DOI 10.1007/s11253-018-1527-7
Ukrainian Mathematical Journal, Vol. 70, No. 5, October, 2018 (Ukrainian Original Vol. 70, No. 5, May, 2018)

DIVERGENCE THEOREM IN THE L_2-VERSION. APPLICATION TO THE DIRICHLET PROBLEM

Yu. V. Bogdanskii

UDC 517.98+517.954

We propose an L_2-version of the divergence theorem. The Green and Poisson operators associated with the infinite-dimensional version of the Dirichlet problem are investigated.

1. Preliminary Information

Let H be a separable real Hilbert space $(\dim H \leq \infty)$, let μ be a nonnegative finite Borel measure on H, and let G be a bounded domain in H with the boundary $S = \partial G$.

By $C_b = C_b(H)$ we denote the space of all continuous and bounded real functions on H, by $C_b(H; H)$ we denote the space of all vector fields $\mathbf{X}\colon H \to H$ continuous and bounded on H, and by $C_b^1 = C_b^1(H)$ (resp., $C_b^1(H; H)$) we denote the space of all functions $f \in C_b$ (resp., vector fields $\mathbf{X} \in C_b(H; H)$) Fréchet differentiable at every point $x \in H$ whose derivatives $f'(\cdot)$ (resp., $\mathbf{X}'(\cdot)$) are bounded and continuous on H. Furthermore, by $C^1(\overline{G})$ we denote the family of all functions on $\overline{G}$ admitting extensions to functions from the class C_b^1 on H and by $C_0^1(G)$ we denote a family of functions from $C^1(\overline{G})$ equal to zero in a certain ε-neighborhood of the boundary S. Similarly, we denote $C(\overline{G})$, $C(\overline{G}; H)$, and $C^1(\overline{G}; H)$.

By $L^2(G) = L^2(G; \mu)$ we denote a space of functions square integrable on G with respect to the measure $\mu|_G$. Similarly, by $L^2(G; H) = L^2(G; H; \mu)$ we denote a space of square integrable vector fields on G. We define the norm in $L^2(G; H)$ by the fromula

$$|||\mathbf{Z}|||^2 = \int\limits_G \|\mathbf{Z}(x)\|^2 \, d\mu$$

(the integrability of a vector field is understood in a sense of the Bochner structure).

Let $\Phi_t = \Phi_t^{\mathbf{Z}}$ be a flow of the vector field $\mathbf{Z} \in C_b^1(H; H)$, and let μ_t be a shift of the measure μ along the vector field $\mathbf{Z}$ $(\mu_t(A) = \mu(\Phi_t A)$ for each $A \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ is a Borel σ-algebra in H. Recall that the differentiability of the measure μ along the field $\mathbf{Z}$ in a strict (Fomin) sense means the existence of the limit

$$\nu(A) = \lim_{t \to 0} \frac{1}{t}\big(\mu_t(A) - \mu(A)\big)$$

for each $A \in \mathcal{B}(H)$. In this case, $\nu = d_{\mathbf{Z}}\mu$ (the derivative of the measure μ along the field $\mathbf{Z}$) is a Borel alternating measure absolutely continuous with respect to the measure μ. The corresponding density $\dfrac{d\nu}{d\mu}$ is conventionally called the logarithmic derivative of the measure μ along the field $\mathbf{Z}$ or the divergence of the field $\mathbf{Z}$ (with respect to the measure μ):

$$\rho = \rho_\mu^{\mathbf{Z}} = \operatorname{div} \mathbf{Z} = \operatorname{div}_\mu \mathbf{Z} = \frac{d\nu}{d\mu}.$$

Institute of Applied Systems Analysis, “Sikorski Kiev Polytechnic Institute” Ukrainian National Technical University, Kiev, Ukraine.

Translated from Ukrains’kyi Matematychnyi Zhurnal, Vol. 70, No. 5, pp. 611–624, May, 2018. Original article submitted July 6, 2017.

 0041-5995/18/7005–0702

Assume that the boundary S of the domain G is a smooth submanifold of codimension 1 embedded in H and that the field of unit outer normal of the boundary S can be extended to the vector field $\mathbf{n} \in C_b^1(H; H)$. We also assume that the measure μ is differentiable in a strong sense along the field $\mathbf{n}$. We postulate the existence of a field $\mathbf{n}$ with the above-mentioned properties and speak about the *consistency of S (or G) with the measure μ*.

For $\varepsilon > 0$, by A_ε we denote the ε-neighborhood of the set A. In [1], we proved that, in the case of consistency of S with the measure μ, the equality $\mu(S_\varepsilon) = O(\varepsilon)$ $(\varepsilon \to 0)$ is true and, according to Proposition 1 in [2], $C_0^1(G)$ is dense in $L_2(G)$.

The measure μ consistent with S induces a surface measure on S [1, 2]. We denote it by μ_S or σ. If u is a bounded continuous function on S and $\widehat{u}$ is its extension to a function continuous and bounded on H, then the surface measure σ is correctly defined by the following relation valid for all bounded and continuous functions on S:

$$\int\limits_S u\, d\sigma = \frac{d}{dt}\bigg|_{t=0} \int\limits_{\Phi_t^{\mathbf{n}} G} \widehat{u}\, d\mu.$$

Moreover, σ is independent of the choice of extension $\mathbf{n}$ of the field of unit outer normal to S (see [1–3]).

If $u \in C_b^1$, then the following relation is true (see [2]):

$$\int\limits_S u\, d\sigma = \int\limits_G (\mathbf{grad}\, u, \mathbf{n})\, d\mu + \int\limits_G u \cdot \rho_\mu^{\mathbf{n}}\, d\mu. \tag{1}$$

By using the results obtained in [1], we can determine the measure μ_S also in the case where the measure μ is differentiable along the field $\mathbf{Z} \in C_b^1(H; H)$ strictly transverse to the surface S. In terms of the inner product, the last condition means that

$$\inf\Big\{ \big|\big(\mathbf{Z}(x), \mathbf{n}(x)\big)\big| \mid x \in S \Big\} > 0.$$

In this case, equality (1) for $u \in C_b^1$ is reduced to the form

$$\frac{d}{dt}\bigg|_{t=0} \int\limits_{\Phi_t^{\mathbf{Z}} G} u\, d\mu = \int\limits_S (\mathbf{Z}, \mathbf{n}) u\, d\sigma = \int\limits_G (\mathbf{grad}\, u, \mathbf{Z})\, d\mu + \int\limits_G u \cdot \operatorname{div}_\mu \mathbf{Z}\, d\mu, \tag{2}$$

and we can also speak about the consistency of S (or G) with the measure μ.

We set $C^1(S) = \{ u\big|_S \mid u \in C_b^1 \}$. Note that $C^1(S)$ is dense in $L_2(S, \sigma)$ (see [4]).

Consider an operator

$$\mathbf{grad} = \mathbf{grad}_G : L_2(G) \to L_2(G; H)$$

with natural domain of definition $C^1(\overline{G})$ $(C^1(\overline{G}) \ni u \mapsto \mathbf{grad}\, u \in C(\overline{G}; H))$. For the correct definition of this operator, the following condition must be satisfied:

$$\big(u, v \in C^1(\overline{G});\ u = v \,(\operatorname{mod} \mu)\big) \Longrightarrow \big(\mathbf{grad}\, u = \mathbf{grad}\, v \,(\operatorname{mod} \mu)\big).$$

This requirement is true, in particular, for measures μ sarisfying the the inequality $\mu(U) > 0$ for any nonempty open set $U \subset H$ (completeness of the support of the measure μ). The last condition is satisfied for quasiinvariant measures μ, i.e., for measures whose sets of quasiinvariant shifts h $(\mu_h(A) := \mu(A + h);\ \mu_h \sim \mu)$ contain linear submanifolds dense in H. As an example of this measure, we can mention a Gaussian measure whose kernel correlation operator has a dense image in H

Further, we impose the following restrictions on the measure μ and the domain G:

(a) the operator

$$\mathbf{grad}\colon\ L_2(G) \supset C^1(\overline{G}) \ni u \mapsto \mathbf{grad}\, u \in L_2(G; H)$$

is well defined and admits the closure $\overline{\mathbf{grad}} = \overline{\mathbf{grad}}_G$;

(b) $\operatorname{div}_\mu \mathbf{n}\,\big|_G \in L_\infty(G)$.

A model example of measure consistent with the surface S for which conditions (a) and (b) are simultaneously satisfied was proposed in [5]. The corresponding measure μ_φ is given by the formula

$$\mu_\varphi(A) = \int\limits_{\mathbb{R}} \varphi(t)\mu(\Phi_t^{\mathbf{n}} A)\, dt,$$

where μ is a Gaussian measure with nonsingular kernel correlation operator, $A \in \mathcal{B}(H)$, $\varphi \in C_b^1(\mathbb{R})$, $\varphi \geq 0$, and

$$\int\limits_{\mathbb{R}} \varphi(t)\, dt < \infty.$$

There exists a constant C such that the inequality $\big|\varphi'(s)\big| \leq C\varphi(s)$ is true for all $s \in \mathbb{R}$.

The simultaneous validity of conditions (a) and (b) enables one to correctly introduce the trace operator

$$\gamma\colon\ L_2(G) \to L_2(S) = L_2(S, \sigma)$$

with the domain of definition $D\big(\,\overline{\mathbf{grad}}\,\big)$ (see [2]). In addition, for functions $u \in C^1(\overline{G})$, we assume that

$$\gamma(u) = u\,|_S.$$

In [2], it is proved that there is a constant C_1 such that the inequality

$$\|u\,|_S\,\|_{L_2(S)} \leq C_1(\|u\|_{L_2(G)} + |||\mathbf{grad}\, u|||)$$

holds for all $u \in C^1(\overline{G})$. We correctly extend the operator $C^1(\overline{G}) \ni u \mapsto u\,|_S \in L_2(S)$ onto $D\big(\,\overline{\mathbf{grad}}\,\big)$ as a bounded operator γ from the Banach space $D\big(\,\overline{\mathbf{grad}}\,\big)$ (in the norm of the graph in $L_2(S)$) independent of the extension $\mathbf{n}$ of the field of unit outer normal to S.

In [3], it was shown that $\operatorname{Ker}\gamma$ coincides with the closure of $C_0^1(G)$ in the norm of the graph of the operator $\overline{\mathbf{grad}}$.

We specify the operator $\operatorname{div}\colon\ L_2(G; H) \to L_2(G)$ by the formula

$$\operatorname{div} = -\Big(\overline{\mathbf{grad}}\,\big|_{\operatorname{Ker}\gamma}\Big)^* = -\Big(\mathbf{grad}\,\big|_{C_0^1(G)}\Big)^*. \tag{3}$$

The arguments for the introduction of the operator div by formula (3) readily follow from (2) if we substitute $u \in C_0^1(G)$.

We introduce the Laplacian operator with respect to the measure in the L_2-version as follows:

$$\triangle = \Delta_G = \operatorname{div} \circ \overline{\mathbf{grad}}.$$

If Δ is regarded as an operator from $D(\overline{\mathbf{grad}})$ into $L_2(G)$, then it is closed and, hence, $\operatorname{Ker} \triangle$ is closed in $D(\overline{\mathbf{grad}})$.

The space $D(\overline{\mathbf{grad}})$ is an infinite-dimensional analog of the classical Sobolev space $H^1(G)$. It is a Hilbert space with respect to the scalar product

$$(u, v)_\Gamma = \int\limits_G uv\, d\mu + \int\limits_G (\overline{\mathbf{grad}}\, u, \overline{\mathbf{grad}}\, v)\, d\mu$$

(in our subsequent presentation, we always use this assumption). The space $\operatorname{Im} \gamma$ is algebraically isomorphic to the space

$$D(\overline{\mathbf{grad}})/_{\operatorname{Ker}\gamma} \cong D(\overline{\mathbf{grad}}) \ominus \operatorname{Ker}\gamma$$

and can be equipped with the induced scalar product

$$\big(\gamma(u), \gamma(v)\big)_\gamma = (u, v)_\Gamma,$$

where $u, v \in D(\overline{\mathbf{grad}}) \ominus \operatorname{Ker}\gamma$. As a result, this space turns into a Hilbert space with respect to the indicated scalar product.

In [1], the Gauss–Ostrogradski formula was obtained for the "classical" version of the operator div, namely, for the operator $\mathbf{Z} \mapsto \operatorname{div}_\mu \mathbf{Z}$. In the present paper, we investigate the L_2-version of the operator div given by relation (3).

2. Weak Surface Operator

The next lemma is similar to Lemma 1 in [3]:

Lemma 1. *Suppose that* $u, v \in D(\overline{\mathbf{grad}})$. *Then*

$$\frac{d}{dt}\bigg|_{t=0} \int\limits_{\Phi_t^{\mathbf{n}} G} uv\, d\mu = \int\limits_S \gamma(u)\gamma(v)\, d\sigma$$

exists.

Proof. Assume that the sequences $u_m, v_m \in C^1(\overline{G})$ converge to u and v, respectively, in the norm of the graph of the space $D(\overline{\mathbf{grad}})$. The inequality

$$\int\limits_G |u_m v_m - uv|\, d\mu \le \|u_m\|_{L_2(G)} \|v_m - v\|_{L_2(G)} + \|v\|_{L_2(G)} \|u_m - u\|_{L_2(G)} \tag{4}$$

yields the convergence $u_m v_m \to uv$ in $L_1(G)$.

The convergence $u_m\,\mathbf{grad}\,v_m \to u\,\overline{\mathbf{grad}}\,v$ in $L_1(G;H)$ follows from the inequality

$$\int\limits_G \|u_m\,\mathbf{grad}\,v_m - u\,\overline{\mathbf{grad}}\,v\|\,d\mu \le \|u_m - u\|_{L_2(G)}\,|||\mathbf{grad}\,v_m||| + \|u\|_{L_2(G)}\,\Big|\Big|\Big|\mathbf{grad}\,v_m - \overline{\mathbf{grad}}\,v\Big|\Big|\Big|\,.$$

Similarly, we arrive at the convergence $v_m\,\mathbf{grad}\,u_m \to v\,\overline{\mathbf{grad}}\,u$ in $L_1(G;H)$. Hence,

$$\mathbf{grad}(u_m v_m) \to u\,\overline{\mathbf{grad}}\,v + v\,\overline{\mathbf{grad}}\,u \quad \text{in} \quad L_1(G;H). \tag{5}$$

For $t \le 0,$ we denote

$$g_m(t) = \int\limits_{\Phi_t^{\mathbf{n}}G} u_m v_m\,d\mu, \qquad g(t) = \int\limits_{\Phi_t^{\mathbf{n}}G} uv\,d\mu.$$

Inequality (4) yields the uniform convergence of the sequence of functions g_m to g on $(-\infty;0]$.

By virtue of (2), the equality

$$g_m'(t) = \int\limits_{\Phi_t^{\mathbf{n}}G} \big(\mathbf{grad}(u_m v_m), \mathbf{n}\big)\,d\mu + \int\limits_{\Phi_t^{\mathbf{n}}G} u_m v_m\,\mathrm{div}_\mu\,\mathbf{n}\,d\mu, \qquad \mathrm{div}_\mu\,\mathbf{n} \in L_\infty(G),$$

is true for $t \le 0$. Hence, relations (4) and (5) imply the continuity of the functions g_m' and the uniform convergence of the sequence of functions g_m' to the function

$$\int\limits_{\Phi_t^{\mathbf{n}}G} \big((u\,\overline{\mathbf{grad}}\,v + v\,\overline{\mathbf{grad}}\,u, \mathbf{n}) + uv\cdot\mathrm{div}_\mu\,\mathbf{n}\big)\,d\mu$$

on $(-\infty;0]$.

By virtue of the classical theorem of analysis, we conclude that the function

$$\frac{d}{dt}\int\limits_{\Phi_t^{\mathbf{n}}G} uv\,d\mu = \lim_{m\to\infty} g_m'(t)$$

exists on $(-\infty;0]$. Moreover, the equality

$$\frac{d}{dt}\int\limits_{\Phi_t^{\mathbf{n}}G} uv\,d\mu = \int\limits_{\Phi_t^{\mathbf{n}}G} \big((u\,\overline{\mathbf{grad}}\,v + v\,\overline{\mathbf{grad}}\,u, \mathbf{n}) + uv\cdot\mathrm{div}_\mu\,\mathbf{n}\big)\,d\mu$$

is true.

Since the sequence $\gamma(u_m) = u_m\,|_S$ converges to the function $\gamma(u)$ in $L_2(S;\sigma)$ and, similarly, $\gamma(v_m) \to \gamma(v)$, we conclude that $\gamma(u_m v_m) = \gamma(u_m)\cdot\gamma(v_m)$ converges to $\gamma(u)\cdot\gamma(v)$ in $L_1(S;\sigma)$. Since

$$\int\limits_S \gamma(u_m)\cdot\gamma(v_m)\,d\sigma = g_m'(0),$$

we get

$$\int_S \gamma(u)\cdot\gamma(v)\,d\sigma = \lim_{m\to\infty} g'_m(0) = \frac{d}{dt}\bigg|_{t=0} \int_{\Phi_t^{\mathbf{n}}G} uv\,d\mu.$$

Lemma 1 is proved.

Thus, the trace γ associates the function $u \in D(\,\overline{\mathbf{grad}}\,)$ with a functional on $\operatorname{Im}\gamma$ continuous with respect to the norm of $L_2(S)$.

We fix $f \in L_2(G)$ and assume that, for any function $u \in D(\,\overline{\mathbf{grad}}\,)$, the number

$$\varphi(u) = \varphi_f(u) = \frac{d}{dt}\bigg|_{t=0} \int_{\Phi_t^{\mathbf{n}}G} uf\,d\mu \tag{6}$$

exists.

By virtue of Lemma 5 in [3], a function $u \in \operatorname{Ker}\gamma$ satisfies the equality

$$\int_{G\setminus\Phi_t^{\mathbf{n}}G} u^2\,d\mu = o(t^2), \qquad t \to 0-0.$$

Hence, the value $\varphi_f(u)$ depends only on $\gamma(u)$.

For fixed $t < 0$, the functional

$$\varphi_t\colon\, u \mapsto \frac{1}{t}\int_{G\setminus\Phi_t^{\mathbf{n}}G} uf\,d\mu$$

is continuous on $D(\,\overline{\mathbf{grad}}\,)$ with respect to the norm of $L_2(G)$ and, hence, also with respect to the norm of $D(\,\overline{\mathbf{grad}}\,)$. By

$$\Phi\colon\, \operatorname{Im}\gamma \to D(\,\overline{\mathbf{grad}}\,) \ominus \operatorname{Ker}\gamma$$

we denote a unique isomorphism introducing the structure of Hilbert space in $\operatorname{Im}\gamma$ (see Sec. 1). By the Banach–Steinhaus theorem, we conclude that the functional $\alpha_f(h) = \varphi_f(\Phi h)$ is continuous in the Hilbert space $\operatorname{Im}\gamma$.

Assume that the function $\widehat{\gamma}(f) \in \operatorname{Im}\gamma$ is defined, according to the Riesz theorem, by the equality

$$\big(\widehat{\gamma}(f), h\big)_\gamma = \alpha_f(h),$$

which is true for any function $h \in \operatorname{Im}\gamma$.

Definition 1. *By $D(\widehat{\gamma})$ we denote the set of functions from $L_2(G)$ such that, for all $u \in D(\,\overline{\mathbf{grad}}\,)$, the number $\varphi_f(u)$ is given by relation (6) and independent of the extension $\mathbf{n}$ of the field of unit outer normal to S. For $f \in D(\widehat{\gamma})$ and $u \in D(\,\overline{\mathbf{grad}}\,)$, the number*

$$w\int_S uf\,d\sigma := \frac{d}{dt}\bigg|_{t=0} \int_{\Phi_t^{\mathbf{n}}G} uf\,d\mu$$

is called a weak surface integral of the function uf on S.

The linear operator $\widehat{\gamma}\colon\ D(\widehat{\gamma}) \to \operatorname{Im}\gamma$ is connected with weak surface integral by the formula

$$w\int\limits_S uf\,d\sigma = \big(\widehat{\gamma}(f), \gamma(u)\big)_\gamma. \tag{7}$$

Moreover, $D\big(\,\overline{\mathbf{grad}}\,\big) \subset D(\widehat{\gamma})$ and $\operatorname{Ker}\widehat{\gamma} \cap D\big(\,\overline{\mathbf{grad}}\,\big) = \operatorname{Ker}\gamma$.

3. Main Theorem

Theorem 1. *Suppose that* $\boldsymbol{X} \in D(\operatorname{div})$, $\mathbf{n} \in C_b^1(H;H)$, *is the field of unit outer normal to* S *extended to* H. *Then* $(\boldsymbol{X}, \mathbf{n}) \in D(\widehat{\gamma})$ *and, for any function* $u \in D\big(\,\overline{\mathbf{grad}}\,\big)$, *the equality*

$$w\int\limits_S u(\boldsymbol{X}, \mathbf{n})\,d\sigma = \int\limits_G (u \cdot \operatorname{div}\boldsymbol{X} + (\,\overline{\mathbf{grad}}\,u, \boldsymbol{X}))\,d\mu \tag{8}$$

is true.

Proof. *Step 1.* We specify the function $t(\cdot)$ by the formula $\Phi\big(t(x), x\big) \in S$ (here, $\Phi(t,x) = \Phi_t(x) = \Phi_t^{\mathbf{n}}(x)$). The function $t(\cdot)$ is defined in a certain ε-neighborhood S_ε of the surface S; $t(\cdot) \in C^1(S_\varepsilon)$ and (see [2])

$$\mathbf{grad}\,t(x) = -\left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^* \mathbf{n}(\Phi\big(t(x), x\big)). \tag{9}$$

Differentiating the identity $t(\Phi(s,x)) = t(x) - s$ with respect to s, for $s = 0$, we obtain the equality $(\mathbf{grad}\,t(x), \mathbf{n}(x)) = -1$ everywhere in S_ε. Hence,

$$\mathbf{grad}\,t(x) = -a(x)\mathbf{n}(x) + \boldsymbol{\xi}(x), \tag{10}$$

where $\boldsymbol{\xi}(x) \perp \mathbf{n}(x)$ and $a(x) = \dfrac{1}{\|\mathbf{n}(x)\|^2}$.

Taking into account the fact that the equality

$$\mathbf{n}(x) = \left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^{-1} \mathbf{n}(\Phi\big(t(x), x\big)) \tag{11}$$

is true in S_ε and $\big\|\mathbf{n}(\Phi\big(t(x), x\big))\big\| = 1$, we derive the following estimate from (9)–(11):

$$\begin{aligned}\big\|\boldsymbol{\xi}(x)\big\| &\le \left\| \frac{1}{\|\mathbf{n}(x)\|^2}\left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^{-1} - \left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^* \right\| \\ &\le \left\| \left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^{-1} - \left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^* \right\| + \frac{\big|\|\mathbf{n}(x)\|^2 - 1\big|}{\|\mathbf{n}(x)\|^2} \left\| \left(\frac{\partial\Phi}{\partial x}\big(t(x), x\big)\right)^{-1} \right\|. \end{aligned} \tag{12}$$

For any fixed $x \in H$, the one-parameter operator family

$$Y(t) = \frac{\partial}{\partial x}\,\Phi_t x$$

satisfies the equation

$$\frac{d}{dt}\,Y(t) = A(t)Y(t),$$

where $A(t) = \mathbf{n}'(\Phi_t x)$, and the initial condition $Y(0) = I$.

For sufficiently small t, the operator $Y(t)^{-1}$ exists and the equalities

$$\begin{aligned}
&\frac{d}{dt}\,(Y(t))^* = (Y(t))^*(A(t))^*,\\
&\frac{d}{dt}\,Y(t)^{-1} = -Y(t)^{-1}\frac{d}{dt}\,Y(t)Y(t)^{-1} = -Y(t)^{-1}A(t),\\
&\frac{d}{dt}\,(Y(t)^{-1} - Y(t)^*) = -Y(t)^*A(t)^* - Y(t)^{-1}A(t)
\end{aligned} \tag{13}$$

are true. There exists $\alpha > 0$ such that the inequalities $\|Y(t)\| \le 2$ and $\|(Y(t))^{-1}\| \le 2$ hold for all $t \in (-\alpha, \alpha)$ and $x \in H$ (see [6, 7]). Setting $K = \sup_{x\in H}\|\mathbf{n}'(x)\| < +\infty$ for all $x \in H$ and $t \in (-\alpha, \alpha)$, we deduce the following estimate from (13):

$$\left\|\frac{d}{dt}\left(\left(\frac{\partial\Phi}{\partial x}\,(t,x)\right)^{-1} - \left(\frac{\partial\Phi}{\partial x}\,(t,x)\right)^*\right)\right\| \le 4K.$$

For all $x \in G \setminus \Phi_{-\delta}G$ ($\delta \in (0,\alpha)$), this yields the inequality

$$\left\|\left(\frac{\partial\Phi}{\partial x}\,(t,x)\right)^{-1} - \left(\frac{\partial\Phi}{\partial x}\,(t,x)\right)^*\right\| \le 4K\delta. \tag{14}$$

Since $\dfrac{d}{dt}\,\mathbf{n}(\Phi_t x) = \mathbf{n}'(\Phi_t x)\mathbf{n}(\Phi_t x)$ and $\|\mathbf{n}(\Phi_0 x)\| = 1$ for $x \in S$, we arrive at the estimate

$$\|\mathbf{n}(\Phi_t x)\| \le e^{K|t|} \qquad \text{for} \quad x \in S.$$

Decreasing (if necessary) $\alpha > 0$, for $\delta \in (0,\alpha)$ and $x \in G \setminus \Phi_{-\delta}G$, we arrive at the estimates

$$\begin{aligned}
&\frac{1}{2} \le \|\mathbf{n}(x)\| \le 2,\\
&\big|\|\mathbf{n}(x)\| - 1\big| \le e^{K\delta} - 1 \le 2K\delta.
\end{aligned} \tag{15}$$

Hence, for $x \in G \setminus \Phi_{-\delta}G$, the estimate

$$\frac{\big|\|\mathbf{n}(x)\|^2 - 1\big|}{\|\mathbf{n}(x)\|^2}\left\|\left(\frac{\partial\Phi}{\partial x}\big(t(x),x\big)\right)^{-1}\right\| \le 48K\delta \tag{16}$$

is true.

Inequalities (12), (14), and (16) now yield the following equality:

$$\|\boldsymbol{\xi}(x)\| = O(t(x)) \qquad \text{as} \quad t(x) \to 0. \tag{17}$$

Step 2. Assume that, for $\delta > 0$, a continuous function $\beta_\delta \colon \mathbb{R} \to \mathbb{R}$ is given by the conditions

$$\beta_\delta(t) = 0 \quad \text{for} \quad t \le 0, \qquad \beta_\delta(t) = \frac{1}{\delta} t \quad \text{for} \quad t \in [0, \delta], \qquad \text{and} \qquad \beta_\delta(t) = 1 \quad \text{for} \quad t \ge \delta.$$

Then $\beta_\delta'(t) = 0$ for $t \in (-\infty, 0) \cup (\delta, +\infty)$ and $\beta_\delta'(t) = \dfrac{1}{\delta}$ for $t \in (0, \delta)$.

Let $\delta \in (0, \varepsilon)$ and let the function v_δ be defined in $\overline{G}$ by the condition that $v_\delta(x) = \beta_\delta(t(x))$ for all x for which the function $t(x)$ is defined and by the condition $v_\delta(x) = 1$ for all other $x \in G$. We now show that $v_\delta \in \operatorname{Ker} \gamma$.

We take a sequence of functions $\varphi_m \in C^1(\mathbb{R})$ satisfying the conditions

$$\varphi_m(t) = 0 \quad \text{for} \quad t \le \frac{1}{m} < \frac{\delta}{3}, \qquad \varphi_m(t) = 1 \quad \text{for} \quad t \ge \delta, \qquad \varphi_m' \to \beta_\delta' \quad \text{in} \quad L_2(\mathbb{R}),$$

and the sequence $\{\varphi_m'\}$ is uniformly bounded. Then $\varphi_m \circ t \in C_0^1(G)$. As $\{\varphi_m\}$, we can use a sequence of functions

$$\varphi_m(t) = \int\limits_{-\infty}^{t} \omega_m(s)\, ds,$$

where

$$\omega_m(s) = 0 \qquad \text{for} \quad s \in \left(-\infty, \frac{1}{m}\right] \cup [\delta, +\infty),$$

$$\omega_m(s) = \frac{m^2}{m\delta - 2} \left(s - \frac{1}{m}\right) \qquad \text{for} \quad s \in \left[\frac{1}{m}, \frac{2}{m}\right],$$

$$\omega_m(s) = -\frac{m^2}{m\delta - 2} (s - \delta) \qquad \text{for} \quad s \in \left[\delta - \frac{1}{m}, \delta\right],$$

and

$$\omega_m(s) = \frac{m}{m\delta - 2} \qquad \text{for} \quad s \in \left[\frac{2}{m}, \delta - \frac{1}{m}\right].$$

Since the inequality

$$\left|\varphi_m(s) - \beta_\delta(s)\right| \le \int\limits_0^s \left|\varphi_m'(s) - \beta_\delta'(s)\right| ds \le \sqrt{\delta}\, \left\|\varphi_m' - \beta_\delta'\right\|_{L_2(\mathbb{R})}$$

is true for $s \in [0, \delta]$, the sequence of functions $\varphi_m \circ t$ uniformly converges to v_δ on $\overline{G}$.

It remains to prove that the sequence $\mathbf{grad}(\varphi_m \circ t)$ converges to $L_2(G;H)$ in $\overline{G}$.

We set

$$\mathbf{Z} = (\beta'_\delta \circ t)\,\mathbf{grad}\, t(\cdot) \in L_2(G;H).$$

Here and in what follows, we use the fact that $\mu(S) = \mu(\Phi_{-\delta}S) = 0$ (see [1]). Thus, by virtue of (9) and the Lebesgue theorem on bounded convergence,

$$\int\limits_G \left\| \mathbf{grad}(\varphi_m \circ t) - \mathbf{Z} \right\|^2 d\mu = \int\limits_G \left\| \varphi'_m(t(x)) \cdot \mathbf{grad}(t(x)) - \beta'_\delta(t(x)) \cdot \mathbf{grad}(t(x)) \right\|^2 d\mu \to 0.$$

Hence, $v_\delta \in \operatorname{Ker}\gamma$ and

$$\overline{\mathbf{grad}}\, v_\delta = (\beta'_\delta \circ t) \cdot \mathbf{grad}\, t.$$

Step 3. Let $u \in D\big(\,\overline{\mathbf{grad}}\,\big)$ and let $u_m \in C^1(\overline{G})$ be such that $u_m \to u$ in $D\big(\,\overline{\mathbf{grad}}\,\big)$. Since $\mathbf{X} \in D(\mathrm{div})$, we have $u_m\mathbf{X} \in D(\mathrm{div})$ and, in addition (see [5]),

$$\mathrm{div}(u_m\mathbf{X}) = u_m \cdot \mathrm{div}\,\mathbf{X} + (\mathbf{grad}\, u_m, \mathbf{X}).$$

Passing to the limit as $m \to \infty$ on both sides of the equality

$$\int\limits_G \big(u_m \cdot \mathrm{div}\,\mathbf{X} + (\mathbf{grad}\, u_m, \mathbf{X})\big) v\, d\mu = -\int\limits_G u_m(\mathbf{X}, \overline{\mathbf{grad}}\, v)\, d\mu$$

valid for all $v \in \operatorname{Ker}\gamma$ [see (3)], we conclude that, for sufficiently small $\delta > 0$, the equality

$$\int\limits_G \big(u \cdot \mathrm{div}\,\mathbf{X} + (\,\overline{\mathbf{grad}}\, u, \mathbf{X})\big) v_\delta\, d\mu = -\int\limits_G u(\mathbf{X}, \overline{\mathbf{grad}}\, v_\delta)\, d\mu \tag{18}$$

holds in view of the fact that $v_\delta \in \operatorname{Ker}\gamma$.

The left-hand side of equality (18) tends to

$$\int\limits_G \big(u \cdot \mathrm{div}\,\mathbf{X} + (\,\overline{\mathbf{grad}}\, u, \mathbf{X})\big)\, d\mu$$

as $\delta \to 0$.

Since

$$\big(\,\overline{\mathbf{grad}}\, v_\delta\big)(x) = \beta'_\delta(t(x))\bigg(-\frac{1}{\|\mathbf{n}(x)\|^2}\,\mathbf{n}(x) + \boldsymbol{\xi}(x)\bigg),$$

the right-hand side of equality (18) is decomposed into the sum

$$-\int\limits_G (u\mathbf{X}, \overline{\mathbf{grad}}\, v_\delta)\, d\mu = \frac{1}{\delta}\int\limits_{G\setminus\Phi_{-\delta}G} \bigg(u\mathbf{X}, \frac{1}{\|\mathbf{n}(\cdot)\|^2}\,\mathbf{n}(\cdot)\bigg) d\mu - \frac{1}{\delta}\int\limits_{G\setminus\Phi_{-\delta}G} (u\mathbf{X}, \boldsymbol{\xi})\, d\mu. \tag{19}$$

By virtue of (17) and the inequality

$$\left|\frac{1}{\delta}\int\limits_{G\setminus\Phi_{-\delta}G}(u\mathbf{X},\boldsymbol{\xi})\,d\mu\right|\leq\frac{1}{\delta}\sup_{G\setminus\Phi_{-\delta}G}\|\boldsymbol{\xi}(\cdot)\|\,d\mu\int\limits_{G\setminus\Phi_{-\delta}G}|u|\|\mathbf{X}(\cdot)\|\,d\mu,$$

the second term on the right-hand side of equality (19) approaches zero as $\delta\to 0$ and, hence, the limit

$$\lim_{\delta\to 0+}\frac{1}{\delta}\int\limits_{G\setminus\Phi_{-\delta}G}\left(u\mathbf{X},\frac{1}{\|\mathbf{n}(\cdot)\|^2}\,\mathbf{n}(\cdot)\right)d\mu$$

exists.

We now show that the last limit coincides with $\lim_{\delta\to 0+}\frac{1}{\delta}\int_{G\setminus\Phi_{-\delta}G}(u\mathbf{X},\mathbf{n})\,d\mu.$ We get

$$\left|\frac{1}{\delta}\int\limits_{G\setminus\Phi_{-\delta}G}\left(u\mathbf{X},\left(1-\frac{1}{\|\mathbf{n}(\cdot)\|^2}\right)\mathbf{n}(\cdot)\right)d\mu\right|\leq\frac{1}{\delta}\sup_{G\setminus\Phi_{-\delta}G}\frac{\big|\|\mathbf{n}(x)\|^2-1\big|}{\|\mathbf{n}(x)\|}\int\limits_{G\setminus\Phi_{-\delta}G}|u|\,\|\mathbf{X}(\cdot)\|\,d\mu\;\to\;0$$

$$\text{as}\quad\delta\;\to\;0$$

because the equality

$$\sup_{G\setminus\Phi_{-\delta}G}\frac{\big|\|\mathbf{n}(x)\|^2-1\big|}{\|\mathbf{n}(x)\|}=O(\delta),\quad\delta\to 0+,$$

follows from (15). Thus, for $u\in D(\overline{\textbf{grad}}\,),\ \mathbf{X}\in D(\mathrm{div}),$ we obtain the equality

$$\frac{d}{dt}\bigg|_{t=0}\int\limits_{\Phi_t^{\mathbf{n}}G}u(\mathbf{X},\mathbf{n})\,d\mu=\int\limits_{G}\big(u\cdot\mathrm{div}\,\mathbf{X}+\big(\overline{\textbf{grad}}\,u,\mathbf{X}\big)\big)\,d\mu.\tag{20}$$

Step 4. Since the right-hand side of equality (20) is independent of the choice of the field $\mathbf{n},$ to prove that the function $(\mathbf{X},\mathbf{n})$ belongs to the domain of definition $D(\widehat{\gamma})$ of the operator $\widehat{\gamma},$ it is sufficient to check the equality

$$\int\limits_{G\setminus\Phi_{-t}^{\mathbf{n}}G}u(\mathbf{X},\mathbf{n}-\mathbf{n}_1)\,d\mu=o(t),\quad t\to 0,\tag{21}$$

where $\mathbf{n},\mathbf{n}_1\in C_b^1(H;H)$ are two different extensions of the field of unit outer normal to the surface $S.$

Equality (21) directly follows from the Lipschitz property of the function $\|\mathbf{n}(\cdot)-\mathbf{n}_1(\cdot)\|,$ the equality

$$\|\mathbf{n}(x)-\mathbf{n}_1(x)\|=0\qquad\text{for}\quad x\in S,$$

and the estimate

$$\left|\int\limits_{G\setminus\Phi_{-t}^{\mathbf{n}}G}u(\mathbf{X},\mathbf{n}-\mathbf{n}_1)\,d\mu\right|\leq\sup_{G\setminus\Phi_{-t}^{\mathbf{n}}G}\|\mathbf{n}(\cdot)-\mathbf{n}_1(\cdot)\|\int\limits_{G\setminus\Phi_{-t}^{\mathbf{n}}G}|u|\|\mathbf{X}(\cdot)\|\,d\mu.$$

Theorem 1 is proved.

4. Gauss–Ostrogradskii Formula. Green Formulas

By virtue of relation (7), we can represent relation (8) in the form

$$\big(\gamma(u), \widehat{\gamma}((\mathbf{X}, \mathbf{n}))\big)_\gamma = \int\limits_G \big(u \cdot \operatorname{div} \mathbf{X} + \big(\,\overline{\mathbf{grad}}\, u, \mathbf{X}\big)\big)\, d\mu.$$

In the case where $u, v \in D\big(\,\overline{\mathbf{grad}}\,\big)$, in view of Lemma 1 and equality (6), we arrive at the identities

$$\big(\widehat{\gamma}(u), \gamma(v)\big)_\gamma = \big(\gamma(u), \gamma(v)\big)_{L_2(S)} = \big(\gamma(u), \widehat{\gamma}(v)\big)_\gamma. \tag{22}$$

Hence, in the case where $\operatorname{div} \mathbf{X} \in D\big(\,\overline{\mathbf{grad}}\,\big)$, relation (8) takes the form

$$\int\limits_S \gamma(u)\gamma((\mathbf{X}, \mathbf{n}))\, d\sigma = \int\limits_G \big(u \cdot \operatorname{div} \mathbf{X} + \big(\,\overline{\mathbf{grad}}\, u, \mathbf{X}\big)\big)\, d\mu. \tag{23}$$

The following statements are direct corollaries of Theorem 1:

Corollary 1 (Gauss–Ostrogradskii formula). *Let* $\boldsymbol{X} \in D(\operatorname{div})$. *Then* $(\boldsymbol{X}, \mathbf{n}) \in D(\widehat{\gamma})$ *and the equality*

$$\big(1, \widehat{\gamma}((\boldsymbol{X}, \mathbf{n}))\big)_\gamma = w\int\limits_S (\boldsymbol{X}, \mathbf{n})\, d\sigma = \int\limits_G \operatorname{div} \boldsymbol{X}\, d\mu$$

is true.

If $(\mathbf{X}, \mathbf{n}) \in D\big(\,\overline{\mathbf{grad}}\,\big)$, then, by using (23), we arrive at the equality

$$\int\limits_S \gamma((\mathbf{X}, \mathbf{n}))\, d\sigma = \int\limits_G \operatorname{div} \mathbf{X}\, d\mu.$$

For $u \in D\big(\,\overline{\mathbf{grad}}\,\big)$, we introduce the notation

$$\frac{\partial u}{\partial \mathbf{n}} := (\,\overline{\mathbf{grad}}\, u, \mathbf{n}).$$

By virtue of Theorem 1, if $u \in D(\Delta)$, then $\dfrac{\partial u}{\partial \mathbf{n}} \in D(\widehat{\gamma})$.

Corollary 2 (first Green formula). *Let* $v \in D(\Delta)$ *and* $u \in D\big(\,\overline{\mathbf{grad}}\,\big)$. *Then the equality*

$$\left(\gamma(u), \widehat{\gamma}\left(\frac{\partial v}{\partial \mathbf{n}}\right)\right)_\gamma = w\int\limits_S u \frac{\partial v}{\partial \mathbf{n}}\, d\sigma = \int\limits_G \big(u \cdot \Delta v + \big(\,\overline{\mathbf{grad}}\, u, \overline{\mathbf{grad}}\, v\big)\big)\, d\mu \tag{24}$$

is true.

In the case where $\dfrac{\partial v}{\partial \mathbf{n}} \in D(\,\overline{\mathbf{grad}}\,)$, equality (24) takes the form

$$\int_S \gamma(u)\,\gamma\left(\frac{\partial v}{\partial \mathbf{n}}\right) d\sigma = \int_G \big(u \cdot \Delta v + (\,\overline{\mathbf{grad}}\,u, \overline{\mathbf{grad}}\,v)\big)\,d\mu.$$

Corollary 3 (second Green formula)**.** *Let* $u, v \in D(\Delta)$. *Then the equality*

$$\left(\gamma(u), \widehat{\gamma}\left(\frac{\partial v}{\partial \mathbf{n}}\right)\right)_\gamma - \left(\gamma(v), \widehat{\gamma}\left(\frac{\partial u}{\partial \mathbf{n}}\right)\right)_\gamma = w\int_S \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right) d\sigma = \int_G (u \cdot \Delta v - v \cdot \Delta u)\,d\mu$$

is true.

5. Green and Poisson Operators

Since $\operatorname{Ker}\gamma \supset C_0^1(G)$, $\operatorname{Ker}\gamma$ is dense in $L_2(G)$ (see [2]). The operator $\overline{\mathbf{grad}}\,\big|_{\operatorname{Ker}\gamma}$ is closed. Therefore, by the von Neumann theorem (see [8]), the operator

$$\Delta\big|_{\operatorname{Ker}\gamma} = -\left(\overline{\mathbf{grad}}\,\big|_{\operatorname{Ker}\gamma}\right)^* \overline{\mathbf{grad}}\,\big|_{\operatorname{Ker}\gamma}$$

is a self-adjoint negative-definite operator in $L_2(G)$.

Further, we replace condition (a) in Sec. 1 by condition (a′):

(a′) the operator

$$\mathbf{grad} = \mathbf{grad}_H : L_2(H;\mu) \to L_2(H;H;\mu)$$

with the domain of definition $D(\mathbf{grad}) = C_b^1(H)$ is well defined, admits the closure $\overline{\mathbf{grad}}$, and possesses the property

$$\big(\overline{\mathbf{grad}}\,u = 0\,(\operatorname{mod}\mu)\big) \Longrightarrow \big(u = \operatorname{const}(\operatorname{mod}\mu)\big).$$

In [4], we present examples of measures simultaneously satisfying conditions (a′) and (b). In [9], it is proved that condition (a′) implies condition (a).

Under conditions (a′) and (b), by virtue of the maximum principle (see [4]), 0 is not an eigenvalue of the operator $\Delta\big|_{\operatorname{Ker}\gamma}$. Therefore, $\operatorname{Im}\left(\Delta\big|_{\operatorname{Ker}\gamma}\right)$ is dense in $L_2(G)$ and there exists a densely defined operator (Green operator) $\mathcal{G} = \left(\Delta\big|_{\operatorname{Ker}\gamma}\right)^{-1}$ in $L_2(G)$.

For $f \in D(\mathcal{G})$, the function $v = \mathcal{G}f$ is a (unique) solution of the boundary-value problem $\Delta v = f$, $\gamma(v) = 0$, in the domain G.

Let $v \in \operatorname{Ker}\gamma$ and $\Delta u = 0$. Then

$$\int_G (\,\overline{\mathbf{grad}}\,u, \overline{\mathbf{grad}}\,v)\,d\mu = -\int_G \Delta u \cdot v\,d\mu = 0,$$

and we arrive at the following relation from (24):

$$\left(\gamma(u), \widehat{\gamma}\left(\frac{\partial v}{\partial \mathbf{n}}\right)\right)_\gamma = \int_G u \cdot \Delta v\,d\mu.$$

By virtue of the equality $\Delta u = 0,$ we get the equality

$$\left(\gamma(u), \widehat{\gamma}\left(\frac{\partial}{\partial \mathbf{n}}\mathcal{G}f\right)\right)_\gamma = \int\limits_G u \cdot f \, d\mu, \tag{25}$$

which is true for every function $f \in D(\mathcal{G})$.

Conversely, let $u \in D(\Delta)$ and let equality (25) be true for all $f \in D(\mathcal{G})$. Then we derive the following equality from (24):

$$\int\limits_G \Delta u \cdot \mathcal{G}f \, d\mu = -\int\limits_G \left(\overline{\mathbf{grad}}\, u, \overline{\mathbf{grad}}\,(\mathcal{G}f)\right) d\mu = 0,$$

which is true for all $f \in D(\mathcal{G})$. Since $\operatorname{Im}\mathcal{G}$ is dense in $L_2(G)$, we find $\Delta u = 0$. Thus, we have proved the following lemmas:

Lemma 2. *Suppose that $u \in D(\Delta)$. Then $\Delta u = 0$ if and only if equality (25) holds for any function $f \in D(\mathcal{G})$.*

By virtue of the maximum principle [4], the problem $\Delta u = 0,\ \gamma(u) = g,$ has at most one solution. The mapping $g \mapsto u$ specifies a linear operator $\mathcal{P}$ (Poisson operator) that maps from $\operatorname{Im}\gamma$ into $L_2(G)$.

By virtue of (25), the equality

$$\left(g, \widehat{\gamma}\left(\frac{\partial}{\partial \mathbf{n}}\mathcal{G}f\right)\right)_\gamma = \int\limits_G \mathcal{P}g \cdot f \, d\mu$$

is true. This equality yields the following relation between the Green and Poisson operators:

$$\mathcal{P} \subset \left(\widehat{\gamma} \circ \frac{\partial}{\partial \mathbf{n}} \circ \mathcal{G}\right)^*. \tag{26}$$

Here and in what follows, $\operatorname{Im}\gamma$ is a Hilbert space with respect to the scalar product $(\cdot,\cdot)_\gamma$.

Note that the (unique) solution of the problem

$$\Delta u = f \in D(\mathcal{G}), \quad \gamma(u) = g \in D(\mathcal{P}),$$

is given by the formula $u = \mathcal{G}f + \mathcal{P}g$.

6. Bounded Green Operator

In the case where the Green operator is bounded, it is defined on the entire $L_2(G)$ (0 is a regular value of the operator $\Delta\big|_{\operatorname{Ker}\gamma}$).

Lemma 3. $\gamma(D(\Delta)) = \operatorname{Im}\gamma$.

Proof. Consider a problem

$$\Delta u = u, \qquad \gamma(u) = g.$$

It suffices to prove that this problem has a solution for any function $g \in \operatorname{Im}\gamma$.

The function u satisfies the equation $\Delta u = u$ if and only if, for each $v \in \operatorname{Ker}\gamma$, the equality

$$(u,v)_\Gamma = \int\limits_G \big(uv + \big(\,\overline{\mathbf{grad}}\,u, \overline{\mathbf{grad}}\,v\big)\big)\,d\mu = 0$$

is true.

If $\widehat{g} \in \gamma^{-1}(\{g\})$, then we get the required solution u as an orthogonal projection of the function $\widehat{g}$ in $D\big(\overline{\mathbf{grad}}\big)$ onto $D\big(\overline{\mathbf{grad}}\big) \ominus \operatorname{Ker}\gamma$.

Lemma 3 is proved.

Let $g \in \operatorname{Im}\gamma$ and let the function $\widehat{g} \in D(\Delta)$ be such that $\gamma(\widehat{g}) = g$. If the Green operator is bounded, then $D(\mathcal{G}) = L_2(G)$. Hence, there exists $v \in \operatorname{Ker}\gamma$ for which $\Delta v = \Delta\widehat{g}$. In this case, $\widehat{g} - v \in \operatorname{Ker}\Delta$ and, hence,

$$\gamma(\operatorname{Ker}\Delta) = \operatorname{Im}\gamma.$$

Therefore, $D(\mathcal{P}) = \operatorname{Im}\gamma$, and inclusion (26) implies the boundedness of the operators $\mathcal{P}$ and

$$\widehat{\gamma} \circ \frac{\partial}{\partial \mathbf{n}} \circ \mathcal{G} \colon\ \mathcal{P} \in \mathcal{L}(\operatorname{Im}\gamma, L_2(G)), \qquad \widehat{\gamma} \circ \frac{\partial}{\partial \mathbf{n}} \circ \mathcal{G} \in \mathcal{L}(L_2(G), \operatorname{Im}\gamma)$$

and their mutual adjointness:

$$\mathcal{P} = \left(\widehat{\gamma} \circ \frac{\partial}{\partial \mathbf{n}} \circ \mathcal{G}\right)^*. \tag{27}$$

In the case where, for $g \in L_2(G)$, the function $\dfrac{\partial}{\partial \mathbf{n}}(\mathcal{G}g)$ belongs to $D\big(\,\overline{\mathbf{grad}}\,\big)$, relations (27) and (22) yield the equality

$$(\mathcal{P}h, g)_{L_2(G)} = \left(h, \gamma\left(\frac{\partial}{\partial \mathbf{n}}\,\mathcal{G}g\right)\right)_{L_2(S)}, \tag{28}$$

which is true for all $h \in \operatorname{Im}\gamma$.

Equality (28) corresponds to the known relationship between the Green and Poisson operators in the case of a finite-dimensional space H and the invariant Lebesgue measure μ.

7. Sufficient Condition of Boundedness of the Green Operator

Theorem 2 (Friedrichs inequality)**.** *Suppose that there exists a field* $\mathbf{Z} \in C^1(H;H)$ *along which the measure* μ *is differentiable and the condition* $\operatorname{div}_\mu \mathbf{Z}\,\big|_V \in L_\infty(V,\mu)$ *is satisfied on balls* V. *Assume that there exists* $t_0 \in \mathbb{R}$ *for which* $\Phi_{t_0} G \cap G = \varnothing$ *(here,* Φ_t *is the flow of the field* $\mathbf{Z}$*). Then there exists* $C > 0$ *such that the inequality*

$$\|v\|_{L_2(G)} \le C\ \big|\big|\big|\,\overline{\mathbf{grad}}\,v\big|\big|\big| \tag{29}$$

is true for all $v \in \operatorname{Ker}\gamma$.

Proof. It is sufficient to prove inequality (29) only for functions $v \in C_0^1(G)$. Further, without loss of generality, we can assume that $t_0 > 0$.

We extend the definition of the function $v \in C_0^1(G)$ to the outside of $\overline{G}$ by zero. Thus, for every point $x \in G$, the equality

$$-v^2(x) = v^2(\Phi_{t_0}x) - v^2(x) = \int_0^{t_0} \frac{d}{dt}\, v^2(\Phi_t x)\, dt = 2\int_0^{t_0} v(\Phi_t x)\big(\mathbf{grad}\, v(\Phi_t x), \mathbf{Z}(\Phi_t x)\big)\, dt$$

holds. This yields the inequality

$$\int_H v^2\, d\mu \le 2\int_H d\mu \int_0^{t_0} |v(\Phi_t x)|\, \|\mathbf{grad}\, v(\Phi_t x)\|\, \|\mathbf{Z}(\Phi_t x)\|\, dt$$

$$= 2\int_0^{t_0} dt \int_H |v(x)|\, \|\mathbf{grad}\, v(x)\|\, \|\mathbf{Z}(x)\|\, \frac{d\mu_{-t}}{d\mu}\, d\mu. \tag{30}$$

Since $\mathbf{Z} \in C_b^1(H; H)$, there exists a ball $V \subset H$ for which

$$\Phi_{[0,t_0]}G := \{\Phi_t x \mid x \in G;\ t \in [0, t_0]\} \subset V.$$

Let $K = \|\operatorname{div}_\mu \mathbf{Z}\|_{L_\infty(V,\mu)}$. Thus, for all $t \in [0, t_0]$, the inequality

$$\left.\frac{d\mu_{-t}}{d\mu}\right|_V \le e^{Kt}\, (\operatorname{mod} \mu)$$

is true (see [3], Lemma 2). By using (30), we derive the inequality

$$\int_G v^2\, d\mu \le 2t_0\, e^{Kt_0} \sup_H \|\mathbf{Z}(\cdot)\|\, \|v\|_{L_2(G)}\, |||\mathbf{grad}\, v|||\,,$$

which yields the assertion of the theorem with a constant

$$C = 2t_0\, e^{Kt_0} \sup_H \|\mathbf{Z}(\cdot)\|.$$

Theorem 2 is proved.

Under the conditions of the proved theorem, the operator $\overline{\mathbf{grad}}\big|_{\operatorname{Ker}\gamma}$ has a bounded inverse operator. This yields the boundedness of the operator $\mathcal{G} = \left(\Delta\big|_{\operatorname{Ker}\gamma}\right)^{-1}$.

Note that, for $\dim H < \infty$ and the invariant Lebesgue measure μ, the conditions of Theorem 2 are obviously satisfied.

REFERENCES

1. Yu. V. Bogdanskii, "Banach manifolds with bounded structure and the Gauss–Ostrogradskii formula," *Ukr. Mat. Zh.*, **64**, No. 10, 1299–1313 (2012); ***English translation:*** *Ukr. Math. J.*, **64**, No. 10, 1475–1494 (2013).
2. Yu. V. Bogdanskii, "Laplacian with respect to a measure on a Hilbert space and an L_2-version of the Dirichlet problem for the Poisson equation," *Ukr. Mat. Zh.*, **63**, No. 9, 1169–1178 (2011); ***English translation:*** *Ukr. Math. J.*, **63**, No. 9, 1339-1348 (2012).

3. Yu. V. Bogdanskii, "Boundary trace operator in a domain of Hilbert space and the characteristic property of its kernel," *Ukr. Mat. Zh.*, **67**, No. 11, 1450–1460 (2015); ***English translation:*** *Ukr. Math. J.*, **67**, No. 11, 1629–1642 (2016).
4. Yu. V. Bogdanskii, "Maximum principle for the Laplacian with respect to a measure in a domain of the Hilbert space," *Ukr. Mat. Zh.*, **68**, No. 4, 460–468 (2016); ***English translation:*** *Ukr. Math. J.*, **68**, No. 4, 515–525 (2016).
5. Yu. V. Bogdanskii and Ya. Yu. Sanzharevskii, "The Dirichlet problem with Laplacian with respect to a measure in the Hilbert space," *Ukr. Mat. Zh.*, **66**, No. 6, 733–739 (2014); ***English translation:*** *Ukr. Math. J.*, **66**, No. 6, 818–826 (2014).
6. Yu. L. Daletskii and M. G. Krein, *Stability of the Solutions of Differential Equations in Banach Spaces* [in Russian], Nauka, Moscow (1970).
7. Yu. V. Bogdanskii and A. Yu. Potapenko, "Laplacian with respect to a measure on the Riemannian manifold and the Dirichlet problem. I," *Ukr. Mat. Zh.*, **68**, No. 7, 897–907 (2016); ***English translation:*** *Ukr. Math. J.*, **68**, No. 7, 1021–1033 (2016).
8. K. Yosida, *Functional Analysis*, Springer, Berlin (1965).
9. Yu. V. Bogdanskii and A. Yu. Potapenko, "Laplacian with respect to a measure on the Riemannian manifold and the Dirichlet problem. II," *Ukr. Mat. Zh.*, **68**, No. 11, 1443–1449 (2016); ***English translation:*** *Ukr. Math. J.*, **68**, No. 11, 1665–1672 (2017).