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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF ORDINARY SECOND-ORDER DIFFERENTIAL EQUATIONS WITH RAPIDLY VARYING NONLINEARITIES

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We establish conditions for the existence of one class of solutions of two-term nonautonomous second-order differential equations with rapidly varying nonlinearities and obtain the asymptotic representations for these solutions and their first-order derivatives as $t \uparrow \omega$ $(\omega \le +\infty)$.

1. Introduction

Consider a differential equation

$$y'' = \alpha_0 p(t)\varphi(y), \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $p\colon [a,\omega[\, \longrightarrow\,]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, and

$$\varphi\colon\ \Delta_{Y_0} \longrightarrow\,]0,+\infty[$$

is a twice continuously differentiable function such that

$$\varphi'(y) \ne 0 \quad \text{for} \quad y \in \Delta_{Y_0}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0,\\ \text{or} & +\infty, \end{cases} \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{1.2}$$

where Y_0 is equal either to zero or to $\pm\infty$ and Δ_{Y_0} is one-sided neighborhood of Y_0.

It follows from the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)'}{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)^2} + 1 \quad \text{for} \quad y \in \Delta_{Y_0}$$

and conditions (1.2) that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \quad \text{as} \quad y \to Y_0\ (y \in \Delta_{Y_0}) \qquad \text{and} \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty. \tag{1.3}$$

Hence, in the considered equation, the function φ and its first-order derivative are rapidly varying as $y \to Y_0$ (see [1, pp. 91, 92], Chap. 3, Sec. 3.4, Lemmas 3.2 and 3.3).

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Under conditions (1.2), the asymptotic behavior of solutions of the differential equation (1.1) was investigated in the monograph [1, pp. 90–99] (Chap. 3, Sec. 3.4) in a special case where $\alpha_0 = 1,\ \omega = +\infty,\ Y_0 = 0,$ and p is a regularly varying function as $t \to +\infty$. In the general case, this problem was investigated in [2]. However, the class of solutions studied in [2] was expressed via a function φ, which is not natural. Indeed, it is more natural to study for Eq. (1.1) the same class of solutions as earlier (see, e.g., [3]) in the case of a function φ regularly varying as $y \to Y_0$.

Definition 1.1. *A solution y of the differential equation (1.1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions*

$$y(t) \in \Delta_{Y_0} \quad \textit{for} \quad t \in [t_0, \omega[,$$

$$\lim_{t\uparrow\omega} y(t) = Y_0, \qquad \lim_{t\uparrow\omega} y'(t) = \begin{cases} \textit{either} & 0, \\ \textit{or} & \pm\infty, \end{cases} \qquad \lim_{t\uparrow\omega} \frac{y'^2(t)}{y''(t)y(t)} = \lambda_0.$$

The aim of the present paper is to establish necessary and sufficient conditions for the existence of $P_\omega(Y_0, \lambda_0)$-solutions of Eq. (1.1) in the nonsingular case where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, as well as the asymptotic representations for these solutions and their first-order derivatives as $t \uparrow \omega$.

2. Some Auxiliary Statements

We first recall a series of important properties of a class of twice continuously differentiable functions

$$f\colon\ \Delta_{Y_0} \longrightarrow \mathbb{R} \setminus \{0\},$$

where Y_0 is equal either to zero or to $\pm\infty$ and Δ_{Y_0} is a one-sided neighborhood of Y_0, each of which satisfies the conditions

$$f'(y) \ne 0 \quad \text{for} \quad y \in \Delta_{Y_0}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y)f''(y)}{f'^2(y)} = 1. \tag{2.1}$$

In what follows, without loss of generality, we assume that

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[& \text{if} \quad \Delta_{Y_0} \ \text{is a left neighborhood of} \quad Y_0, \\ ]Y_0, y_0] & \text{if} \quad \Delta_{Y_0} \ \text{is a right neighborhood of} \quad Y_0, \end{cases} \tag{2.2}$$

where $y_0 \in \mathbb{R}$ is such that $|y_0| < 1$ for $Y_0 = 0$ and $y_0 > 1$ ($y_0 < -1$) for $Y_0 = +\infty$ (for $Y_0 = -\infty$).

In addition to the asymptotic relations (1.3) with φ replaced by f, these functions satisfy the following assertion:

Lemma 2.1. *If a twice continuously differentiable function $f\colon\ \Delta_{Y_0} \longrightarrow \mathbb{R} \setminus \{0\}$, where Y_0 is equal either to zero or to $\pm\infty$ and Δ_{Y_0} is a one-sided neighborhood of Y_0, satisfies conditions (2.1), then*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f^2(y)}{f'(y)\displaystyle\int_Y^y f(x)\,dx} = 1, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\left[\displaystyle\int_Y^y f(x)\,dx\right]^2}{f(y)\displaystyle\int_Y^y \left(\int_Y^x f(u)\,du\right) dx} = 1, \tag{2.3}$$

where

$$Y = \begin{cases} y_0 & \textit{for} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = +\infty, \\ Y_0 & \textit{for} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = 0. \end{cases}$$

Proof. By virtue of the asymptotic relations (1.3) with φ replaced by f and the choice of the limit of integration Y, each integral in (2.3) tends either to zero or to $\pm\infty$ as $y \to Y_0$.

In view of this fact, we first prove the validity of the first limit relation in (2.3). We set

$$z(y) = \frac{f^2(y)}{f'(y) \displaystyle\int_Y^y f(x)\,dx}. \tag{2.4}$$

Then

$$z'(y) = \frac{2f(y)}{\displaystyle\int_Y^y f(x)\,dx} - \frac{f^2(y) f''(y)}{f'^2(y) \int_Y^y f(x)\,dx} - \frac{f^3(y)}{f'(y) \left[\int_Y^y f(x)\,dx\right]^2}$$

$$= \frac{f(y)}{\int_Y^y f(x)\,dx} \left[2 - \frac{f''(y) f(y)}{f'^2(y)} - z(y)\right],$$

i.e., function (2.4) is a solution of the differential equation

$$z' = \frac{f(y)}{\displaystyle\int_Y^y f(x)\,dx} \left[2 - \frac{f''(y) f(y)}{f'^2(y)} - z\right]. \tag{2.5}$$

We now write the following function corresponding to this equation:

$$F(y, c) = \frac{f(y)}{\displaystyle\int_Y^y f(x)\,dx} \left[2 - \frac{f''(y) f(y)}{f'^2(y)} - c\right].$$

By virtue of conditions (2.1), for any real $c \neq 1$, this function preserves sign in a certain neighborhood of Y_0 contained in Δ_{Y_0}. Thus, by Lemma 2.1 in [3], for any solution of the differential equation (2.5) defined in a neighborhood of Y_0 contained in Δ_{Y_0} and, hence, also for function (2.4), there exists the limit (finite or equal to $\pm\infty$) as $y \to Y_0$. We now show that this limit is equal to one. Assume the contrary. Then

$$\text{either} \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y) = c = \text{const} \neq 1 \qquad \text{or} \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y) = \pm\infty.$$

In the first case, in view of (2.1), it follows from (2.5) that

$$z'(y) = \frac{f(y)}{\displaystyle\int_Y^y f(x)\,dx} [1 - c + o(1)] \qquad \text{as} \quad y \to Y_0.$$

Integrating this relation from y_1 to y, where y_1 is an arbitrary interior point of the segment with endpoints y_0 and Y_0 and taking into account that $\int_Y^y f(x)\,dx$ tends either to zero or to $\pm\infty$ as $y \to Y_0$ and $c \neq 1$, we obtain

$$z(y) - z(y_1) = [1 - c + o(1)] \ln \left| \int\limits_Y^y f(x)\,dx \right| \longrightarrow \pm\infty \qquad \text{as} \quad y \to Y_0.$$

However, this is impossible because the expression on the left-hand side has a finite limit as $y \to Y_0$.

We now assume that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y) = \pm\infty.$$

In this case, in view of (2.4), we rewrite the expression for $z'(y)$ in the form

$$z'(y) = \frac{f'(y)}{f(y)} z(y) \left[2 - \frac{f''(y) f(y)}{f'^2(y)} - z(y) \right].$$

In view of the last condition in (2.1) and our assumption, we get

$$z'(y) = -\frac{f'(y)}{f(y)} z^2(y) \big[1 + o(1)\big] \qquad \text{as} \quad y \to Y_0.$$

Since $f(y)$ tends either to zero or to $+\infty$ as $y \to Y_0$, we can divide both sides of this relation by $z^2(y)$ and then integrate from y_1 to y. This yields

$$-\frac{1}{z(y)} + \frac{1}{z(y_1)} = \big[1 + o(1)\big] \ln f(y) \longrightarrow \pm\infty \qquad \text{as} \quad y \to Y_0.$$

Hence, we arrive at a contradiction because the limit of the expression on the left-hand side as $y \to Y_0$ is equal to a constant $\dfrac{1}{z(y_1)}$.

In view of the contradictions obtained in the analyzed two cases, we conclude that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y) = 1$$

and, hence, the first limit relation in (2.3) is true.

Similarly, by using the already established first limit in (2.3), we prove the second limit.

The lemma is proved.

By virtue of this lemma and Theorem 3.10.8 in [5, p. 178], a twice continuously differentiable function

$$f\colon \Delta_{Y_0} \longrightarrow\,]0, +\infty[$$

satisfying conditions (2.1) belongs, for $Y_0 = +\infty$ and under the condition $\lim_{y \to +\infty} f(y) = +\infty$, to the class of functions Γ introduced by Hahn (see, e.g., [5, p. 175]).

Definition 2.1. *The class* Γ *is formed by measurable nondecreasing and right-continuous functions*

$$f\colon\ [y_0,+\infty[\ \longrightarrow\ ]0,+\infty[$$

for each of which there exists a measurable function $g\colon\ [y_0,+\infty[\ \longrightarrow\ ]0,+\infty[$ *complementary for the function* f *such that*

$$\lim_{y\to+\infty}\frac{f\left(y+ug(y)\right)}{f(y)}=e^u \qquad \text{for any} \quad u\in\mathbb{R}.$$

The functions from the class Γ satisfy, in particular, the following assertions (see [5, pp. 174–178]):

Lemma 2.2.

1. *If* $f\in\Gamma$ *with a complementary function* g, *then*

$$\lim_{y\to+\infty}\frac{g(y)}{y}=0.$$

2. *If* $f\in\Gamma$ *with a complementary function* g, *then, for any function* $u\colon\ [y_0,+\infty[\ \longrightarrow\mathbb{R}$ *satisfying the conditions*

$$\lim_{y\to+\infty}u(y)=u_0\in[-\infty,+\infty], \qquad \lim_{y\to+\infty}f(y+u(y)g(y)]=+\infty,$$

the limit relation

$$\lim_{y\to+\infty}\frac{f\left(y+u(y)g(y)\right)}{f(y)}=e^{u_0}$$

is true.

3. *For* $f\in\Gamma$, *the complementary function is unique to within functions equivalent as* $y\to+\infty$ *and, e.g., the function* $\dfrac{\int_{y_0}^{y}f(x)\,dx}{f(y)}$ *can be chosen as one of these functions.*

4. *The conditions* $f\in\Gamma$ *and*

$$\lim_{y\to+\infty}\frac{\left[\int_{y_0}^{y}f(x)\,dx\right]^2}{f(y)\int_{y_0}^{y}\left(\int_{y_0}^{x}f(u)\,du\right)dx}=1$$

are equivalent, i.e., the first condition implies the second condition, and vice versa.

By the change of variables, the class Γ can be easily extended to a class $\Gamma_{Y_0}(Z_0)$ of functions

$$f\colon\ \Delta_{Y_0}\longrightarrow\ ]0,+\infty[,$$

where Y_0 is equal either to zero or to $\pm\infty$ and Δ_{Y_0} is a one-sided neighborhood of Y_0 for which

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}f(y)=Z_0=\begin{cases}\text{either} & 0,\\ \text{or} & +\infty.\end{cases}$$

Definition 2.2. *We say that a function* $f:\ \Delta_{Y_0} \longrightarrow]0,+\infty[$ *belongs to the class of functions* $\Gamma_{Y_0}(Z_0)$ *if the following functions belong to the class* Γ :

(1) the function $f_0(y) = \dfrac{1}{f(y)}$ *for* $Y_0 = +\infty$ *and* $Z_0 = 0$;

(2) the function $f_0(y) = f(-y)$ *for* $Y_0 = -\infty$ *and* $Z_0 = +\infty$;

(3) the function $f_0(y) = f\left(\dfrac{1}{y}\right)$ *for* $Y_0 = 0$ *in the case where* Δ_{Y_0} *is a right neighborhood of zero and* $Z_0 = +\infty$;

(4) the function $f_0(y) = \dfrac{1}{f\left(\dfrac{1}{y}\right)}$ *for* $Y_0 = 0$ *in the case where* Δ_{Y_0} *is a right neighborhood of zero and* $Z_0 = 0$;

(5) the function $f_0(y) = f\left(-\dfrac{1}{y}\right)$ *for* $Y_0 = 0$ *in the case where* Δ_{Y_0} *is a left neighborhood of zero and* $Z_0 = +\infty$;

(6) the function $f_0(y) = \dfrac{1}{f\left(-\dfrac{1}{y}\right)}$ *for* $Y_0 = 0$ *in the case where* Δ_{Y_0} *is a left neighborhood of zero and* $Z_0 = 0$;

(7) the function $f_0(y) \equiv f(y)$ *for* $Y_0 = +\infty$ *and* $Z_0 = +\infty$.

By using these two definitions and the first two assertions of Lemma 2.2, we conclude that the function $f \in \Gamma_{Y_0}(Z_0)$ satisfies the limit relation

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y+ug(y))}{f(y)} = e^u \quad \text{for any} \quad u \in \mathbb{R}, \tag{2.6}$$

where, in each case (1)–(7), the function g complementary for f can be expressed via the function g_0 complementary for f_0 as follows:

(1) $g(y) = -g_0(y)$;

(2) $g(y) = -g_0(-y)$;

(3) $g(y) = -y^2 g_0\left(\dfrac{1}{y}\right)$;

(4) $g(y) = y^2 g_0\left(\dfrac{1}{y}\right)$;

(5) $g(y) = y^2 g_0\left(-\dfrac{1}{y}\right)$;

(6) $g(y) = -y^2 g_0\left(-\dfrac{1}{y}\right)$;

(7) $g(y) = g_0(y)$.

Here, by virtue of the third assertion in Lemma 2.2, every function $g_0\colon\ [x_0, +\infty[\ \longrightarrow\ ]0, +\infty[$ is uniquely defined to within functions equivalent as $x \to +\infty$. As one of these functions, we can take, e.g., the function

$$\frac{\int\limits_{x_0}^{x} f_0(s)\,ds}{f_0(x)}.$$

By using the first two assertions of Lemma 2.2, we arrive at the following lemma:

Lemma 2.3.

1. *If $f \in \Gamma_{Y_0}(Z_0)$ with a complementary function g, then*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{g(y)}{y} = 0.$$

2. *If $f \in \Gamma_{Y_0}(Z_0)$ with a complementary function g, then, for any function $u\colon \Delta_{Y_0} \longrightarrow \mathbb{R}$ satisfying the conditions*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} u(y) = u_0 \in \mathbb{R}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y + u(y)g(y)] = Z_0,$$

the limit relation

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f\left(y + u(y)g(y)\right)}{f(y)} = e^{u_0}$$

is true.

If $f \in \Gamma_{Y_0}(Z_0)$ is a function with complementary function g and, in addition, it is continuous and strictly monotone, then there exists a continuous strictly monotone inverse function $f^{-1}\colon \Delta_{Z_0} \longrightarrow \Delta_{Y_0}$ such that

$$\Delta_{Z_0} = \begin{cases} \text{either} & [z_0, Z_0[, \\ \text{or} &]Z_0, z_0], \end{cases} \qquad z_0 = f(y_0), \quad Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y).$$

By virtue of Theorems 3.1.16 and 3.10.4 in [5, pp. 139, 176] and Definition 2.2, this inverse function has the following properties:

Lemma 2.4. *Suppose that $f \in \Gamma_{Y_0}(Z_0)$ is a function with complementary function g continuous and strictly monotone on the segment Δ_{Y_0}. Then its inverse function $f^{-1}(z)$ slowly varies as $z \to Z_0$ and satisfies the limit relation*

$$\lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}} \frac{f^{-1}(\lambda z) - f(z)}{g(f^{-1}(z))} = \ln\lambda \qquad \textit{for any} \quad \lambda > 0.$$

Moreover, for any $\Lambda > 1$, this limit relation is uniformly true in $\lambda \in \left[\dfrac{1}{\Lambda}, \Lambda\right]$.

Finally, we consider the case where the function $f\colon \Delta_{Y_0} \longrightarrow]0,+\infty[$ is twice continuously differentiable and satisfies conditions (2.1). In this case, any function $f_0\colon [x_0,+\infty[\longrightarrow]0,+\infty[$, where x_0 is a positive number, indicated in Definition 2.2 satisfies the conditions

$$f_0'(x) \neq 0 \quad \text{for} \quad x \in [x_0,+\infty[, \qquad \lim_{x\to+\infty} f_0(x) = +\infty, \qquad \lim_{x\to+\infty} \frac{f_0(x) f_0''(x)}{f_0'^2(x)} = 1.$$

By virtue of these conditions, Lemma 2.1, and the third and fourth assertions of Lemma 2.2, we get the following statement:

Lemma 2.5. *If a twice continuously differentiable function* $f\colon \Delta_{Y_0} \longrightarrow]0,+\infty[$ *satisfies conditions (2.1), then this function belongs to the class* $\Gamma_{Y_0}(Z_0)$ *together with the complementary function* $g\colon \Delta_{Y_0} \longrightarrow \mathbb{R}$, *which is uniquely defined to within the equivalence of functions as* $y \to Y_0$. *As this function, one can take, e.g., one of the functions*

$$\frac{\displaystyle\int_Y^y \left(\int_Y^t f(u)\,du \right) dt}{\displaystyle\int_Y^y f(x)\,dx} \sim \frac{\displaystyle\int_Y^y f(x)\,dx}{f(y)} \sim \frac{f(y)}{f'(y)} \sim \frac{f'(y)}{f''(y)} \qquad \textit{as} \quad y \to Y_0,$$

where the limit of integration Y *is the same as in (2.2).*

Remark 2.1. Lemmas 2.3–2.5 belong to the case where the function $f\colon \Delta_{Y_0} \longrightarrow]0,+\infty[$ (i.e., takes positive values). We say that the function $f\colon \Delta_{Y_0} \longrightarrow]-\infty,0[$ belongs to the class $\Gamma_{Y_0}(Z_0)$ if $(-f) \in \Gamma_{Y_0}(-Z_0)$. Then it is easy to see that Lemmas 2.3–2.5 remain true for this function.

In what follows, in addition to the above-mentioned properties of twice continuously differentiable functions $f\colon \Delta_{Y_0} \longrightarrow \mathbb{R} \setminus \{0\}$ satisfying conditions (2.1), we also need one more auxiliary statement about the *a priori* asymptotic properties of the $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1), which follows from Corollary 10.1 in [6].

Lemma 2.6. *If* $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$, *then, for any* $P_\omega(Y_0,\lambda_0)$*-solution of the differential equation (1.1), the following asymptotic relations are true:*

$$\frac{\pi_\omega(t) y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}\big[1 + o(1)\big], \qquad \frac{\pi_\omega(t) y''(t)}{y'(t)} = \frac{1 + o(1)}{\lambda_0 - 1} \qquad \textit{as} \quad t \uparrow \omega, \tag{2.7}$$

where

$$\pi_\omega(t) = \begin{cases} t & \textit{for} \quad \omega = +\infty, \\ t - \omega & \textit{for} \quad \omega < +\infty. \end{cases} \tag{2.8}$$

3. Main Results

First, we introduce the notation necessary in what follows. Assume that the domain of definition of the function φ in Eq. (1.1) is given by relation (2.2). Further, we set

$$\mu_0 = \operatorname{sign} \varphi'(y), \qquad \nu_0 = \operatorname{sign} y_0, \qquad \nu_1 = \begin{cases} 1 & \text{for} \quad \Delta_{Y_0} = [y_0, Y_0[, \\ -1 & \text{for} \quad \Delta_{Y_0} =]Y_0, y_0], \end{cases}$$

and introduce a function

$$J(t) = \int\limits_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \qquad \Phi(y) = \int\limits_B^y \frac{ds}{\varphi(s)},$$

where π_ω is given by relation (2.8),

$$A = \begin{cases} \omega & \text{for} \quad \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = \text{const}, \\ a & \text{for} \quad \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = \pm\infty, \end{cases} \qquad B = \begin{cases} Y_0 & \text{for} \quad \displaystyle\int_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \text{const}, \\ y_0 & \text{for} \quad \displaystyle\int_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \pm\infty. \end{cases}$$

In view of the definition of a $P_\omega(Y_0, \lambda_0)$-solution of the differential equation (1.1), we note that the numbers ν_0, ν_1, and α_0 determine the signs of any $P_\omega(Y_0, \lambda_0)$-solution and its first and second derivatives, respectively, in a left neighborhood of ω. In this case, it is clear that the conditions

$$\nu_0\nu_1 < 0 \quad \text{for} \quad Y_0 = 0, \qquad \nu_0\nu_1 > 0 \quad \text{for} \quad Y_0 = \pm\infty, \tag{3.1}$$

and

$$\nu_1\alpha_0 < 0 \quad \text{for} \quad \lim_{t\uparrow\omega} y'(t) = 0, \qquad \nu_1\alpha_0 > 0 \quad \text{for} \quad \lim_{t\uparrow\omega} y'(t) = \pm\infty, \tag{3.2}$$

are necessary for the existence of these solutions. Moreover, according to Lemma 2.6, for $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$, we find

$$\nu_0\nu_1 = \operatorname{sign}[\,\lambda_0(\lambda_0 - 1)\pi_\omega(t)], \qquad \nu_1\alpha_0 = \operatorname{sign}[(\lambda_0 - 1)\pi_\omega(t)] \qquad \text{for} \quad t \in [a, \omega[. \tag{3.3}$$

In particular, this yields

$$\alpha_0\nu_0\lambda_0 > 0. \tag{3.4}$$

We now mention some properties of the function Φ. It preserves sign on the segment Δ_{y_0}, tends either to zero or to $\pm\infty$ as $y \to Y_0$, and is increasing on Δ_{Y_0} because

$$\Phi'(y) = \frac{1}{\varphi(y)} > 0$$

in this segment. Therefore, this function possesses the inverse function

$$\Phi^{-1} \colon \Delta_{Z_0} \longrightarrow \Delta_{Y_0},$$

where, by virtue of the second condition in (1.2) and the fact that Φ^{-1} monotonically increases,

$$Z_0 = \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \Phi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \qquad \Delta_{Z_0} = \begin{cases} [z_0, Z_0[& \text{for} \quad \Delta_{Y_0} = [y_0, Y_0[, \\ ]Z_0, z_0] & \text{for} \quad \Delta_{Y_0} =]Y_0, y_0], \end{cases} \qquad z_0 = \varphi(y_0). \tag{3.5}$$

By virtue of the L'Hospital rule in the Stolz form and the last condition in (1.2), we find

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\Phi(y)}{\dfrac{1}{\varphi'(y)}} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\dfrac{1}{\varphi(y)}}{-\dfrac{\varphi''(y)}{\varphi'^2(y)}} = -\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi'^2(y)}{\varphi''(y)\varphi(y)} = -1.$$

Hence,

$$\Phi(y) \sim -\frac{1}{\varphi'(y)} \quad \text{as} \quad y\to Y_0 \qquad \text{and} \qquad \operatorname{sign}\Phi(y) = -\mu_0 \quad \text{for} \quad y\in\Delta_{Y_0}. \tag{3.6}$$

By using the first relation, we get

$$\frac{\Phi'(y)}{\Phi(y)} = \frac{\dfrac{1}{\varphi(y)}}{\Phi(y)} \sim -\frac{\varphi'(y)}{\varphi(y)}, \qquad \frac{\Phi''(y)\Phi(y)}{\Phi'^2(y)} = \frac{-\dfrac{\varphi'(y)}{\varphi^2(y)}\Phi(y)}{\dfrac{1}{\varphi^2(y)}} \sim 1 \qquad \text{as} \quad y\to Y_0. \tag{3.7}$$

Therefore, according to Lemma 2.5, $\Phi \in \Gamma_{Y_0}(Z_0)$ with complementary function. As this function, we can take one of the following equivalent functions:

$$\frac{\Phi'(y)}{\Phi''(y)} \sim \frac{\Phi(y)}{\Phi'(y)} \sim -\frac{\varphi(y)}{\varphi'(y)} \qquad \text{as} \quad y\to Y_0. \tag{3.8}$$

In addition to the notation introduced above, we also consider auxiliary functions

$$q(t) = \frac{\alpha_0(\lambda_0-1)\pi_\omega^2(t)\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))},$$

$$H(t) = \frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\varphi'\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}.$$

The following statement is true for Eq. (1.1):

Theorem 3.1. *Suppose that $\lambda_0 \in \mathbb{R}\backslash\{0;1\}$. Then, for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1), it is necessary that, parallel with (3.4), the following conditions be satisfied:*

$$\alpha_0\mu_0(\lambda_0-1)J(t) < 0 \qquad \textit{for} \quad t\in]a,\omega[, \tag{3.9}$$

$$\alpha_0(\lambda_0-1)\lim_{t\uparrow\omega} J(t) = Z_0, \qquad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)} = \pm\infty, \qquad \lim_{t\uparrow\omega} q(t) = \frac{\lambda_0}{\lambda_0-1}. \tag{3.10}$$

Moreover, each of these solutions admits the asymptotic representations

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0-1)J(t)\big)\left[1+\frac{o(1)}{H(t)}\right] \qquad \textit{as} \quad t\uparrow\omega, \tag{3.11}$$

$$y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \frac{\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)}{\pi_\omega(t)} \big[1 + o(1)\big] \qquad as \quad t \uparrow \omega. \tag{3.12}$$

Theorem 3.2. *Suppose that* $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$ *and, parallel with (3.4), (3.9), and (3.10), the following finite (or equal to* $\pm\infty$*) limit exists:*

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)'}{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)^2} \sqrt{\left|\frac{y\varphi'(y)}{\varphi(y)}\right|}. \tag{3.13}$$

Then the following assertions are true:

(1) If

$$(\lambda_0 - 1)J(t) < 0 \qquad for \quad t \in \,]a, \omega[\qquad and \qquad \lim_{t \uparrow \omega} \left[\frac{\lambda_0}{\lambda_0 - 1} - q(t)\right] |H(t)|^{1/2} = 0, \tag{3.14}$$

then there exists a one-parameter family of $P_\omega(Y_0, \lambda_0)$*-solutions of the differential equation (1.1) with representations (3.12) and (3.13) such that their derivative satisfies the asymptotic relation*

$$y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \frac{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))}{\pi_\omega(t)} \Big[1 + |H(t)|^{-1/2} o(1)\Big] \qquad as \quad t \uparrow \omega. \tag{3.15}$$

(2) If

$$(\lambda_0 - 1)J(t) > 0 \quad for \quad t \in \,]a, \omega[,$$

$$\lim_{t \uparrow \omega} \left[\frac{\lambda_0}{\lambda_0 - 1} - q(t)\right] |H(t)|^{1/2} \left(\int\limits_{t_0}^{t} \frac{|H(\tau)|^{1/2}\, d\tau}{\pi_\omega(\tau)}\right)^2 = 0, \tag{3.16}$$

and

$$\lim_{t \uparrow \omega} \frac{\displaystyle\int_{t_0}^{t} \frac{|H(\tau)|^{1/2}\, d\tau}{\pi_\omega(\tau)}}{|H(t)|^{1/2}} = 0,$$

$$\lim_{t \uparrow \omega} |H(t)|^{1/2} \left(\int\limits_{t_0}^{t} \frac{|H(\tau)|^{1/2}\, d\tau}{\pi_\omega(\tau)}\right) \left.\frac{\left(\dfrac{y\varphi'(y)}{\varphi(y)}\right)'}{\left(\dfrac{y\varphi'(y)}{\varphi(y)}\right)^2}\right|_{y = \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))} = 0, \tag{3.17}$$

where t_0 *is a number from the interval* $[a, \omega[$*, then, for* $\omega = +\infty$*, Eq. (1.1) possesses a single* $P_\omega(Y_0, \lambda_0)$*-solution admitting the asymptotic representations*

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big) \left[1 + \left(H(t) \int\limits_{t_0}^{t} \frac{|H(\tau)|^{1/2}\, d\tau}{\pi_\omega(\tau)}\right)^{-1} o(1)\right] \qquad as \quad t \uparrow \omega, \tag{3.18}$$

$$y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \frac{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))}{\pi_\omega(t)} \left[1 + \left(\int_{t_0}^{t} \frac{|H(\tau)|^{1/2}\, d\tau}{\pi_\omega(\tau)} \right)^{-1} o(1) \right] \qquad as \quad t \uparrow \omega \tag{3.19}$$

and, for $\omega < +\infty$, the analyzed equation possesses a two-parameter family of $P_\omega(Y_0, \lambda_0)$-solutions with the indicated representations.

Proof of Theorem 3.1. Let $y\colon [t_0, \omega[\longrightarrow \mathbb{R}$ be an arbitrary $P_\omega(Y_0, \lambda_0)$-solution of the differential equation (1.1). Then, by Lemma 2.6, the asymptotic relations (2.7) are true. By virtue of these relations and Eq. (1.1), this solution and its first and second derivatives preserve signs on a certain interval $[t_1, \omega[\subset [t_0, \omega[$. Moreover, equalities (3.2) are true for these signs and imply condition (3.4). In addition, it follows from Eq. (1.1) and the second asymptotic relation in (2.7) that

$$\frac{y'(t)}{\varphi(y(t))} = \alpha_0(\lambda_0 - 1)\pi_\omega(t)p(t)\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega. \tag{3.20}$$

Integrating this relation from t_0 to t, we obtain

$$\int_{y(t_0)}^{y(t)} \frac{ds}{\varphi(s)} = \alpha_0(\lambda_0 - 1) \int_{t_0}^{t} \pi_\omega(\tau)p(\tau)\big[1 + o(1)\big]\, d\tau \qquad \text{as} \quad t \uparrow \omega.$$

Since, according to the definition of a $P_\omega(Y_0, \lambda_0)$-solution, we have $y(t) \longrightarrow Y_0$ as $t \uparrow \omega$, the improper integrals

$$\int_{y(t_0)}^{Y_0} \frac{ds}{\varphi(s)} \qquad \text{and} \qquad \int_{t_0}^{\omega} \pi_\omega(\tau)p(\tau)\, d\tau$$

are simultaneously convergent or divergent. In view of this fact and the rule used to choose the limits of integration A and B in the functions J and Φ introduced at the beginning of this section, the relation established above can be rewritten in the form

$$\Phi(y(t)) = \alpha_0(\lambda_0 - 1)J(t)\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega. \tag{3.21}$$

Thus, in view of the second condition in (3.6), we conclude that inequality (3.9) is true and the first condition in (3.10) is satisfied. By virtue of the first condition in (3.6), the relation

$$\frac{y'(t)\varphi'(y(t))}{\varphi(y(t))} = -\frac{\pi_\omega(t)p(t)}{J(t)}\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega$$

follows from (3.20) and (3.21). By using the first asymptotic relation in (2.7), we obtain

$$\frac{y(t)\varphi'(y(t))}{\varphi(y(t))} = -\frac{(\lambda_0 - 1)\pi_\omega(t)p(t)}{\lambda_0 J(t)}\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega.$$

By virtue of (1.3) and the definition of $P_\omega(Y_0, \lambda_0)$-solution, this relation directly implies the second limit condition in (3.10).

Thus, by using (3.21), we get

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda - 1)J(t)\big[1 + o(1)\big]\big) \qquad \text{as} \quad t \uparrow \omega. \tag{3.22}$$

As shown above, the function Φ belongs to the class $\Gamma_{Y_0}(Z_0),$ where

$$Z_0 = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi(y),$$

and its complementary function can be chosen in the form

$$g(y) = -\frac{\varphi(y)}{\varphi'(y)}.$$

Hence, by virtue of the conditions

$$\alpha_0(\lambda_0 - 1)\lim_{t\uparrow\omega} J(t) = Z_0 \qquad \text{and} \qquad \alpha_0(\lambda_0 - 1)J(t) \in \Delta_{Z_0} \qquad \text{for} \quad t \in [t_0, \omega[,$$

which follow from (3.20) and (3.5), according to Lemma 2.4, we obtain

$$\lim_{t\uparrow\omega} \frac{\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big[1 + o(1)\big]\big) - \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))}{-\dfrac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))}} = \lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} \frac{\Phi^{-1}(z(1 + o(1))) - \Phi^{-1}(z)}{-\dfrac{\varphi(z)}{\varphi'(z)}} = 0.$$

This yields

$$\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big[1 + o(1)\big]\big)$$
$$= \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))}o(1) \qquad \text{as} \quad t \uparrow \omega.$$

By using this relation and (3.22), we arrive at the asymptotic representation (3.11). In view of the fact that

$$\lim_{t\uparrow\omega} \frac{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\varphi'\big(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\big)}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))} = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty,$$

we can rewrite (3.11) in the form

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega.$$

Thus, by virtue of the first asymptotic relation in (2.7), we get the asymptotic representation (3.12).

Further, in view of representation (3.11), it follows from (1.1) that

$$y''(t) = \alpha_0 p(t)\varphi\left(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) + \frac{\varphi(\alpha_0(\lambda_0 - 1)J(t))}{\varphi'(\alpha_0(\lambda_0 - 1)J(t))}o(1)\right) \qquad \text{as} \quad t \uparrow \omega. \tag{3.23}$$

Since $\varphi \in \Gamma_{Y_0}(Z_0)$, where $Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y)$, which, according to the second condition in (1.2), is equal either to zero or to $+\infty$, and as a complementary function, we can choose the function

$$g(y) = \frac{\varphi(y)}{\varphi'(y)},$$

by virtue of Lemma 2.3 and the conditions

$$\lim_{t\uparrow\omega} \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) = Y_0 \qquad \text{and} \qquad \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) \in \Delta_{Y_0} \qquad \text{for} \quad t \in [t_0, \omega[,$$

we obtain

$$\lim_{t\uparrow\omega} \frac{\varphi\left(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) + \dfrac{\varphi(\alpha_0(\lambda_0 - 1)J(t))}{\varphi'(\alpha_0(\lambda_0 - 1)J(t))} o(1)\right)}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi\left(y + \dfrac{\varphi(y)}{\varphi'(y)} o(1)\right)}{\varphi(y)} = 1.$$

Therefore, as $t \uparrow \omega$, we find

$$\varphi\left(\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big) + \frac{\varphi\Big(\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\Big)}{\varphi'\Big(\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\Big)} o(1)\right) = \varphi\Big(\Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\Big)[1 + o(1)]$$

and we can rewrite the asymptotic relation (3.23) in the form

$$y''(t) = \alpha_0 p(t)\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\big)\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega.$$

By virtue of this representation and (3.12), we get

$$\frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{\alpha_0(\lambda_0 - 1)\pi_\omega^2(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\big)}{\lambda_0\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))}\big[1 + o(1)\big] \qquad \text{as} \quad t \uparrow \omega,$$

whence, in view of the second asymptotic relation in (2.7), we arrive at the third condition in (3.10).

Theorem 3.1 is proved.

Proof of Theorem 3.2. Assume that limit (3.13) (finite or equal to $\pm\infty$) exists and, for some $\lambda_0 \in \mathbb{R}\setminus\{0, 1\}$, conditions (3.4), (3.9), and (3.10) and one of the following conditions: either (3.14) or (3.16) and (3.17), are satisfied. Under these conditions, we establish the existence of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1.1) admitting the asymptotic representations (3.11) and (3.12) and find the number of these solutions.

First, in view of the existence of limit (3.13) (finite or equal to $\pm\infty$), we show that this limit is equal to zero. Assume the contrary. Then the following relation is true:

$$\frac{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)'}{\left|\dfrac{\varphi'(y)}{\varphi(y)}\right|^{3/2}} = \frac{z(y)}{|y|^{1/2}},$$

where the function $z\colon \Delta_{Y_0} \longrightarrow \mathbb{R}$ is continuous and such that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y) = \begin{cases} \text{either} & c = \text{const} \neq 0, \\ \text{or} & \pm\infty. \end{cases} \tag{3.24}$$

Integrating this relation from y_0 to $y,$ we find

$$-2\mu_0 \left| \frac{\varphi'(y)}{\varphi(y)} \right|^{-1/2} = c_0 + \int_{y_0}^{y} \frac{z(s)}{|s|^{1/2}} ds, \tag{3.25}$$

where c_0 is a constant.

If

$$\int_{y_0}^{Y_0} \frac{z(s)\, ds}{|s|^{1/2}} = \pm\infty,$$

then, as a result of the division by $|y|^{1/2},$ we get

$$-2\mu_0 \left| \frac{y\varphi'(y)}{\varphi(y)} \right|^{-1/2} = \frac{\displaystyle\int_{y_0}^{y} \frac{z(s)ds}{|s|^{1/2}}}{|y|^{1/2}} \big[1 + o(1)\big] \qquad \text{as} \quad y \to Y_0.$$

Here, by virtue of (1.3), the expression on the left-hand side tends to zero as $y \to Y_0$, whereas the expression on the right-hand side tends either to a nonzero constant or to $\pm\infty$ by virtue of condition (3.24) because, according to the L'Hospital rule in the Stolz form, we have

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\displaystyle\int_{y_0}^{y} \frac{z(s)ds}{|s|^{1/2}}}{|y|^{1/2}} = 2\mu_0 \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} z(y),$$

which is impossible.

If

$$\int_{y_0}^{Y_0} \frac{z(s)\, ds}{|s|^{1/2}}$$

converges, which is possible only in the case where $Y_0 = 0,$ then we can rewrite (3.25) in the form

$$-2\mu_0 \left| \frac{\varphi'(y)}{\varphi(y)} \right|^{-1/2} = c_1 + \int_{0}^{y} \frac{z(s)\, ds}{|s|^{1/2}},$$

where

$$c_1 = c_0 + \int_{y_0}^{0} \frac{z(s)\, ds}{|s|^{1/2}}.$$

We now prove that, in the analyzed case, $c_1 = 0$. Indeed, if $c_1 \neq 0$, then this relation implies that

$$\frac{\varphi'(y)}{\varphi(y)} = \frac{4\mu_0}{c_1^2} + o(1) \qquad \text{as} \quad y \to 0.$$

Integrating this expression from y_0 to y, we find

$$\ln|\varphi(y)| = \text{const} + o(1) \qquad \text{as} \quad y \to 0,$$

which contradicts the second condition in (1.2). Hence, $c_1 = 0$ and we get

$$-2\mu_0 \left|\frac{\varphi'(y)}{\varphi(y)}\right|^{-1/2} = \int\limits_0^y \frac{z(s)\,ds}{|s|^{1/2}}.$$

Dividing both sides of this equality by $|y|^{1/2}$, we note that the left-hand side of the obtained relation tends to zero as $y \to 0$ by virtue of conditions (1.3), whereas the right-hand side, according to the L'Hospital rule and relation (3.24), tends either to a nonzero constant or to $\pm\infty$.

The contradictions obtained in both possible cases imply that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)'}{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)^2} \sqrt{\left|\frac{y\varphi'(y)}{\varphi(y)}\right|} = 0. \tag{3.26}$$

Further, by applying the transformation

$$\begin{aligned} y(t) &= \Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\left[1 + \frac{y_1}{H(t)}\right], \\ y'(t) &= \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)} \Phi^{-1}\big(\alpha_0(\lambda_0 - 1)J(t)\big)\big[1 + y_2(t)\big] \end{aligned} \tag{3.27}$$

to Eq. (1.1), we obtain a system of differential equations

$$\begin{aligned} y_1' &= \frac{H(t)}{\pi_\omega(t)} \left[\frac{\lambda_0}{\lambda_0 - 1} - q(t) + h(t)y_1 + \frac{\lambda_0}{\lambda_0 - 1} y_2\right], \\ y_2' &= \frac{1}{\pi_\omega(t)} \left[1 - \frac{\lambda_0 - 1}{\lambda_0} q(t) + \frac{q(t)}{\lambda_0} y_1 + (1 - q(t))\, y_2 + \frac{1}{\lambda_0} q(t) R(t, y_1)\right], \end{aligned} \tag{3.28}$$

where

$$h(t) = q(t) \left. \frac{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)'}{\left(\dfrac{\varphi'(y)}{\varphi(y)}\right)^2} \right|_{y = \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))},$$

$$R(t,y_1)=\frac{\varphi\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))+\dfrac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}y_1\right)}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}-1-y_1.$$

We consider this system on the set

$$\Omega=[t_0,\omega[\,\times D_1\times D_2,$$

where $D_i=\{y_i\colon |y_i|\le 1\},\ i=1,2,$ and the number $t_0\in[a,\omega[$ chosen with regard for conditions (1.3), (3.5), (3.6), (3.9), and (3.10) is such that

$$\alpha_0(\lambda_0-1)J(t)\in\Delta_{Z_0}\qquad\text{for}\quad t\in[t_0,\omega[,$$

$$\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))+\frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}y_1\in\Delta_{Y_0}\qquad\text{for}\quad t\in[t_0,\omega[\quad\text{and}\quad |y_1|\le 1.$$

In this set, the right-hand sides of the system of differential equations (3.28) are continuous and the function R has continuous partial derivatives on the set $[t_0,\omega[\,\times D_1$ with respect to the variable y_1 up to the second order, inclusively. Thus, we get

$$R'_{y_1}(t,y_1)=\frac{\varphi'\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))+\dfrac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}y_1\right)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}-1.$$

Here, $\varphi'\in\Gamma_{Y_0}(Z_0)$ with the complementary function $g(y)=\dfrac{\varphi(y)}{\varphi'(y)}$. Therefore,

$$\lim_{t\uparrow\omega}\frac{\varphi'\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))+\dfrac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}y_1\right)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi'\left(y+y_1\dfrac{\varphi(y)}{\varphi'(y)}\right)}{\varphi'(y)}=e^{y_1}.$$

By virtue of this limit relation and Lemma 2.3, we find

$$R'_{y_1}(t,y_1)=e^{y_1}\big[1+r(t,y_1)\big]-1,$$

where

$$\lim_{t\uparrow\omega}r(t,y_1)=0\qquad\text{uniformly in}\quad y_1\in[-1,1].$$

Hence, for any $\varepsilon>0$, there exist $t_1\in[t_0,\omega[$ and $\delta>0$ such that

$$|R'_{y_1}(t,y_1)|\le\varepsilon\qquad\text{for}\quad t\in[t_1,\omega[\quad\text{and}\quad y_1\in D_{1\delta}=\{y_1\colon |y_1|\le\delta\le 1\}.$$

This means that, on the set $[t_1,\omega[\,\times D_{1\delta}$, the function R satisfies the Lipschitz condition with respect to the variable y_1 with Lipschitz constant ε. By virtue of the identity $R(t,0)\equiv 0$, this yields the estimate

$$|R(t,y_1)|\le\varepsilon|y_1|\qquad\text{for}\quad t\in[t_1,\omega[\quad\text{and}\quad y_1\in D_{1\delta}. \tag{3.29}$$

If, for fixed $t \in [t_0, \omega[$, we expand the function R in the Maclaurin series with Lagrange remainder up to the terms of the second order, then we get

$$R(t, v_1) = \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'^2\,(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))} \times \varphi''\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\xi\right) y_1^2, \tag{3.30}$$

where $|\xi| < |y_1|$. Here, by virtue of the last condition in (1.2), we obtain

$$\varphi''\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\xi\right) = \frac{\varphi'^2\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\xi\right)}{\varphi\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\xi\right)}\big[1 + r_1(t, y_1)\big],$$

where $\lim_{t\uparrow\omega} r_1(t, y_1) = 0$ uniformly in $y_1 \in D_1$. Thus, in view of the fact that the functions $\varphi, \varphi' \in \Gamma_{Y_0}(Z_0)$ and have the complementary function

$$g(y) = \frac{\varphi(y)}{\varphi'(y)},$$

we obtain

$$\varphi''\left(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\xi\right) = \frac{\varphi'^2\,\big(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\big)}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}e^{\xi}\big[1 + r_2(t, y_1)\big],$$

where $\lim_{t\uparrow\omega} r_2(t, y_1) = 0$ uniformly in $y_1 \in D_1$. Hence, relation (3.25) can be rewritten in the form

$$R(t, y_1) = e^{\xi}\big[1 + r_1(t, y_1)\big]\big[1 + r_2(t, y_1)\big]y_1^2.$$

It is clear that, for any $\varepsilon > 0$, there exist $\delta > 0$ and $t_1 \in [t_0, \omega[$ such that

$$|R(t, y_1)| \le (1+\varepsilon)|y_1|^2 \qquad \text{for} \quad t \in [t_1, \omega[\quad \text{and} \quad y_1 \in D_{1\delta} = \{y_1 : |y_1| \le \delta\}. \tag{3.31}$$

Moreover, by virtue of conditions (1.2), (1.3), (3.5), (3.9), and (3.10), in the system of equations (3.28), we have

$$\lim_{t\uparrow\omega} q(t) = \frac{\lambda_0}{\lambda_0 - 1}, \qquad \lim_{t\uparrow\omega} h(t) = 0, \qquad \lim_{t\uparrow\omega} H(t) = \pm\infty. \tag{3.32}$$

According to the results presented above, system (3.28) is a quasilinear system of differential equations. To establish the existence of $P_\omega(Y_0, \lambda_0)$-solutions of Eq. (1.1) admitting the asymptotic representations (3.11) and (3.12), according to transformation (3.27), it is necessary to prove the existence of solutions of the system of differential equations (3.28) approaching zero as $t \uparrow \omega$. In order to use the available results on the solutions of quasilinear systems of differential equations vanishing at a singular point, we reduce system (3.28) to the form that admits their application.

Applying an additional transformation

$$y_1 = v_1, \qquad y_2 = |H(t)|^{-1/2} v_2 \tag{3.33}$$

to system (3.28), we obtain the following system of differential equations:

$$\begin{aligned} v_1' &= \frac{|H(t)|^{1/2}}{\pi_\omega(t)} \big[f_1(t) + c_{11}(t) v_1 + c_{12}(t) v_2 \big], \\ v_2' &= \frac{|H(t)|^{1/2}}{\pi_\omega(t)} \big[f_2(t) + c_{21}(t) v_1 + c_{22}(t) v_2 + V(t, v_1) \big], \end{aligned} \tag{3.34}$$

where

$$f_1(t) = \left[\frac{\lambda_0}{\lambda_0 - 1} - q(t) \right] |H(t)|^{1/2} \operatorname{sign} H(t), \qquad f_2(t) = 1 - \frac{\lambda_0 - 1}{\lambda_0} q(t),$$

$$c_{11}(t) = h(t) |H(t)|^{1/2} \operatorname{sign} H(t), \qquad c_{12}(t) = \frac{\lambda_0}{\lambda_0 - 1} \operatorname{sign} H(t),$$

$$c_{21}(t) = \frac{q(t)}{\lambda_0}, \qquad c_{22}(t) = |H(t)|^{-1/2} \left(1 - \frac{q(t)}{2} + \frac{h(t)}{2} |H(t)|^{1/2} \operatorname{sign} H(t) \right),$$

$$V(t, v_1) = \frac{1}{\lambda_0} q(t) R(t, v_1).$$

We choose an arbitrary number $\varepsilon > 0$ and find, in view of the above-mentioned properties of the function R, the numbers $\delta > 0$ and $t_1 \in [t_0, \omega[$ such that inequality (3.31) is true. Consider system (3.34) on the set

$$\Omega_1 = \big\{ (t, v_1, v_2) \in \mathbb{R}^3 : \ t \in [t_1, \omega[, \ v_1 \in [-\delta, \delta], \ v_2 \in [-1, 1] \big\}.$$

By virtue of (3.31), the replacement of y_1 with v_1, and the first condition in (3.32), we conclude that

$$\lim_{v_1 \to 0} \frac{V(t, v_1)}{|v_1|} = 0 \qquad \text{uniformly in} \quad t \in [t_1, \omega[.$$

In addition, in view of conditions (3.32) and (3.26) and the notation introduced at the beginning of this section, we find

$$\lim_{t \uparrow \omega} f_2(t) = 0, \qquad \lim_{t \uparrow \omega} c_{11}(t) = 0, \qquad c_{12}(t) \equiv \frac{\nu_0 \mu_0 \lambda_0}{\lambda_0 - 1}, \qquad \lim_{t \uparrow \omega} c_{21}(t) = \frac{1}{\lambda_0 - 1}, \tag{3.35}$$

$$\lim_{t \uparrow \omega} c_{22}(t) = 0, \qquad \int_{t_1}^{\omega} \frac{|H(\tau)|^{1/2}}{\pi_\omega(\tau)} d\tau = \pm\infty. \tag{3.36}$$

In particular, this implies that the limit matrix of the coefficients of v_1 and v_2 in the square brackets in system (3.34) has the form

$$C = \begin{pmatrix} 0 & \dfrac{\nu_0\mu_0\lambda_0}{\lambda_0 - 1} \\ \dfrac{1}{\lambda_0 - 1} & 0 \end{pmatrix}$$

and

$$\rho^2 - \frac{\nu_0\mu_0\lambda_0}{(\lambda_0 - 1)^2} = 0 \tag{3.37}$$

is its characteristic equation. Here, by virtue of conditions (3.4) and (3.9), we have

$$\operatorname{sign}(\nu_0\mu_0\lambda_0) = -\operatorname{sign}[(\lambda_0 - 1)J(t)] \qquad \text{for} \quad t \in]a, \omega[.$$

Further, we assume that conditions (3.14) are satisfied. In this case, the algebraic equation (3.37) has two real roots with opposite signs and, parallel with (3.35) and (3.36), we get

$$\lim_{t \uparrow \omega} f_1(t) = 0.$$

This implies that the system of differential equations (3.34) satisfies all conditions of Theorem 2.2 in [7]. According to this theorem, the system of differential equations (3.34) possesses a one-parameter family of solutions

$$(v_1, v_2)\colon\ [t_*, \omega[\longrightarrow \mathbb{R}^2 \qquad (t_* \in [t_1, \omega[)$$

vanishing as $t \uparrow \omega$. By virtue of changes (3.27) and (3.33), each of these solutions is associated with the solution

$$y\colon\ [t_*, \omega[\longrightarrow \mathbb{R}$$

admitting the asymptotic representations (3.11) and (3.15).

Now let conditions (3.16) and (3.17) be satisfied. In this case, by virtue of the first condition in (3.16), the algebraic equation (3.37) has pure imaginary roots. To find the solutions of the system of equations (3.34) vanishing as $t \uparrow \omega$, we use the results obtained in [8]. To this end, by the change of the independent variable

$$v_1(t) = z_1(x), \qquad v_2(t) = z_2(x), \qquad x = \int_{t_1}^{t} \frac{|H(\tau)|^{1/2}\,d\tau}{|\pi_\omega(\tau)|}, \tag{3.38}$$

we reduce the system of equations (3.34) to the following system:

$$\begin{aligned} z_1' &= q_1(x) + b_1(x)z_1 + \frac{\beta\nu_0\mu_0\lambda_0}{\lambda_0 - 1} z_2, \\ z_2' &= q_2(x) + \frac{\beta}{\lambda_0 - 1} z_1 + b_2(x)z_2 + Z(x, z_1), \end{aligned} \tag{3.39}$$

where

$$q_1(x(t)) = \beta\nu_0\mu_0\left[\frac{\lambda_0}{\lambda_0-1} - q(t)\right]|H(t)|^{1/2}, \qquad q_2(x(t)) = \beta\left[1 - \frac{\lambda_0-1}{\lambda_0}q(t)\right],$$

$$b_1(x(t)) = \beta\nu_0\mu_0 h(t)\big|H(t)\big|^{1/2}, \qquad b_2(x(t)) = \beta\big|H(t)\big|^{-1/2}\left(1 - \frac{q(t)}{2} + \frac{\nu_0\mu_0 h(t)}{2}\big|H(t)\big|^{1/2}\right),$$

$$Z(x(t), z_1) = \frac{\beta q(t)}{\lambda_0}R(t, z_1), \qquad \beta = \operatorname{sign}\pi_\omega(t).$$

Since $x'(t) > 0$ for $t \in]t_0, \omega[$ and $\lim_{t\uparrow\omega} x(t) = +\infty$ by virtue of the third condition in (3.32), the system of equations (3.39) is defined on the set

$$G = \{(x, z_1, z_2) \in \mathbb{R}^3 \colon\, x \in [0, +\infty[,\ |z_1| \le \delta,\ |z_2| \le 1\}$$

and, in view of (3.32), (3.16), (3.17), and (3.31), we find

$$\lim_{x\to+\infty} x^2 q_i(x) = \lim_{t\uparrow\omega}\left(\int_{t_1}^{t}\frac{|H(\tau)|^{1/2}\,d\tau}{|\pi_\omega(t)|}\right)^2 q_i(x(t)) = 0, \quad i = 1, 2,$$

$$\lim_{x\to+\infty} x b_i(x) = \lim_{t\uparrow\omega}\left(\int_{t_1}^{t}\frac{|H(\tau)|^{1/2}\,d\tau}{|\pi_\omega(t)|}\right) b_i(x(t)) = 0, \quad i = 1, 2,$$

$$\lim_{z_1\to 0}\frac{x^2 Z\left(x, \dfrac{z}{x}\right)}{z_1} = \lim_{z_1\to 0}\frac{x^2(t)q(t)R\left(t, \dfrac{z_1}{x(t)}\right)}{\lambda_0 z_1} = 0 \qquad \text{uniformly in} \quad x \in [0, +\infty[.$$

In this case, the characteristic equation of the limit matrix of coefficients of the linear part of the system is the algebraic equation (3.37), which, in the analyzed case, has pure imaginary roots.

This implies that the system of differential equations (3.39) satisfies all conditions of Theorem 2.2 in [8] (for $r = \varepsilon = 1$). According to this theorem, the system of differential equations (3.39) with $\omega < +\infty$ has a two-parameter family of solutions vanishing at infinity $(z_1, z_2) \colon\, [x_0, +\infty[\longrightarrow \mathbb{R}^2$ $(x_0 \ge 0)$ of the form

$$z_i(x) = o\left(\frac{1}{x}\right) \qquad \text{as} \quad x \to +\infty, \quad i = 1, 2.$$

At the same time, for $\omega = +\infty$, this system has at least one solution with these representations (this solution is unique because the function R satisfies the Lipschitz condition with respect to the variable z_1). In view of changes (3.27), (3.33), and (3.38), each solution of this kind is associated with a $P_\omega(Y_0, \lambda_0)$-solution

$$y \colon\, [t_2, \omega[\longrightarrow \mathbb{R} \qquad (t_2 \in [a, \omega[)$$

admitting asymptotic representations of the forms (3.18) and (3.19).

Theorem 3.2 is proved.

4. Conclusions

In the present paper, for an equation of the form (1.1) with nonlinearity φ rapidly varying as $y \to Y_0$, where Y_0 is equal either to zero or to $\pm\infty$, we establish, for the first time, the conditions for the existence of $P_\omega(Y_0, \lambda_0)$-solutions in the nonsingular case $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$ and the asymptotic representations of these solutions and their first-order derivatives as $t \uparrow \omega$ $(\omega \le +\infty)$. Earlier, the problem of existence of solutions from the class of $P_\omega(Y_0, \lambda_0)$-solutions and their asymptotics was fairly completely investigated for the nonlinearity φ regularly varying as $y \to Y_0$.

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