoods of points of the form $(x,0)$ is formed by the sets $U \cup \{(x,0)\}$, where U is an open ball tangential to the straight line $\mathbb{R} \times \{0\}$ at the point $(x,0)$.

Note that, for any $p \in X$, the one-point set $\{p\}$ is functionally closed in X because every continuous function on $\mathbb{R} \times [0,+\infty)$ is continuous on X. Then the family $\mathcal{F} = (\{(x,0)\}\colon x \in \mathbb{Q})$ consists of functionally closed subsets of the space X. We assume that the union $F = \bigcup \mathcal{F}$ is functionally closed in X and choose a continuous function $f\colon X \to [0,1]$ such that $F = f^{-1}(0)$. For all $(x,y) \in X$ and $n \in \mathbb{N}$, we set

$$f_n(x,y) = \begin{cases} f(x,y), & y \ge \dfrac{1}{n}, \\ f\left(x, \dfrac{1}{n}\right), & 0 \le y < \dfrac{1}{n}. \end{cases}$$

Then the function $f_n\colon \mathbb{R} \times [0,+\infty) \to [0,1]$ is continuous and $\lim_{n\to\infty} f_n(x,y) = f(x,y)$ for any $(x,y) \in X$. Since

$$F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}\left(\left[0, \frac{1}{k}\right)\right),$$

we conclude that F is a set of the type G_δ in $\mathbb{R} \times [0,+\infty)$. A contradiction.

Definition 3.1. *A family $(A_i\colon i \in I)$ of subsets A_i of the topological space X is called functionally locally finite in X if there exists a family $(U_i\colon i \in I)$ locally finite in X of functionally open sets in X such that $U_i \supseteq A_i$ for each $i \in I$. A family $(A_i\colon i \in I)$ of subsets A_i of the topological space X is called σ-functionally locally finite if there exists a partition*

$$I = \bigsqcup_{n\in\omega} I_n$$

such that each family $(A_i\colon i \in I_n)$ is functionally locally finite in X.

Theorem 3.1. *Suppose that X, Y, and Z are topological spaces, $\alpha \in [0,\omega_1)$, $(\mathcal{A}_n)_{n=1}^{\infty}$ is a sequence of σ-functionally locally finite coverings $\mathcal{A}_n = (A_{i,n}\colon i \in I_n)$ of the space X by the sets $A_{i,n} \in \mathcal{A}_\alpha(X)$, $\big((x_{i,n}\colon i \in I_n)\big)_{n=1}^{\infty}$ is a sequence of families of points $x_{i,n} \in X$, and $F\colon X \times Y \multimap Z$ is a compact-valued (multivalued) mapping satisfying the following conditions:*

1. *for all $(x,y) \in X \times Y$ and an arbitrary sequence $(i_n)_{n\in\omega}$ of indices $i_n \in I_n$ such that $x \in A_{i_n,n}$, the sequence $(F(x_{i_n,n}, y))_{n\in\omega}$ converges to $F(x,y)$ in the Vietoris topology;*
2. *$F^x \in \mathrm{L}^f_\alpha(Y,Z)$ $\big(F^x \in \mathrm{U}^f_\alpha(Y,Z)\big)$ for every x from the set $D = \{x_{i,n}\colon n \in \mathbb{N},\ i \in I_n\}$.*

Then $F \in \mathrm{U}^f_{\alpha+1}(X \times Y, Z)$ $\big(F \in \mathrm{L}^f_{\alpha+1}(X \times Y, Z)\big)$.

Proof. Consider the case where F is a compact-valued mapping. For all $n\in\omega$ and $i\in I_n$, we set $F_{i,n}=F^{x_{i,n}}$. Let $W\subseteq Z$ be a functionally closed set and let $\varphi\colon Z\to[0,1]$ be a continuous function such that $W=\varphi^{-1}(0)$. For each $n\in\omega$, we denote

$$W_n=\varphi^{-1}\left(\left[0,\frac{1}{n}\right]\right)\qquad\text{and}\qquad G_n=\varphi^{-1}\left(\left[0,\frac{1}{n}\right)\right).$$

For all $m,n\in\omega$, we set

$$C_{m,n}=\bigcup_{i\in I_n}\Big(A_{i,n}\times F^-_{i,n}(G_m)\Big)\qquad\text{and}\qquad C=\bigcap_{m\in\omega}\bigcup_{n\geq m}C_{n,m}.$$

Since $A_{i,n}\in\mathcal{A}_\alpha(X)$ and $F^-_{i,n}(G_m)\in\mathcal{A}_\alpha(Y)$ by condition (2), we find

$$A_{i,n}\times F^-_{i,n}(G_m)\in\mathcal{A}_\alpha(X\times Y)$$

for all $m,n\in\omega$ and $i\in I_n$. Thus, by Lemma 3.1, $C_{m,n}\in\mathcal{A}_\alpha(X\times Y)$. Hence, $C\in\mathcal{M}_{\alpha+1}(X\times Y)$.

It remains to show that $C=F^-(W)$. Let $(x_0,y_0)\in F^-(W)$. We fix $m\in\omega$ and note that $(x_0,y_0)\in F^-(G_m)$. Consider a neighborhood

$$O=\big\{B\subseteq Z\colon\ B\cap G_m\neq\varnothing\big\}$$

of the set $F(x_0,y_0)$ in the Vietoris topology. Under condition (1), there exists $n_0\geq m$ such that, for any $n\geq n_0$ for which $i\in I_n$, the inclusion $x_0\in A_{i,n}$ implies that $F(x_{i,n},y_0)\in O$, i.e., $(x_{i,n},y_0)\in F^-(G_m)$. In particular, for some $i\in I_{n_0}$, we conclude that $x_0\in A_{i,n_0}$ and $y_0\in F^-_{i,n_0}(G_m)$. Thus, $(x_0,y_0)\in C_{m,n_0}$. Therefore, $(x_0,y_0)\in C$.

Now let $(x_0,y_0)\notin F^-(W)$. Then

$$F(x_0,y_0)\subseteq Z\setminus W=\bigcup_{m\in\omega}(Z\setminus W_m).$$

Since the set $F(x_0,y_0)$ is compact, there exists $m_0\in\omega$ such that $F(x_0,y_0)\subseteq Z\setminus W_{m_0}$. Consider a neighborhood

$$O_1=\big\{B\subseteq Z\colon\ B\cap W_{m_0}=\varnothing\big\}$$

of the set $F(x_0,y_0)$ in the Vietoris topology. It follows from property (1) that there exists a number $n_0\in\omega$ such that, for all $n\geq n_0$, the inclusion $x_0\in A_{i,n}$ yields the inclusion $F(x_{i,n},y_0)\in O_1$. Hence,

$$F(x_{i,n},y_0)\subseteq Z\setminus W_{m_0}\subseteq Z\setminus W_m$$

and $y_0\notin F^-_{i,n}(G_m)$ for all $m\geq m_0$, $n\geq n_0$, and $i\in I_n$ such that $x_0\in A_{i,n}$. This implies that $(x_0,y_0)\notin C_{n,m}$ for all $n\geq n_0$ and $m\geq m_0$. Thus, $(x_0,y_0)\notin C$.

Now let the mapping F be multivalued and let $F^x\in \mathrm{U}^f_\alpha(Y,Z)$ for all $x\in D$. Reasoning as in the previous case and using similar notation, for all $m,n\in\omega$, we get

$$C_{m,n}=\bigcup_{i\in I_n}\Big(A_{i,n}\times F^+_{i,n}(G_m)\Big).$$

According to Lemma 3.1, we have $C_{m,n} \in \mathcal{A}_\alpha(X \times Y)$ and

$$C = \bigcap_{m\in\omega} \bigcup_{n\geq m} C_{n,m} \in \mathcal{M}_{\alpha+1}(X \times Y).$$

Further, we show that $C = F^+(W)$. Let $(x_0, y_0) \in F^+(W)$ and $m \in \omega$. Then $(x_0, y_0) \in F^+(G_m)$. In view of property (1), there exist $n \geq m$ and $i \in I_n$ such that $x_0 \in A_{i,n}$ and $F(x_{i,n}, y_0) \subseteq G_m$. Hence, $(x_0, y_0) \in C_{m,n}$ and, therefore, $(x_0, y_0) \in C$.

Now let $(x_0, y_0) \notin F^+(W)$. Then $F(x_0, y_0) \cap (Z \setminus W) \neq \varnothing$ and there exist a number $m_0 \in \omega$ such that

$$F(x_0, y_0) \cap (Z \setminus W_{m_0}) \neq \varnothing.$$

In view of property (1), there exists a number $n_0 \in \omega$ such that, for all $n \geq n_0$, the inclusion $x_0 \in A_{i,n}$ implies that $F(x_{i,n}, y_0) \cap (Z \setminus W_{m_0}) \neq \varnothing$. Hence, $(x_0, y_0) \notin C_{n,m}$ for all $n \geq n_0$ and $m \geq m_0$. Thus, $(x_0, y_0) \notin C$.

Theorem 3.1 is proved.

Remark 3.1. The multivalued mapping $F\colon (X, d) \times Y \multimap Z$ in Theorem 1.1 satisfies conditions (1) and (2) in Theorem 3.1. For all $u \in D$ and $n \in \omega$, we set

$$A_{u,n} = A(u) \cap \left\{ v \in X\colon d(u, v) < \frac{1}{n} \right\} \qquad \text{and} \qquad x_{u,n} = u.$$

Then the sequences of families $(A_{u,n}\colon u \in D)$ and $(x_{u,n}\colon u \in D)$ satisfy condition (1). In addition, condition (2) is equivalent to condition (e). Thus, Theorem 3.1 generalizes Theorem 1.1.

Definition 3.2. *A topological space X is called a (strong) PP-space if, for any dense set $D \subseteq X$, there exist a sequence $(\mathcal{U}_n)_{n=1}^\infty$ of locally finite coverings $\mathcal{U}_n = (U_{i,n}\colon i \in I_n)$ of the space X and a sequence $\big((x_{i,n}\colon i \in I_n)\big)_{n=1}^\infty$ of families of points from the space X (from the set D) such that, for any $x \in X$ and any neighborhood U of the point x, there exists a number $n_0 \in \omega$ such that, for all $n \geq n_0$ and $i \in I_n$, the inclusion $x_{i,n} \in U$ follows from the inclusion $x \in U_{i,n}$.*

The notion of PP-space was introduced in [14]. It is closely connected with the notion of metrically quarter-stratifiable spaces (see [1]). In [13], it was shown that metrically quarter-stratifiable spaces coincide with the Hausdorff PP-spaces. It is clear that every strong PP-space is a PP-space.

For topological spaces X, Y, and Z and ordinal $\alpha \in [0, \omega_1)$, by $\mathrm{CU}_\alpha^f(X, Y, Z)$ $\big(\mathrm{CL}_\alpha^f(X, Y, Z)\big)$ we denote a family of all multivalued mappings $F\colon X \times Y \multimap Z$ that are continuous in the first variable and belong to the upper (lower) functional Lebesgue class α with respect to the second variable. Similarly, by $\overline{\mathrm{C}}\mathrm{U}_\alpha^f(X, Y, Z)$ $\big(\overline{\mathrm{C}}\mathrm{L}_\alpha^f(X, Y, Z)\big)$ we denote a family of all multivalued mappings $F\colon X \times Y \multimap Z$ that are continuous with respect to the first variable and such that, for any dense set $D \subseteq X$, every multivalued mapping F^x belongs to the upper (lower) functional Lebesgue class α. The families $\mathrm{CU}_\alpha(X, Y, Z)$, $\mathrm{CL}_\alpha^f(X, Y, Z)$ $\overline{\mathrm{C}}\mathrm{U}_\alpha^f(X, Y, Z)$, and $\overline{\mathrm{C}}\mathrm{L}_\alpha^f(X, Y, Z)$ are defined similarly.

Corollary 3.1. *Let X be a PP-space, let Y and Z be topological spaces, and let $\alpha \in [0, \omega_1)$. Then*

$$\mathrm{CU}_\alpha^f(X, Y, Z) \subseteq \mathrm{L}_{\alpha+1}^f(X \times Y, Z) \quad \textit{and} \quad \mathrm{CL}_\alpha^f(X, Y, Z) \subseteq \mathrm{U}_{\alpha+1}^f(X \times Y, Z).$$

Proof. Let $(\mathcal{U}_n)_{n=1}^\infty$ and $((x_{i,n}\colon i \in I_n))_{n=1}^\infty$ be sequences from Definition 3.2 and let $A_{i,n} = U_{i,n}$ for all $n \in \omega$ and $i \in I_n$. It remains to use Theorem 3.1.

Similarly, we can prove the following result for strong PP-spaces:

Corollary 3.2. *Let X be a strong PP-space, let Y and Z be topological spaces, and let $\alpha \in [0, \omega_1)$. Then*

$$\mathrm{C}\overline{\mathrm{U}}^f_\alpha(X,Y,Z) \subseteq \mathrm{L}^f_{\alpha+1}(X \times Y, Z) \quad \textit{and} \quad \mathrm{C}\overline{\mathrm{L}}^f_\alpha(X,Y,Z) \subseteq \mathrm{U}^f_{\alpha+1}(X \times Y, Z).$$

4. Lebesgue Classification of Multivalued Mappings of Two Variables

We first generalize Theorem 3.30 in [11].

Theorem 4.1. *Suppose that X is a perfect space, $\alpha \in [0, \omega_1)$, and $(A_i\colon i \in I)$ is a locally finite family of sets from the additive (multiplicative) class α in X. Then the set*

$$A = \bigcup_{i \in I} A_i$$

belongs to the additive (multiplicative) class α in X.

Proof. We proceed by induction on α. It is known that the assertion of the theorem is true for $\alpha = 0$.

Let $(A_i\colon i \in I)$ be a locally finite family of F_σ-sets $A_i \subseteq X$ and let $\big((B_{i,n})_{n\in\omega}\colon i \in I\big)$ be a sequence of families of closed subsets of the space X such that

$$A_i = \bigcup_{n \in \omega} B_{i,n} \quad \text{for each} \quad i \in I.$$

Note that each family $(B_{i,n}\colon i \in I)$ is locally finite. Thus, all sets $B_n = \bigcup_{i\in I} B_{i,n}$ are closed and $A = \bigcup_{n\in\omega} B_n$ is an F_σ-set.

Let $(A_i\colon i \in I)$ be a locally finite family of G_δ-sets $A_i \subseteq X$ and let $\big((B_{i,n})_{n\in\omega}\colon i \in I\big)$ be a sequence of families of open sets $B_{i,n}$ such that

$$A_i = \bigcap_{n \in \omega} B_{i,n} \quad \text{for each} \quad i \in I.$$

For all $i \in I$, we set $F_i = \overline{A_i}$. It is clear that the family $(F_i\colon i \in I)$ is locally finite and the set $F = \bigcup_{i\in I} F_i$ is closed in X. For every $x \in F$, we set

$$I(x) = \{i \in I\colon x \in F_i\} \qquad \text{and} \qquad n(x) = |I(x)|.$$

In addition, let $K_n = \{x \in F_{I(x)}\colon n(x) > n\}$ for each $n \in \omega$. Since the family $(F_i\colon i \in I)$ is locally finite, every set K_n is closed.

We now consider the set $C = F \setminus A$ and show that this is a set of the type F_σ. For any $n \in \omega$, we set

$$C_n = \{x \in C\colon |n(x)| = n\}, \qquad \mathcal{J}_n = \{J \subseteq I\colon |J| = n\}$$

and

$$C_{J,n} = \{x \in C_n\colon I(x) = J\}$$

for all $J \in \mathcal{J}_n$. Let us show that each family $\mathcal{C}_n = (C_{J,n}\colon J \in \mathcal{J}_n)$ is locally finite. We fix $x \in X$ and choose a neighborhood U of the point x in X such that the set $I_1 = \{i \in I : U \cap F_i \neq \varnothing\}$ is finite. As a result, we obtain

$$\mathcal{I}_1 = \{J \in \mathcal{J}_n : U \cap C_{J,n} \neq \varnothing\} \subseteq \{J \in \mathcal{J}_n : J \subseteq I_1\}.$$

Thus, the set $\mathcal{I}_1$ is finite and the family $\mathcal{C}_n$ is locally finite. We now show that

$$C_{J,n} = \left(\bigcap_{i \in J} F_i\right) \setminus \left(K_n \cup \bigcup_{i \in J} A_i\right)$$

for all $n \in \omega$ and $J \in \mathcal{J}_n$. Since $C_{J,n} \subseteq \bigcap_{i \in J} F_i$ and $C_{J,n} \cap (K_n \cup \bigcup_{i \in J} A_i) = \varnothing$, we get

$$C_{J,n} \subseteq \left(\bigcap_{i \in J} F_i\right) \setminus \left(K_n \cup \bigcup_{i \in J} A_i\right).$$

Conversely, let $x \in \bigcap_{i \in J} F_i$, $x \notin K_n$ and $x \notin \bigcup_{i \in J} A_i$. Then $n(x) \geq |J| = n$ and $n(x) \leq n$. Therefore,

$$n(x) = n \qquad \text{and} \qquad I(x) = J.$$

Hence, $x \notin F_i$ for all $i \in I \setminus J$. This yields

$$x \notin \left(\bigcup_{i \in J} A_i\right) \bigcup \left(\bigcup_{i \in I \setminus J} F_i\right) \supseteq A.$$

Since the sets $\bigcap_{i \in J} F_i$ and K_n are closed and $\bigcup_{i \in J} A_i$ is a set of the type G_δ, the set $C_{J,n}$ is an F_σ-set. Hence, each set C_n is an F_σ-type set as a locally finite union of F_σ-sets. Thus, C is also a set of the type F_σ.

Assume that the lemma is true for all $\alpha < \beta$, where $\beta \in [1, \omega_1)$. Let $(A_i : i \in I)$ be a locally finite family of sets of the additive class β in X. Consider the case where $\beta = \alpha + 1$ for some $\alpha < \omega_1$. Then, for each $i \in I$, there exists a sequence $(B_{i,n})_{n \in \omega}$ of sets of the multiplicative class α in X such that $A_i = \bigcup_{n \in \omega} B_{i,n}$ for all $i \in I$. By induction, each set $B_n = \bigcup_{i \in I} B_{i,n}$ belongs to the multiplicative class α in X. Hence, the set $A = \bigcup_{n \in \omega} B_n$ belongs to the additive class β in X.

We now consider the case of the limiting ordinal β. We choose an increasing sequence of ordinals $\alpha_n < \beta$ such that $\sup_{n \in \omega} \alpha_n = \beta$. For each $i \in I$, there exists a sequence $(B_{i,n})_{n \in \omega}$ of sets of the multiplicative class α_n in X such that $A_i = \bigcup_{n \in \omega} B_{i,n}$ for all $i \in I$. Then each set $B_n = \bigcup_{i \in I} B_{i,n}$ belongs to the multiplicative class α in X, which implies that $A = \bigcup_{n \in \omega} B_n$ is a set of the additive class β.

We now consider the case where a set belongs to the multiplicative class $\beta \geq 2$. Let $(A_i : i \in I)$ be a local finite family of sets of the multiplicative class β in X and, moreover, $\beta = \alpha + 1$ for some $\alpha < \omega_1$ and let $\big((B_{i,n})_{n \in \omega} : i \in I\big)$ be a sequence of families of sets from the additive class α such that $A_i = \bigcap_{n \in \omega} B_{i,n}$ for all $i \in I$. For any $i \in I$, we set $F_i = \overline{A_i}$. The family $(F_i : i \in I)$ is locally finite. Further, for any $n \in \omega$ and $i \in I$, we denote $A_{i,n} = B_{i,n} \cap F_i$. Since $\alpha \geq 1$, each set $A_{i,n}$ belongs to the additive class α. Then $B_n = \bigcup_{i \in I} A_{i,n}$ is also a set from the additive class α for each n. The same is true for the set $A = \bigcup_{n \in \omega} B_n$. For the limit β, the reasoning is similar.

Theorem 4.1 is proved.

Remark 4.1. For a perfect paracompact set X, Theorem 4.1 is known [see, e.g., [3] (4.5.8)]. On the other hand, Theorem 4.1 cannot be generalized to arbitrary topological spaces. Thus, in particular, the question whether a locally finite union of G_δ-sets is a G_δ-set is discussed on the site http://mathoverflow.net. In this connection, Banakh showed that there exists a functionally Hausdorff (irregular) space X with the second axiom of countability that contains a closed discrete subset D which is not a subset of the type G_δ in X. At the same time, Baillif constructed an example of zero-dimensional Hausdorff space X with the first axiom of countability and cardinality $|X| = \omega_1$ that contains a closed discrete subspace D not of the type G_δ in X. Then $\big(\{x\} \colon x \in D\big)$ is a locally finite family of compact G_δ-sets in X whose union F is not a set of the type G_δ in X.

The next statement can be proved by analogy with the corresponding result in the previous section.

Theorem 4.2. *Suppose that X is a topological space, Y is a perfect space, Z is a perfectly normal space, $\alpha \in [0, \omega_1)$, $(\mathcal{A}_n)_{n=1}^{\infty}$ is a sequence of σ-locally finite coverings $\mathcal{A}_n = (A_{i,n} \colon i \in I_n)$ of the space X by sets $A_{i,n}$ from the additive class α in X, $\big((x_{i,n} \colon i \in I_n)\big)_{n=1}^{\infty}$ is a sequence of the families of points $x_{i,n} \in X$, and $F \colon X \times Y \multimap Z$ is a compact-valued (multivalued) mapping with the following properties:*

(1) for all $(x, y) \in X \times Y$ and an arbitrary sequence $(i_n)_{n\in\omega}$ of indices $i_n \in I_n$, it follows from the condition $x \in A_{i_n,n}$ that the sequence $\big(F(x_{i_n,n}, y)\big)_{n\in\omega}$ converges to $F(x, y)$ in the Vietoris topology;

(2) $F^x \in \mathrm{L}_\alpha(Y, Z)$ $\big(F^x \in \mathrm{U}_\alpha(Y, Z)\big)$ for any $x \in D = \{x_{i,n} \colon n \in \mathbb{N},\ i \in I_n\}$.

Then $F \in \mathrm{U}_{\alpha+1}(X \times Y, Z)$ $\big(F \in \mathrm{L}_{\alpha+1}(X \times Y, Z)\big)$.

Corollary 4.1. *Let X be a PP-space, let Y be a perfect space, let Z be a perfectly normal space, and let $\alpha \in [0, \omega_1)$. Then $\mathrm{CU}_\alpha(X, Y, Z) \subseteq \mathrm{L}_{\alpha+1}(X \times Y, Z)$ and $\mathrm{CL}_\alpha(X, Y, Z) \subseteq \mathrm{U}_{\alpha+1}(X \times Y, Z)$.*

Corollary 4.2. *Let X be a strong PP-space, let Y and Z be topological spaces, and let $\alpha \in [0, \omega_1)$. Then $\mathrm{C\overline{U}}_\alpha(X, Y, Z) \subseteq \mathrm{L}_{\alpha+1}(X \times Y, Z)$ and $\mathrm{C\overline{L}}_\alpha(X, Y, Z) \subseteq \mathrm{U}_{\alpha+1}(X \times Y, Z)$.*

Remark 4.2. Since every metrizable space is a strong PP-space, Theorem 4.2 generalizes Theorem 1.2. In addition, Theorem 4.2 is a generalization of Theorem 1.1 (it suffices to argue as in Remark 3.1).

The following example shows that the compact-valuedness of the mapping in Theorems 3.1 and 4.2 is essential:

Proposition 4.1. *There exists a separately continuous lower semicontinuous mapping F: $[0,1]^2 \to [0,1]$ which is not (functionally) upper measurable.*

Proof. Let $A \subseteq [0,1]$ be a set, which is not Borel measurable. Consider a continuous function

$$g \colon [0,1]^2 \to [0,1], \qquad g(x,y) = \frac{2(x+1)(y+1)}{(x+1)^2 + (y+1)^2},$$

and a multivalued mapping $F \colon [0,1]^2 \multimap [0,1]$,

$$F(x,y) = \begin{cases} [0, g(x,y)], & (x,y) \in [0,1]^2 \setminus \{(z,z) \colon z \in A\}, \\ [0,1), & (x,y) \in \{(z,z) \colon z \in A\}. \end{cases}$$

We set $\Delta = \{(x,x) \colon x \in [0,1]\}$. Since the function g is continuous, the mapping F is separately continuous at every point of the set $[0,1]^2 \setminus \{(z,z) \colon z \in A\}$ and jointly continuous at every point of the set $[0,1]^2 \setminus \Delta$.

Since the mapping $H(x,y) = [0, g(x,y)]$ is continuous and $H^{-}(G) = F^{-}(G)$ for any open set $G \subseteq [0,1]$, the mapping F is lower semicontinuous. In addition,

$$F(x,y) \subseteq [0,1) \subseteq F(z,z)$$

for all $(x,y) \in [0,1]^2 \setminus \Delta$ and $z \in [0,1]$. Thus, the mapping F is separately continuous at any point of the set Δ. This implies that F is a separately continuous lower semicontinuous mapping. According to Example 2.1, the restriction $F|_{\Delta}$ is not upper Lebesgue measurable. Hence, the mapping F is also not upper Lebesgue measurable.

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