DOI 10.1007/s11253-018-1518-8
Ukrainian Mathematical Journal, Vol. 70, No. 4, September, 2018 (Ukrainian Original Vol. 70, No. 4, April, 2018)

LIMIT THEOREMS FOR THE MAXIMUM OF SUMS OF INDEPENDENT RANDOM PROCESSES

I. K. Matsak[1], **A. M. Plichko**[2], **and A. S. Sheludenko**[1]

UDC 519.21

We study the conditions of weak convergence of the maximum of sums of independent random processes in the spaces $C[0,1]$ and L_p and present examples of applications to the analysis of statistics of the type ω^2.

1. Introduction

Let (ξ_n) be independent identically distributed random variables with $\mathbf{E}\xi_n = 0$ and $\mathbf{D}\xi_n = 1$. In 1946, Erdös and Kac [1] established that

$$\lim_{n\to\infty} \mathbf{P}\big\{ \max(0, \xi_1, \xi_1+\xi_2, \ldots, \xi_1+\xi_2+\ldots+\xi_n) < x\sqrt{n} \,\big\} = \sqrt{\frac{2}{\pi}} \int\limits_0^x \exp(-t^2/2)dt. \tag{1}$$

In addition, the following equality is true for the process of Brownian motion $W(t)$ in $\mathbb{R}$ [2]:

$$\mathbf{P}\left\{ \sup_{0\le t\le 1} W(t) < x \right\} = \sqrt{\frac{2}{\pi}} \int\limits_0^x \exp(-t^2/2)dt.$$

In fact, these relations already contain one of important ideas leading to the construction of the theory of weak convergence of measures in function spaces (see [3, 4]). Clearly, equalities of the form (1) were also considered in the vector case [5, 6]. In the present paper, we study the infinite-dimensional case.

Let $X = \big\{X(s),\ s \in [0,1]\big\}$ be a random process and let $\Gamma = \big\{\Gamma(s),\ s \in [0,1]\big\}$ be a normal random process defined in the probability space $(\Omega, \Sigma, \mathbf{P})$ with values in $\mathbb{R}$ and such that, for any $s, t \in [0,1]$,

$$\mathbf{E}X(s) = \mathbf{E}\Gamma(s) = 0 \qquad \text{and} \qquad \mathbf{E}X(s)X(t) = \mathbf{E}\Gamma(s)\Gamma(t) =: R(s,t). \tag{2}$$

Consider a separable function Banach space $B = \big\{x = x(s),\ s \in [0,1]\big\}$. We say that a random process *belongs to B almost surely* if its sample functions belong to B almost surely.

We assume that Γ belongs to B almost surely and introduce a random function of two variables

$$W(s,t) = \sum_{n=1}^{\infty} \Gamma_n(s) F_n(t), \quad s, t \in [0,1], \tag{3}$$

[1] Shevchenko Kyiv National University, Kyiv, Ukraine.
[2] Kościuzko University of Technology, Kraków, Poland.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 70, No. 4, pp. 506–518, April, 2018. Original article submitted September 29, 2016.

where (Γ_n) is a sequence of independent copies of the process Γ and $F_n(t)$ are peaked Faber–Schauder functions (they are integrals of the Haar functions $H_n(u)$ or, more exactly,

$$F_n(t) = \int_0^t H_{n-1}(u)\,du\Big).$$

It is known (see [7, p. 128; 8]) that series (3) converges both for any $s \in [0,1]$ and with respect to the norm of the space B uniformly in $t \in [0,1]$ almost surely (a.s.). Note that Levy [9] proposed this construction for the representation of the Brownian motion in $\mathbb{R}$ (see also [10]).

For any $s,t \in [0,1]$, $W(s,t)$ is a normally distributed random variable and, for fixed $t \in [0,1]$, $W(\cdot,t)$ is a normally distributed random element in B, i.e., $W(\cdot,t)$ is a normally distributed continuous homogeneous process with independent increments in B. This process is called a *process of Brownian motion (or a Wiener process) with values in* B. In the case where $B = \mathbb{R}^m$, this definition coincides with the classical definition [5, p. 65].

By (X_n) we denote a sequence of independent copies of the process X and set

$$S_n(s) = \sum_{k=1}^n X_k(s), \quad S_0 = 0, \qquad \overline{S}_n(s) = \max_{0\le k\le n} S_k(s), \quad n \ge 1.$$

The following lemma is true:

Lemma 1. *Finite-dimensional distributions of the random process* $\dfrac{\overline{S}_n(s)}{\sqrt{n}}$ *converge to finite-dimensional distributions of the process*

$$\overline{W}(s) = \max_{t\in[0,1]} W(s,t).$$

This statement is a direct corollary of Lemma 3 in [8].

If the random processes $\overline{S}_n(s)$ and $\overline{W}(s)$ belong to the space B a.s., then it is quite natural to study the conditions under which weak convergence is realized in this space:

$$\frac{\overline{S}_n(\cdot)}{\sqrt{n}} \xrightarrow{D} \overline{W}(\cdot) \tag{4}$$

as $n \to \infty$.

In [8], this problem was studied for the spaces $B = L_p$. In the present paper, we consider the space $C[0,1]$ and also weaken the conditions obtained in [8] for L_p.

2. Space $C[0,1]$

This analyzed space consists of functions continuous on the segment $[0,1]$ with uniform norm. We introduce the notation:

$$T_h = \big\{(s,t) \in [0,1]^2 \colon\ |s-t| \le h\big\}, \qquad h > 0,$$

$$d_p(s,t) = \big(\mathbf{E}\big|X(s) - X(t)\big|^p\big)^{\frac{1}{p}}, \qquad s,t \in [0,1], \quad p \ge 2; \qquad d_p(h) = \sup_{T_h} d_p(s,t).$$

Theorem 1. *If separable random processes* X *and* Γ *satisfy condition (2) and, for some* $p \geq 2$,

$$\sum_{n=1}^{\infty} 2^{\frac{n}{p}} d_p(2^{-n}) < \infty, \tag{5}$$

then X, Γ, *and* $\overline{W}$ *belong to* $C[0,1]$ *a.s. and the weak convergence (4) is realized in* $C[0,1]$.

Remark 1. For the validity of the conditions of Theorem 1, it is sufficient that the well-known Kolmogorov condition (see [3, pp. 235–237])

$$\mathbf{E}\big|X(s) - X(s+h)\big|^p \leq K \cdot h^r, \qquad r > 1, \quad p \geq 2,$$

be satisfied. Indeed, by using the definition and the Kolmogorov condition, we immediately obtain

$$d_p(2^{-n}) = \sup_{T_{2^{-n}}} \big(\mathbf{E}|X(s) - X(t)|^p\big)^{\frac{1}{p}} \leq K^{\frac{1}{p}} 2^{-\frac{rn}{p}}.$$

It is clear that, for $r > 1$, this estimate guarantees convergence of series (5).

Note that, in the investigation of continuity of the random processes, the Kolmogorov condition is considered in the cases $p > 0$ or $p > 1$. However, in Theorem 1, it is necessary to set $p \geq 2$.

Proof of Theorem 1. It is well known that condition (5) guarantees the a.s. continuity of sample functions of a separable process X [11].

It is also clear that condition (5) yields the estimate

$$d_2(s, s+h) \leq d_p(s, s+h) \leq C \cdot h^{\frac{1}{p}}.$$

By using this estimate and (2), we get

$$\mathbf{E}\big|\Gamma(s) - \Gamma(s+h)\big|^2 = R(s,s) - 2R(s, s+h) + R(s+h, s+h)$$

$$= d_2^2(s, s+h) \leq C \cdot h^{\frac{2}{p}}.$$

This inequality guarantees continuity of sample functions from the separable normal process Γ [12, p. 192].

We now show that, under the conditions of Theorem 1, the random process $\overline{W}(s)$ also belongs to $C[0,1]$ a.s. To this end, we set

$$W_n(s,t) = \sum_{k=1}^{n} \Gamma_k(s) F_k(t), \quad n \in \mathbb{N}.$$

By using the elementary number inequality

$$\left| \max_{1 \leq k \leq n} a_k - \max_{1 \leq k \leq n} b_k \right| \leq \max_{1 \leq k \leq n} |a_k - b_k| \tag{6}$$

and the triangle inequality, we get

$$\sup_{|s-s'|<h} \left|\overline{W}(s) - \overline{W}(s')\right| \le \sup_{|s-s'|\le h} \sup_{t\in[0,1]} \left|W(s,t) - W(s',t)\right| \le 2D_1 + D_2, \tag{7}$$

where

$$D_1 = \sup_{s,t\in[0,1]} \left|W(s,t) - W_n(s,t)\right|,$$

$$D_2 = \sup_{|s-s'|\le h,\, t\in[0,1]} \left|W_n(s,t) - W_n(s',t)\right|.$$

As shown above, Γ_n belongs to the space $C[0,1]$ a.s. and, hence, series (3) uniformly converges with respect to t in the norm of $C[0,1]$. Thus, for any $\varepsilon > 0$, there exists $n = n(\varepsilon, \omega)$ such that

$$D_1 < \varepsilon. \tag{8}$$

For chosen n, the function $W_n(s,t)$ is uniformly continuous on $[0,1]^2$. Hence, there exists $h = h(n,\varepsilon,\omega)$ such that

$$D_2 < \varepsilon. \tag{9}$$

Since ε is arbitrary, estimates (7)–(9) mean that $\overline{W}(s)$ belong to $C[0,1]$ a.s.

We now proceed to the proof of the weak convergence (4). According to the results obtained in [3, pp. 482–483], in order that this convergence occur in the space $C[0,1]$, it is necessary and sufficient that the following conditions be satisfied:

(i) finite-dimensional distributions of the processes $\dfrac{\overline{S}_n(s)}{\sqrt{n}}$ converge to finite-dimensional distributions of the process $\overline{W}(s)$;

(ii) for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{T_h} \frac{1}{\sqrt{n}}\left|\overline{S}_n(s) - \overline{S}_n(t)\right| > \varepsilon\right\} \to 0 \qquad \text{as} \quad h \to 0. \tag{10}$$

The first condition directly follows from Lemma 1. Therefore, we focus our attention on the proof of the second condition. To simplify calculations, we set

$$Y_n(s,t) = \frac{|S_n(s) - S_n(t)|}{\sqrt{n}}$$

and first show that

$$\limsup_{n\to\infty} \mathbf{E} \sup_{T_h} Y_n(s,t) \to 0 \qquad \text{as} \quad h \to 0. \tag{11}$$

By using the estimates from [11] (Theorem 1A), we prove a more exact result

$$\mathbf{E}\sup_{T_h} Y_n(s,t) \le C_p \sum_{2^{-m}<h} \left| 2^{\frac{m}{p}} d_p(2^{-m}) \right|. \tag{12}$$

Here and in what follows, the constant C_p depends only on p and is not necessarily the same in different relations.

By using condition (5) and inequality (12), we immediately obtain equality (11).

We set

$$J = \left\{ k2^{-m} :\ m > 1,\ 0 \le k \le 2^m \right\}$$

and

$$\alpha_{nm} = \sup_{1\le k\le 2^m} Y_n\left(k2^{-m},\ [k-1]2^{-m}\right).$$

As shown in [11], for any $s, s' \in J,\ |s-s'| < h$,

$$Y_n(s,s') \le 2\sum_{2^{-m}<h} \alpha_{nm} \qquad \text{a.s.} \tag{13}$$

Since the inequality

$$\alpha_{nm}^p \le \sum_{k=1}^{2^m} \left| Y_n(k2^{-m},\ [k-1]2^{-m}) \right|^p$$

is true, by using estimate (13), we get

$$\begin{aligned} \mathbf{E} \sup_{|s-s'|<h,\, s,s'\in J} |Y_n(s,s')| &\le 2\sum_{2^{-m}<h} |\mathbf{E}\alpha_{nm}^p|^{\frac{1}{p}} \\ &\le 2\sum_{2^{-m}<h} \left(\sum_{k=1}^{2^m} \mathbf{E}\left|Y_n\left(k2^{-m},\ [k-1]2^{-m}\right)\right|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{14}$$

To estimate the terms in sum (14), we need the following lemma:

Lemma 2 [13]. *Suppose that $\xi_1, \xi_2, \dots, \xi_n$ are independent random variables, $\mathbf{E}\xi_i = 0$, and $2 \le p < \infty$. Then*

$$\mathbf{E}\left|\sum_{i=1}^n \xi_i\right|^p \le C_p \left(\sum_{i=1}^n \mathbf{E}|\xi_i|^p + \left(\sum_{i=1}^n \mathbf{E}|\xi_i|^2 \right)^{\frac{p}{2}} \right).$$

By using Lemma 2, we get

$$\begin{aligned} \mathbf{E}\left|Y_n\left(k2^{-m},\ [k-1]2^{-m}\right)\right|^p &= \mathbf{E}\left| \frac{1}{\sqrt{n}} S_n(k2^{-m}) - \frac{1}{\sqrt{n}} S_n([k-1]2^{-m}) \right|^p \\ &\le C_p\Big(n^{1-\frac{p}{2}} \mathbf{E}|X(k2^{-m}) - X\big([k-1]2^{-m}\big)|^p \end{aligned}$$

$$+ \left(\mathbf{E}\big|X(k2^{-m}) - X\big([k-1]2^{-m}\big)\big|^2\right)^{\frac{p}{2}}\Big)$$

$$\leq C_p \mathbf{E}\left|X(k2^{-m}) - X([k-1]2^{-m})\right|^p. \tag{15}$$

It is easy to see that

$$\sum_{k=1}^{2^m} \mathbf{E}\left|X(k2^{-m}) - X([k-1]2^{-m})\right|^p$$

$$\leq 2^m \sup_{T_{2^{-m}}} \mathbf{E}\big|X(s) - X(t)\big|^p = 2^m \big|d_p(2^{-m})\big|^p. \tag{16}$$

Combining estimates (14)–(16), we find

$$\mathbf{E} \sup_{|s-s'|<h,\, s,s' \in J} \big|Y_n(s,s')\big| \leq C_p \sum_{2^{-m}<h} 2^{\frac{m}{p}} d_p(2^{-m}). \tag{17}$$

Since the process $S_n(t)$ is continuous, inequality (12) is a direct corollary of inequality (17).

We now prove the implication (11) $\Rightarrow$ (10). To this end, we use inequality (6). We get

$$\mathbf{P}\left\{\sup_{T_h}\left|\max_{1\leq k\leq n} S_k(s) - \max_{1\leq k\leq n} S_k(t)\right| > \varepsilon\right\}$$

$$\leq \mathbf{P}\left\{\sup_{T_h}\max_{1\leq k\leq n} |S_k(s) - S_k(t)| > \varepsilon\right\}$$

$$= \mathbf{P}\left\{\max_{1\leq k\leq n}\sup_{T_h} |S_k(s) - S_k(t)| > \varepsilon\right\}. \tag{18}$$

Further, by $C(T_h)$ we denote a Banach space of continuous functions $x(s,t),\ (s,t) \in T_h,$ with uniform norm. Consider random functions

$$X_n'(s,t) = X_n(s) - X_n(t)$$

as elements of the space $C(T_h),\ \mathbf{E}X_n'(s,t) = 0,\ S_n'(s,t) = \sum_{k=1}^n X_k'(s,t)$. Let

$$\eta_n = \|S_n'\|_{C(T_h)}.$$

Then the sequence (η_n) forms a submartingale. Indeed, for $k < n,$ we get

$$\mathbf{E}_k\eta_n = \mathbf{E}_k\|S_n'\|_{C(T_h)} \geq \|\mathbf{E}_k S_n'\|_{C(T_h)} = \|S_k'\|_{C(T_h)} = \eta_k.$$

Here, $\mathbf{E}_k\eta$ denotes the conditional expectation of the random variable η for fixed random functions $X_i'(t,s),$ $i = \overline{1,k}.$

The submartingale η_k satisfies the inequality [3, p. 78]

$$\mathbf{P}\left\{\max_{1\le k\le n}\eta_k^+\ge\varepsilon\right\}\le\frac{1}{\varepsilon}\mathbf{E}\eta_n^+.$$

In addition, it is clear that

$$\eta_k=\sup_{T_h}\bigl|S_k(s)-S_k(t)\bigr|.$$

Hence,

$$\mathbf{P}\left\{\max_{1\le k\le n}\sup_{T_h}|S_k(s)-S_k(t)|>\varepsilon\right\}\le\frac{1}{\varepsilon}\mathbf{E}\left\{\sup_{T_h}|S_n(s)-S_n(t)|\right\}.$$

By using this result and relations (11) and (18), we obtain (10).

3. Space L_p

Consider the space $([0,1],\Lambda,\mu)$, where Λ is a σ-algebra of Borel sets for the segment $[0,1]$ and μ is the Lebesgue measure. By $L_p=L_p[0,1]$, $1\le p<\infty$, we denote a Banach space of (classes of) measurable functions $x(t)$ in the space $([0,1],\Lambda,\mu)$ with the norm

$$\|x\|_p=\left(\int_0^1|x(t)|^p\mu(dt)\right)^{1/p}.$$

To obtain relation (4), the following conditions were imposed in [8] on the random process $X(s)$ in the space L_p by using the notation $p^*=2$ for $p<2$ and $p^*=p$ for $p\ge 2$:

$$\sup_{0\le s\le 1}\mathbf{E}|X(s)|^{p^*}<\infty$$

and

$$\exists\,\varepsilon>0:\ \ \mathbf{E}|X(s)|^{p+\varepsilon}<\infty\quad\forall s\in[0,1].$$

According to the next theorem, these conditions can be replaced by a weaker condition. We set

$$\mathfrak{S}_p=\bigl(\sigma_p(s),s\in[0,1]\bigr),\qquad\sigma_p(s)=|\mathbf{E}|X(s)|^p|^{1/p}.$$

Theorem 2. *If the measurable random processes X and Γ satisfy condition (2) and the inequality*

$$\|\mathfrak{S}_{p^*}\|_{p^*}=\left(\int_0^1\sigma_{p^*}^{p^*}(s)ds\right)^{1/p^*}<\infty\tag{19}$$

is true, then X, Γ, and $\overline{W}$ belong to L_p a.s. and the weak convergence (4) in L_p is realized.

Proof. First, we assume that $p \geq 2.$ Then $p^* = p.$ By the Fubini theorem, (19) implies that

$$\int\limits_0^1 |X(s)|^p \, ds < \infty \quad \text{a.s.}$$

This means that X belongs to L_p a.s. and, moreover, we can assume that it is a random element in L_p [3, pp. 390–392]. It follows from the inequality [8]

$$\big(\mathbf{E}|\Gamma(s)|^p\big)^{\frac{1}{p}} \leq C_p\big(\mathbf{E}|X(s)|^p\big)^{\frac{1}{p}}$$

that the random process Γ also belongs to L_p a.s. Under the conditions of the theorem, the random processes Γ_n are measurable. Hence, $\overline{W}(s)$ is also measurable. In addition, for fixed $s,$ we have

$$W(s,t) \stackrel{D}{=} \sigma_2(s) \sum_{n=1}^{\infty} \gamma_n F_n(t),$$

where (γ_n) is a sequence of normal independent random variables, $\mathbf{E}\gamma_n = 0,$ $\mathbf{D}\gamma_n = 1,$ and the notation $\xi \stackrel{D}{=} \eta$ means that the distributions of the random variables ξ and η coincide.

Then

$$\overline{W}(s) \stackrel{D}{=} \sigma_2(s)|\gamma_1|$$

and, hence,

$$\mathbf{E}\big|\overline{W}(s)\big|^p = C_p \sigma_2^p(s) \leq C_p \sigma_p^p(s).$$

By using the last estimate and (19), we conclude that the random process $\overline{W}(s)$ belongs to L_p a.s.

Further, we use the well-known result from [14] (Theorem 7 and the remark made after this theorem).

Let $Z_n = \big\{Z_n(s),\ s \in [0,1]\big\}$ $n \geq 1,$ and $Z = \big\{Z(s),\ s \in [0,1]\big\}$ be measurable random processes. Then the following conditions are sufficient for the weak convergence $Z_n \stackrel{D}{\to} Z$ as $n \to \infty$ in L_p:

(i) the finite-dimensional distributions of the random processes Z_n converge to the finite-dimensional distributions of Z;

(ii) for any $\varepsilon > 0,$

$$\limsup_{n\to\infty} \mathbf{P}\left\{\int\limits_0^1 |Z_n(s)|^p I\big(|Z_n(s)| > L\big)\, ds > \varepsilon\right\} \to 0 \qquad \text{as} \quad L \to \infty.$$

Here and in what follows, by $I(A)$ we denote the indicator of the random event $A.$

By Lemma 1, condition (i) is satisfied. Hence, it remains to check condition (ii). Let

$$Y_n(s) = \frac{\overline{S}_n(s)}{\sqrt{n}}. \tag{20}$$

By the Markov inequality, for the validity of condition (ii), it suffices to show that

$$\limsup_{n\to\infty}\int_0^1 \mathbf{E}\big|Y_n(s)\big|^p I\big(\big|Y_n(s)\big|>L\big)\,ds\to 0\qquad\text{as}\quad L\to\infty. \tag{21}$$

Since the sequence $\big(|S_n(s)|\big)$ forms a positive submartingale with respect to n (see [3, p. 78]), for $p>1$, we find

$$\mathbf{E}\left|\frac{\overline{S}_n(s)}{\sqrt{n}}\right|^p\le\left(\frac{p}{p-1}\right)^p\mathbf{E}\left(\frac{|S_n(s)|}{\sqrt{n}}\right)^p. \tag{22}$$

By using Lemma 2 once again, we obtain

$$\mathbf{E}\left|\frac{S_n(s)}{\sqrt{n}}\right|^p\le C_p\left[n^{1-\frac{p}{2}}\mathbf{E}|X(s)|^p+\big(\mathbf{E}|X(s)|^2\big)^{\frac{p}{2}}\right]\le C_p|\sigma_p(s)|^p. \tag{23}$$

Thus, relations (19), (20) and (22), (23) show that the function

$$m_p(s)=\sup_{n\ge1}\big(\mathbf{E}|Y_n(s)|^p\big)^{\frac{1}{p}}\in L_p.$$

It is clear that the integrand in (21) does not exceed $|m_p(s)|^p$ for any $s\in[0,1]$. If we show that, for every $s\in[0,1]$,

$$\sup_{n\ge1}\mathbf{E}|Y_n(s)|^p I\big(|Y_n(s)|>L\big)\to 0\qquad\text{as}\quad L\to\infty, \tag{24}$$

then, by the Lebesgue theorem on the convergence of integrals, we arrive at equality (21).

We now formulate a lemma required to prove relation (24).

Lemma 3. *Suppose that (ξ_i) are independent identically distributed random variables and, for some $p\ge2$,*

$$\mathbf{E}\xi_n^2=\sigma^2,\qquad\mathbf{E}\xi_i=0,\qquad\mathbf{E}|\xi_i|^p<\infty, \tag{25}$$

$$\mathfrak{s}_n=\sum_{i=1}^n\xi_i,\qquad\textit{and}\qquad\bar{\mathfrak{s}}_n=\max\nolimits_{1\le k\le n}\mathfrak{s}_k.$$

Then

$$\sup_{n\ge1}\mathbf{E}\left|\frac{\bar{\mathfrak{s}}_n}{\sqrt{n}}\right|^p I\left(\left|\frac{\bar{\mathfrak{s}}_n}{\sqrt{n}}\right|>L\right)\to0\qquad\textit{as}\quad L\to\infty.$$

Prior to proving Lemma 3, we present two auxiliary lemmas.

Lemma 4 ([15, p. 68], Theorem 12)**.** *Under the conditions of Lemma 3, for any $x>0$,*

$$\mathbf{P}\left\{\max_{1\le k\le n}\frac{1}{\sqrt{n}}|\mathfrak{s}_k|>x\right\}\le 2\mathbf{P}\left\{\frac{1}{\sqrt{n}}|\mathfrak{s}_n|>x-\sqrt{2\sigma^2}\right\}.$$

Lemma 5. *Suppose that nonnegative random variables ξ and ζ satisfy the following conditions (a.s.):*

(i) $\mathbf{E}\xi^p < \infty$ *and* $\mathbf{E}\zeta^p < \infty$ *for some* $p \geq 1$*;*

(ii) there exist positive constants b and C such that

$$\mathbf{P}(\zeta > x) \leq C\mathbf{P}(\xi > x - b) \quad \forall x > 0.$$

Then there exist constants C_1 and C_2 such that, for any $L > b$,

$$\mathbf{E}\zeta^p I(\zeta > L) \leq C_1 \mathbf{E}\xi^p I(\xi > L - b) + C_2 \mathbf{P}(\xi > L - b), \tag{26}$$

where the constants C_1 and C_2 depend only on p, b, and C.

Proof of Lemma 5. We have

$$\begin{aligned}\mathbf{E}\zeta^p I(\zeta > L) &= \int_L^\infty x^p \, d\mathbf{P}(\zeta < x) \\ &= -\int_L^\infty x^p \, d\mathbf{P}(\zeta > x) \\ &= L^p \mathbf{P}(\zeta > L) + \int_L^\infty \mathbf{P}(\zeta > x) \, dx^p \\ &\leq C\Big(L^p \mathbf{P}(\xi > L - b) + \int_L^\infty \mathbf{P}(\xi > x - b) \, dx^p\Big).\end{aligned} \tag{27}$$

By using the number inequality

$$L^p \leq 2^{p-1}\big[(L-b)^p + b^p\big],$$

we get the following estimate for the first term in (27):

$$\begin{aligned}L^p \mathbf{P}(\xi > L - b) &\leq C_1 |L - b|^p \mathbf{P}(\xi > L - b) + C_2 \mathbf{P}(\xi > L - b) \\ &\leq C_1 \int_{L-b}^\infty x^p \, d\mathbf{P}(\xi < x) + C_2 \mathbf{P}(\xi > L - b).\end{aligned} \tag{28}$$

We now estimate the second term in (27) by the quantity

$$x^p \mathbf{P}(\xi > x - b)\big|_L^\infty + \int_L^\infty x^p \, d\mathbf{P}(\xi < x - b)$$

$$\leq \int\limits_{L-b}^{\infty} (y+b)^p \, d\mathbf{P}(\xi < y)$$

$$\leq C_1 \int\limits_{L-b}^{\infty} y^p \, d\mathbf{P}(\xi < y) + C_2 \mathbf{P}(\xi > L - b). \tag{29}$$

Since

$$\mathbf{E}\xi^p I(\xi > L - b) = \int\limits_{L-b}^{\infty} x^p \, d\mathbf{P}(\xi < x),$$

we obtain estimate (26) from (27)–(29).

Proof of Lemma 3. Choosing

$$\xi = \frac{\mathfrak{s}_n}{\sqrt{n}} \quad \text{and} \quad \zeta = \frac{\bar{\mathfrak{s}}_n}{\sqrt{n}},$$

in view of Lemmas 4 and 5, we find

$$\mathbf{E}\left|\frac{\bar{\mathfrak{s}}_n}{\sqrt{n}}\right|^p I\left(\frac{|\bar{\mathfrak{s}}_n|}{\sqrt{n}} > L\right) \leq C_1 \mathbf{E}\left|\frac{\mathfrak{s}_n}{\sqrt{n}}\right|^p I\left(\frac{|\mathfrak{s}_n|}{\sqrt{n}} > L - \sqrt{2\sigma^2}\right) + C_2 \mathbf{P}\left(\frac{|\mathfrak{s}_n|}{\sqrt{n}} > L - \sqrt{2\sigma^2}\right). \tag{30}$$

It is known [15, p. 130] that, under condition (25),

$$\frac{|\mathfrak{s}_n|}{\sqrt{n}} \xrightarrow{D} |\gamma_1|\sigma \tag{31}$$

and

$$\mathbf{E}\left|\frac{\mathfrak{s}_n}{\sqrt{n}}\right|^p \to \mathbf{E}|\gamma_1|^p \sigma^p \quad \text{as} \quad n \to \infty.$$

By using the last relation, we get (see [4, p. 51], Theorem 5.4)

$$\sup_n \mathbf{E}\left|\frac{\mathfrak{s}_n}{\sqrt{n}}\right|^p I\left(\left|\frac{\mathfrak{s}_n}{\sqrt{n}}\right| > L\right) \to 0 \qquad \text{as} \quad L \to \infty,$$

whence, in view of (30), (31), we get the assertion of Lemma 3.

Hence, we have proved Theorem 2 for $p \geq 2$.

The case $p < 2$ is reduced to the case $p = 2$ considered above. Indeed, in this case, $p^* = 2$ and the condition

$$\int\limits_0^1 \sigma_2^2(s) \, ds < \infty$$

is satisfied. Thus, the random processes X, Γ, and $\overline{W}$ belong to L_2 a.s. and, hence, definitely belong to L_p a.s.

As shown above, for $p=2,$ condition (20) is satisfied. Then it is also satisfied for $p<2.$
Theorem 2 is proved.

Theorem 2 yields the following corollary:

Corollary 1. *Suppose that $p \geq 2$ is a Banach function space $B \supset L_p$ and*

$$\|x\|_B \leq \|x\|_{L_p} \quad \forall\, x \in B.$$

If measurable random processes X and Γ satisfy condition (2) and the inequality

$$\|\mathfrak{S}_p\|_p < \infty$$

is true, then $X,$ $\Gamma,$ and $\overline{W}$ belong to B a.s. and the weak convergence (4) takes place in $B.$

We introduce integral functionals of the form

$$f(x(\cdot)) = \int_0^1 \varphi(s, x(s))\, ds,$$

where $\varphi(s,y)$ is a continuous function of two variables such that

$$\sup_{s\in[0,1]} \varphi(s,y) = O\big(|y|^p\big).$$

For these functionals, we obtain the following corollary from Theorem 2 and [14]:

Corollary 2. *If measurable random processes X and Γ satisfy conditions (2) and (21), then the following weak convergence takes place:*

$$f\left(\frac{S_n(\cdot)}{\sqrt{n}}\right) \xrightarrow{D} f(\overline{W}(\cdot)) \qquad \textit{as} \quad n\to\infty.$$

Remark 2. Under the conditions of Theorem 1, we consider the random processes

$$S_n^*(s) = \max_{0\leq k\leq n} \big|S_k(s)\big| \qquad \text{as} \quad W^*(s) = \sup_{0\leq t\leq 1} \big|W(t,s)\big|.$$

It follows from the results presented in [8] that the finite-dimensional distributions of the process $\dfrac{S_n^*(s)}{\sqrt{n}}$ converge to the finite-dimensional distributions of the random process $W^*(s).$

The analysis of the proof of Theorem 1 shows that it also enables us (without any significant changes) to establish the weak convergence in $C[0,1]$:

$$\frac{S_n^*(\cdot)}{\sqrt{n}} \xrightarrow{D} W^*(\cdot). \tag{32}$$

Similar conclusions are true for the space $L_p.$ More exactly, if the conditions of Theorem 2 are satisfied, then relation (32) is true in $L_p.$

4. Examples of Applications

It is clear that, for the application of Theorems 1 and 2, it is necessary to know the distributions of the corresponding limit random variables. Unfortunately, the problem of determination of these distributions is very complicated.

For the space L_p, we denote

$$\zeta_p = \int_0^1 \Big| \sup_{0 \le t \le 1} W(t,s) \Big|^p \, ds,$$

$$\mathfrak{S} = \big\{ \sigma(s),\ s \in [0,1] \big\}, \qquad \sigma(s) = \sigma_2(s), \quad s \in [0,1].$$

The following auxiliary statement gives simple estimates for the first two moments of the quantity ζ_p:

Lemma 6. *Under the conditions of Theorem 2,*

$$\mathbf{E}\zeta_p = \|\mathfrak{S}\|_p^p \cdot \Theta_p,$$

$$\mathbf{D}\zeta_p \le \|\mathfrak{S}\|_{2p}^{2p} \cdot \Theta_{2p} - \|\mathfrak{S}\|_p^{2p} \cdot \Theta_p^2,$$

where

$$\Theta_p = \sqrt{\frac{2^p}{\pi}}\, \mathbf{\Gamma}\Big(\frac{p+1}{2}\Big). \tag{33}$$

Moreover,

$$\Theta_{2k} = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1),$$

$\mathbf{\Gamma}(s)$ *is the gamma-function.*

Proof. As shown above,

$$\overline{W}(s) \overset{d}{=} \sigma(s)|\gamma|, \tag{34}$$

where γ is a standard normal random variable, $\mathbf{E}\gamma = 0$, and $\mathbf{E}\gamma^2 = 1$.

It is known [12, p. 32] that, for $p \ge 1$, we have

$$\mathbf{E}|\gamma|^p = \Theta_p, \tag{35}$$

where Θ_p is given by equality (33).

By using (34) and (35), we get

$$\mathbf{E}\zeta_p = \int_0^1 \mathbf{E}|\overline{W}(s)|^p \, ds = \Theta_p \int_0^1 |\sigma(s)|^p \, ds = \Theta_p \|\mathfrak{S}\|_p^p. \tag{36}$$

Further, we obtain

$$\mathbf{E}\zeta_p^2 \le \int_0^1 \mathbf{E}|\overline{W}(s)|^{2p}\, ds = \|\mathfrak{S}\|_{2p}^{2p} \cdot \Theta_{2p}.$$

By using this result and (36), we immediately arrive at the estimate for the variance $\mathbf{D}\xi_p$.

Let (u_i) be independent identically distributed random variables with the distribution function $F(x) = x$, $x \in [0,1]$, i.e., u_i are uniformly distributed on [0,1]. By

$$F_n^*(s) = \frac{1}{n}\sum_{n=1}^{n} I\big(u_i \in [0,s]\big), \quad s \in \mathbb{R},$$

we denote an empirical distribution function of the random variables $u_i,\ i = \overline{1,n}$.

By analogy with the classical statistics ω_n^2 and Ω_n^2, we consider their modifications:

$$n^{p/2}\omega_n^p = \int_0^1 \left| \frac{\sup_{1\le k\le n} k\big(F_k^*(s) - s\big)}{\sqrt{n}} \right|^p ds,$$

$$n^{p/2}\Omega_n^p = \int_0^1 \left| \frac{\sup_{1\le k\le n} k(F_k^*(s) - s)}{\sqrt{n}} \right|^p \big(s(1-s)\big)^{-1} ds.$$

By $W_0(s),\ s \in [0,1]$, we denote the normal random process for which

$$\mathbf{E}W_0(s) = 0,$$

$$\mathbf{E}W_0(s_1)W_0(s_2) = \min(s_1, s_2) - s_1 s_2.$$

This process is called a *Brownian bridge*. Assume that, in representation (3), for the process $W(t,s)$, we have $\Gamma_n \overset{d}{=} W_0$ and

$$\overline{W}(s) = \sup_{0\le t\le 1} W(t,s).$$

Then Theorem 2 yields the following corollary:

Corollary 3. *As* $n \to \infty$

$$n^{p/2}\omega_n^p \overset{D}{\to} \zeta_p[1] = \int_0^1 \big|\overline{W}(s)\big|^p\, ds, \tag{37}$$

$$n^{p/2}\Omega_n^p \overset{D}{\to} \zeta_p\left[\frac{1}{s(1-s)}\right] = \int_0^1 \frac{\big|\overline{W}(s)\big|^p}{s(1-s)}\, ds. \tag{38}$$

Proof. We set

$$X_i(s) = I\big(u_i \in (0, s)\big) - s.$$

Then

$$\mathbf{E}X_i(s) = 0,$$

$$\mathbf{E}X_i(s_1)X_i(s_2) = R(s_1, s_2) = \min(s_1, s_2) - s_1 s_2,$$

i.e., the correlation functions of the processes $X_i(s)$ and $W_0(s)$ coincide. In order to use Theorem 2, it remains to check condition (21). Thus, we have

$$|\sigma_p(s)|^p = \mathbf{E}|X_i(s)|^p = (1-s)s\big[(1-s)^{p-1} + s^{p-1}\big] \le 1. \tag{39}$$

The last estimate means that condition (21) is satisfied and, hence, (37) is true.

Relation (38) is proved in a similar way. It is necessary to choose

$$X_i(s) = \frac{I(u_i \in (0, s)) - s}{|s(1-s)|^{1/p}} \qquad \text{and} \qquad \Gamma_n(s) \overset{d}{=} \frac{W_0(s)}{|s(1-s)|^{1/p}}.$$

As a result of simple calculations, we conclude that

$$|\sigma_p(s)|^p = |1-s|^{p-1} + s^{p-1}, \qquad \int_0^1 |\sigma_p(s)|^p = \frac{2}{p}. \tag{40}$$

Hence, conditions (2) and (21) are satisfied. This yields (38).

Further, by using Lemma 6, we establish the estimates for the first moments of the limit variables (37) and (38) for $p = 2$.

For $\zeta_2[1]$, we find

$$\mathfrak{S}^2(s) = (\sigma_2(s))^2 = s - s^2,$$

$$\mathbf{E}\zeta_2[1] = \int_0^1 \sigma^2(s)\,ds = \int_0^1 (s - s^2)\,ds = \frac{1}{6},$$

$$\mathbf{D}\zeta_2[1] \le \Theta_4\|\mathfrak{S}\|_4^4 - \|\mathfrak{S}\|_2^4\Theta_2^2 = 3\int_0^1 |\sigma(s)|^4\,ds - \left[\int_0^1 |\sigma(s)|\,ds\right]^2 = \frac{13}{180}.$$

By using equalities (40), for $\zeta_2\left[\dfrac{1}{s(1-s)}\right]$, we obtain

$$\mathbf{E}\,\zeta_2\left[\frac{1}{s(1-s)}\right] = 1,$$

$$\mathbf{D}\,\zeta_2\left[\frac{1}{s(1-s)}\right] \le 2.$$

REFERENCES

1. E. Erdös and M. Kac, "On certain limit theorems in the theory of probability," *Bull. Amer. Math. Soc.*, **52**, 292–302 (1946).
2. P. Bachelier, "Theorie de la speculation," *Ann. Ecol. Norm.*, **17**, 21–86 (1900).
3. I. I. Gikhman and A. V. Skorokhod, *Theory of Random Processes* [in Russian], Vol. 1, Nauka, Moscow (1971).
4. P. Billingsley, *Convergence of Probability Measures* [Russian translation], Nauka, Moscow (1977).
5. A. V. Skorokhod and N. P. Slobodenyuk, *Limit Theorems for Random Walks* [in Russian], Naukova Dumka, Kiev (1970).
6. V. Paulauskas, "On the distribution of maximum for the consecutive sums of independent identically distributed random vectors," *Liet. Mat. Rink.*, **13**, No. 2, 133–138 (1973).
7. J. Lamperti, *Probability* [Russian translation], Nauka, Moscow (1973).
8. I. K. Matsak, "On some limit theorems for the maximum of sums of independent random processes," *Ukr. Mat. Zh.*, **60**, No. 12, 1664–1674 (2008); ***English translation:*** *Ukr. Math. J.*, **60**, No. 12, 1955–1967 (2008).
9. P. Levy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris (1937).
10. Z. Ciesielsky, "Hölder condition for realizations of Gaussian processes," *Trans. Amer. Math. Soc.*, **99**, 403–413 (1961).
11. I. K. Matsak, "Regularity of sampling distribution functions of a random process," *Ukr. Mat. Zh.*, **30**, No. 2, 242–247 (1978); ***English translation:*** *Ukr. Math. J.*, **30**, No. 2, 186–190 (1978).
12. H. Craḿer and M. R. Leadbetter, *Stationary and Related Stochastic Processes* [Russian translation], Mir, Moscow (1969).
13. H. P. Rosenthal, "On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables," *Isr. J. Math.*, **8**, No. 3, 273–303 (1970).
14. A. A. Borovkov and E. A. Pecherskii, "Convergence of the distributions of integral functionals," *Sib. Mat. Zh.*, **16**, No. 5, 899–915 (1975).
15. V. V. Petrov, *Sums of Independent Random Variables* [in Russian], Nauka, Moscow (1972).