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DIFFERENTIAL EQUATIONS WITH SMALL STOCHASTIC TERMS UNDER THE LÉVY APPROXIMATING CONDITIONS

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We propose new methods for the investigation of a model of stochastic evolution with Markov switchings capable of separation of the diffusion component and big jumps of the perturbing process in the limiting equation. Big jumps of this type may describe seldom catastrophic events in various applied problems. We consider the case where the system is perturbed by an impulsive process in the nonclassical approximation scheme. Special attention is given to the asymptotic behavior of the generator of the analyzed evolutionary system.

1. Introduction

Random evolution in the form of a differential equation with stochastic terms is used for the description of a wide class of natural processes in various fields of science. The investigation of the behavior of evolutionary systems of this kind in random media is of very high importance. Numerous works by famous scientists, including Skorokhod, Gikhman, Bogolyubov, and other researchers are devoted to the study of these systems. For the detailed bibliography in this field, see, e.g., the monographs by Korolyuk [2, 3]. We especially mention the work [5] in which the approaches used in the present paper were proposed, including, in particular, the methods aimed at the investigation of stability of evolutionary systems with diffusion perturbations.

In the present paper, we consider the case where perturbations of the system are specified by an impulsive process in the Lévy approximation scheme (for details of the approximation scheme, see [3, 4]). We are mainly interested in the asymptotic behavior of the generator of this scheme. Similar problems were considered earlier by using qualitatively different schemes (see [6] and the references therein). Note that the effect of separation of a deterministic shift from the perturbing impulsive process in the limiting equation established in the present paper was observed earlier, e.g., in Sec. 5.1 of [6]. However, the methods proposed in the present paper enable us to study more complicated problems including Markov switchings corresponding to random media and also to select the diffusion term and big jumps of the perturbing process in the limit equation, which can describe seldom catastrophic events in various applied problems.

The accumulated results enable one to continue our investigations in the following three directions:

1. Proving of the limit functional theorems that describe the behavior of a system on increasing time intervals (see, e.g., [3] and the survey of the methods aimed at proving limit functional theorems in the nonclassical approximation schemes [4]).

2. Proving of the dissipativity of systems, which enables one to investigate their stability, the presence of attractors, etc. (see the monograph [6], where similar problems were considered for the classical approximation schemes and [7, 8]).

3. The analysis of the asymptotic behavior of the normalized control with Markov switchings in the Lévy approximation scheme [9].

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2. Statement of the Problem

Consider a stochastic evolutionary system in an ergodic Markov medium given by the following stochastic differential equation:

$$du^\varepsilon(t) = C\big(u^\varepsilon, x(t/\varepsilon^2)\big)dt + d\eta^\varepsilon(t), \quad u^\varepsilon(t) \in \mathbf{R}, \tag{1}$$

where $x(t)$ is a uniformly ergodic Markov process in the standard phase space $(X, \mathbf{X})$ defined by the generator

$$\mathbf{Q}\varphi(x) = q(x) \int_X P(x, dy)\big[\varphi(y) - \varphi(x)\big]$$

in the Banach space $B(X)$ of real-valued bounded functions $\varphi(x)$ with the supremum-norm

$$\|\varphi\| = \sup_{x \in X} |\varphi(x)|.$$

The stochastic kernel $P(x, B),\ x \in X,\ B \in \mathbf{X}$, defines a uniformly ergodic embedded Markov chain

$$x_n = x(\tau_n), \quad n \geq 0,$$

with a stationary distribution $\rho(B),\ B \in \mathbf{X}$. The stationary distribution $\pi(B),\ B \in \mathbf{X}$, of the Markov process $x(t),\ t \geq 0$, can be found from the relation

$$\pi(dx)q(x) = q\rho(dx), \qquad q = \int_X \pi(dx)q(x).$$

By R_0 we denote a potential operator for the generator $\mathbf{Q}$ given by the equality [3]

$$R_0 = \Pi - (\Pi + \mathbf{Q})^{-1},$$

where

$$\Pi\varphi(x) = \int_X \pi(dy)\varphi(y)1(x)$$

is the projector onto the subspace $N_Q = \{\varphi\colon\ \mathbf{Q}\varphi = 0\}$ of zeros of the operator $\mathbf{Q}$.

3. Impulsive Perturbing Process

In the Lévy approximation scheme, the impulsive perturbing process $\eta^\varepsilon(t),\ t \geq 0$, is specified by the relation

$$\eta^\varepsilon(t) = \int_0^t \eta^\varepsilon\big(ds, x(s/\varepsilon^2)\big), \tag{2}$$

where the collection of processes with independent increments

$$\eta^\varepsilon(t, x), \qquad t \geq 0, \quad x \in X,$$

is defined by the generators

$$\Gamma^{\epsilon}(x)\varphi(w) = \varepsilon^{-2} \int\limits_R \big(\varphi(w+v) - \varphi(w)\big)\Gamma^{\varepsilon}(dv, x), \quad x \in X, \tag{3}$$

and satisfy the Lévy approximating conditions (for details, see [3, 4]):

L1. *Approximation of the means:*

$$\int\limits_R v\Gamma^{\varepsilon}(dv, x) = \varepsilon a_1(x) + \varepsilon^2\big(a_2(x) + \theta_a(x)\big), \qquad \theta_a(x) \to 0, \quad \varepsilon \to 0,$$

and

$$\int\limits_R v^2\Gamma^{\varepsilon}(dv, x) = \varepsilon^2\big(b(x) + \theta_b(x)\big), \qquad \theta_b(x) \to 0, \quad \varepsilon \to 0.$$

L2. *Condition for the distribution function:*

$$\int\limits_R g(v)\Gamma^{\varepsilon}(dv, x) = \varepsilon^2\big(\Gamma_g(x) + \theta_g(x)\big), \qquad \theta_g(x) \to 0, \quad \varepsilon \to 0,$$

for all $g(v) \in C_3(\mathbf{R})$ (the space of real-valued bounded functions such that $g(v)/|v|^2 \to 0,\ |v| \to 0$). Here, the measure $\Gamma_g(x)$ is bounded for all $g(v) \in C_3(\mathbf{R})$ and defined by the following relations (functions from the space $C_3(\mathbf{R})$ separate the measures [1, p. 395])

$$\Gamma_g(x) = \int\limits_R g(v)\Gamma_0(dv, x), \quad g(v) \in C_3(\mathbf{R}).$$

L3. *Uniformly quadratic integrability:*

$$\lim_{c \to \infty} \int\limits_{|v|>c} v^2\Gamma_0(dv, x) = 0.$$

Example 1. The simplest example of a random variable satisfying the Lévy approximating conditions is given by a random variable α:

$$\mathrm{P}\{\alpha = b\} = \varepsilon^2 p,$$

$$\mathrm{P}\{\alpha = \varepsilon a_1 + \varepsilon^2 a_2\} = 1 - \varepsilon^2 p.$$

For the moments of this random variable, we have

$$\mathbf{E}\alpha = \varepsilon a_1 + \varepsilon^2(a_2 + bp) + o(\varepsilon^2),$$

$$\mathbf{E}\alpha^2 = \varepsilon^2(a_1^2 + b^2 p) + o(\varepsilon^2).$$

Assume that the balance condition

$$\hat{a}_1 := \int_X \pi(dx) a_1(x) = 0 \tag{4}$$

is satisfied.

Consider the asymptotic properties of the perturbation process.

Theorem 1. *Under the balance condition (4) and conditions* **L1–L3***, the weak convergence (in a sense of convergence of the corresponding generators)*

$$\eta^\varepsilon(t) \to \eta^0(t), \quad \varepsilon \to 0,$$

is guaranteed.

The limit process $\eta^0(t)$ *is defined by the generator*

$$\Gamma\varphi(w) = \hat{a}_2\varphi'(w) + \frac{1}{2}\sigma^2\varphi''(w) + \int_R \big[\varphi(w+v) - \varphi(w)\big]\hat{\Gamma}_0(dv),$$

where

$$\hat{a}_2 = \int_X \pi(dx)(a_2(x) - a_0(x)),$$

$$\sigma^2 = \int_X \pi(dx)(b(x) - b_0(x)) + 2\int_X \pi(dx) a_1(x) R_0 a_1(x),$$

$$a_0(x) = \int_R v\Gamma_0(dv, x), \qquad b_0(x) = \int_R v^2\Gamma_0(dv, x), \qquad \hat{\Gamma}_0(v) = \int_X \pi(dx)\Gamma_0(v, x).$$

Moreover, it is a Lévy process with three components: deterministic shift, diffusion component, and the Poisson jump part.

Proof of Theorem 1. We first prove several lemmas.

Lemma 1. *Under conditions* **L1–L3***, the generators of processes with independent increments* $\eta^\varepsilon(t, x)$*,* $t \geq 0,\ x \in X$*, on test functions* $\varphi(w) \in C^3(\mathbf{R})$ *admit the asymptotic representation*

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(w) = \varepsilon^{-1}\Gamma_1(x)\varphi(w) + \Gamma_2(x)\varphi(w), \tag{5}$$

where

$$\Gamma_1(x)\varphi(w) = a_1(x)\varphi'(w),$$

$$\Gamma_2(x)\varphi(w) = (a_2(x) - a_0(x))\varphi'(w) + \frac{1}{2}(b(x) - b_0(x))\varphi''(w)$$

$$+ \int_R \big[\varphi(w+v) - \varphi(w)\big]\Gamma_0(dv, x).$$

Proof. By using the expansion of the function $\varphi(w)$ in the Taylor series, we transform generator (3) as follows:

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(w) = \varepsilon^{-2}\int\limits_{R}\big(\varphi(w+v)-\varphi(w)\big)\Gamma^{\varepsilon}(dv,x)$$

$$= \varepsilon^{-2}\int\limits_{R}\left(\varphi(w+v)-\varphi(w)-v\varphi'(w)-\frac{1}{2}v^2\varphi''(w)\right)\Gamma^{\varepsilon}(dv,x)$$

$$+\varepsilon^{-2}\int\limits_{R}v\varphi'(w)\Gamma^{\varepsilon}(dv,x)+\frac{1}{2}v^2\varepsilon^{-2}\int\limits_{R}v^2\varphi''(w)\Gamma^{\varepsilon}(dv,x)$$

$$= \int\limits_{R}\left(\varphi(w+v)-\varphi(w)-v\varphi'(w)-\frac{1}{2}v^2\varphi''(w)\right)\Gamma_0(dv,x)$$

$$+\varepsilon^{-1}a_1(x)\varphi'(w)+a_2(x)\varphi'(w)+\frac{1}{2}b(x)\varphi''(w)+\gamma^{\varepsilon}(x)\varphi(w)$$

$$= \varepsilon^{-1}a_1(x)\varphi'(w)+\big(a_2(x)-a_0(x)\big)\varphi'(w)+\frac{1}{2}\big(b(x)-b_0(x)\big)\varphi''(w)$$

$$+\int\limits_{R}(\varphi(w+v)-\varphi(w))\Gamma_0(dv,x)+\gamma^{\varepsilon}(w)\varphi(w),$$

where the last-but-one equality follows from conditions **L1–L3** $\big($we also note that the function

$$\varphi(w+v)-\varphi(w)-v\varphi'(w)-\frac{1}{2}v^2\varphi''(w)\in C^3(\mathbf{R})$$

because it is bounded in view of the boundedness of $\varphi(w)$ and its derivatives and

$$\left[\varphi(w+v)-\varphi(w)-v\varphi'(w)-\frac{1}{2}v^2\varphi''(w)\right]/|v^2|\to 0,\quad |v|\to 0\big).$$

In view of the fact that $\gamma^{\varepsilon}(w)\varphi(w)=O(\varepsilon^2),\ \varphi(w)\in C^3(\mathbf{R}),$ we arrive at representation (5).

Lemma 2. *The generator of the two-component Markov process* $(\eta^{\varepsilon},\ x(t/\varepsilon^2)),\ t\ge 0,$ *has the form*

$$\hat{\Gamma}^{\varepsilon}(x)\varphi(w,x)=\varepsilon^{-2}\mathbf{Q}\varphi(w,x)+\varepsilon^{-1}\Gamma_1(x)\varphi(w,x)+\Gamma_2(x)\varphi(w,x)+\gamma^{\varepsilon}(x)\varphi(w,x), \tag{6}$$

where the operators $\Gamma_1(x)$ *and* $\Gamma_2(x)$ *are specified in Lemma 1 and the remainder* $\big\|\gamma^{\varepsilon}(x)\varphi(w,x)\big\|\to 0$ *as* $\varepsilon\to 0,\ \varphi(w,\cdot)\in C^3(\mathbf{R}).$

Proof. The assertion of the lemma is obvious if we use the definition of the generator of a Markov process and the form of the corresponding generators of the processes $\eta^{\varepsilon}(t,x)$ and $x(t/\varepsilon^2)$.

A truncated operator has the following structure [8]:

$$\Gamma_0^\varepsilon(x)\varphi(w) = \varepsilon^{-2}\mathbf{Q}\varphi(w,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(w,x) + \Gamma_2(x)\varphi(w,x). \tag{7}$$

Lemma 3. *Under the balance condition (4), the solution of the problem of singular perturbation of the truncated operator (7) on test functions*

$$\varphi^\varepsilon(w,x) = \varphi(w) + \varepsilon\varphi_1(w,x) + \varepsilon^2\varphi_2(w,x)$$

is given by the relations

$$\Gamma_0^\varepsilon(x)\varphi^\varepsilon(w,x) = \Gamma\varphi(w) + \varepsilon\theta_\eta^\varepsilon(x)\varphi(w), \tag{8}$$

where the remainder $\theta_\eta^\varepsilon(x)\varphi(w)$ is uniformly bounded in x.

The limit operator is given by the formula

$$\Gamma = \Pi\Gamma_1(x)R_0\Gamma_1(x)\Pi + \Pi\Gamma_2(x)\Pi. \tag{9}$$

Proof. In order that equality (8) be true, it is necessary that the coefficients of the same powers of ε on the left-hand and right-hand sides be equal. We determine these coefficients

$$\Gamma_0^\varepsilon(x)\varphi^\varepsilon(w,x) = \varepsilon^{-2}\mathbf{Q}\varphi(w) + \varepsilon^{-1}\big[\mathbf{Q}\varphi_1(w,x) + \Gamma_1(x)\varphi(w)\big]$$

$$+ \big[\mathbf{Q}\varphi_2(w,x) + \Gamma_1(x)\varphi_1(w,x) + \Gamma_2(x)\varphi(w)\big]$$

$$+ \varepsilon\big[\Gamma_1(x)\varphi_2(w,x) + \Gamma_2(x)\varphi_1(w,x)\big] + \varepsilon^2\Gamma_2(x)\varphi_2(w,x).$$

The first term gives

$$\mathbf{Q}\varphi(w) = 0 \Leftrightarrow \varphi(w) \in N_{\mathbf{Q}}.$$

It is easy to see that $\varphi(w)$ is independent of x.

The balance condition (4) is the condition of solvability of the equation

$$\mathbf{Q}\varphi_1(w,x) + \Gamma_1(x)\varphi(w) = 0.$$

Therefore,

$$\varphi_1(w,x) = R_0\Gamma_1(x)\varphi(w). \tag{10}$$

In view of (10), we can reduce the equation

$$\mathbf{Q}\varphi_2(w,x) + \Gamma_1(x)\varphi_1(w,x) + \Gamma_2(x)\varphi(w) = \Gamma\varphi(w)$$

to the form

$$\mathbf{Q}\varphi_2(w,x) + \Gamma_1(x)R_0\Gamma_1(x)\varphi(w) + \Gamma_2(x)\varphi(w) = \Gamma\varphi(w).$$

The condition of solvability of the last equation gives the limit operator in the form (9). Therefore,

$$\varphi_2(w,x) = R_0\big[\Gamma_1(x)R_0\Gamma_1(x) + \Gamma_2(x) - \Gamma\big]\varphi(w). \tag{11}$$

By using relations (10) and (11), we rewrite the remaining terms of the expansion in the form

$$\begin{aligned}
&\varepsilon\big[\Gamma_1(x)\varphi_2(w,x) + \Gamma_2(x)\varphi_1(w,x)\big] + \varepsilon^2\Gamma_2(x)\varphi_2(w,x)\\
&\qquad = \varepsilon\big[[\Gamma_1(x)R_0[\Gamma_1(x)R_0\Gamma_1(x) + \Gamma_2(x) - \Gamma] + \Gamma_2(x)R_0\Gamma_1(x)\big]\\
&\qquad\qquad + \varepsilon\big[\Gamma_2(x)R_0[\Gamma_1(x)R_0\Gamma_1(x) + \Gamma_2(x) - \Gamma]\big]\varphi(w)\\
&\qquad = \varepsilon\theta_\eta^\varepsilon(x)\varphi(w).
\end{aligned}$$

The boundedness of $\theta_\eta^\varepsilon(x)\varphi(w)$ follows from the form of the operators $\Gamma_1,\ \Gamma_2,$ and R_0.

By using Lemma 3 and Theorem 4.2 in [3], we complete the proof of the theorem.

4. Behavior of the Dynamical System

We now consider the asymptotic properties of the original evolutionary system (1).

Theorem 2. *Under the balance condition (4), the weak convergence in a sense of convergence of the corresponding generators,*

$$u^\varepsilon(t) \to \hat{u}(t), \quad \varepsilon \to 0,$$

takes place.

The limit process $\hat{u}(t)$ *is specified by the generator*

$$\mathbf{L}\varphi(w) = \hat{C}(u)\varphi'(w) + \Gamma\varphi(w), \tag{12}$$

where

$$\hat{C}(u) = \int_X \pi(dx)C(u,x).$$

Remark 1. The weak convergence of the processes $u^\varepsilon(t) \Rightarrow \hat{u}(t),\ \varepsilon \to 0,$ follows from the convergence of the corresponding generators under the condition of compactness of the prelimit family of the processes $u^\varepsilon(t)$. The corresponding theorems on compactness of the processes with independent increments in the Lévy approximation scheme were proved, e.g., in [4].

Remark 2. The limit process $\hat{u}(t)$ is given by the stochastic differential equation

$$d\hat{u}(t) = \big[\hat{C}(\hat{u}(t)) + \hat{a}_2\big]dt + \sigma dW(t) + \int_R v\tilde{v}(dt,dv),$$

where

$$\mathbf{E}\tilde{\nu}(dt,dv) = dt\tilde{\Gamma}_0(dv).$$

Remark 3. The limit process $\hat{u}(t)$ has three components. The deterministic shift is given by the solution of the differential equation

$$d\hat{u}_d(t) = \big[\hat{C}(\hat{u}_d(t)) + \hat{a}_2\big]dt,$$

where the additional term $\hat{a}_2$ appears due to the accumulation of very small jumps of order ε^2 observed as the normalized time increases: $t/\varepsilon^2,\ \varepsilon \to 0,$ which take place with probability close to 1.

The second diffusion component is determined by the parameter σ and appears due to the accumulation of small jumps of order ε as the normalized time increases: $t/\varepsilon^2,\ \varepsilon \to 0,$ which also occurs with probability close to 1.

The third component reflects seldom large jumps whose probability is close to zero. They are determined via the averaged measure of jumps $\tilde{\Gamma}_0(dv)$ with the generator

$$\Gamma_j\varphi(w) = \int\limits_R \big[\varphi(w+v) - \varphi(w)\big]\tilde{\Gamma}_0(dv).$$

Proof of Theorem 2. We first prove several lemmas.

Lemma 4. *The generator of a two-component Markov process* $\big(u^\varepsilon(t), x(t/\varepsilon^2)\big),\ t \geq 0,$ *has the representation*

$$\mathbf{L}^\varepsilon(x)\varphi(w,x) = \varepsilon^{-2}\mathbf{Q}\varphi(w,x) + \Gamma^\varepsilon(x)\varphi(w,x) + \mathbf{C}(x)\varphi(w,x) + \theta^\varepsilon_w\varphi(w,x),$$

where $\Gamma^\varepsilon(x)$ *is the generator of a collection of impulsive perturbation processes (3)*

$$\mathbf{C}(x)\varphi(w,x) = C(u,x)\varphi'_w(w,x).$$

The remainder $\big\|\theta^\varepsilon_w\varphi(w,x)\big\| \to 0$ *as* $\varepsilon \to 0.$

The proof of the lemma is presented in [7].

Lemma 5. *For an impulsive perturbation process, the generator* $\mathbf{L}^\varepsilon(x)$ *admits the asymptotic representation*

$$\mathbf{L}^\varepsilon(x)\varphi(w,x) = \varepsilon^{-2}\mathbf{Q}\varphi(w,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(w,x)$$

$$+\ \Gamma_2(x)\varphi(w,x) + \mathbf{C}(x)\varphi(w,x) + \hat{\theta}^\varepsilon_w\varphi(w,x),$$

where

$$\hat{\theta}^\varepsilon_w(x) = \gamma^\varepsilon + \theta^\varepsilon_w(x)$$

and $\Gamma_1(x)$ *and* $\Gamma_2(x)$ *are defined in Lemma 1.*

The remainder $\big\|\hat{\theta}^\varepsilon_w\varphi(w,x)\big\| \to 0$ *as* $\varepsilon \to 0.$

Proof. To prove the lemma, we use the representation of operator (5) and Lemma 4.

The truncated operator has the form

$$\mathbf{L}^\varepsilon_0(x)\varphi = \varepsilon^2\mathbf{Q}\varphi + \varepsilon^{-1}\Gamma_1(x)\varphi + \Gamma_2(x)\varphi + \mathbf{C}(x)\varphi. \tag{13}$$

Lemma 6. *Under the balance condition (4), the solution of the problem of singular perturbation for the truncated operator (13) on test functions*

$$\varphi^{\varepsilon}(w,x) = \varphi(w) + \varepsilon\varphi_1(w,x) + \varepsilon^2\varphi_2(w,x)$$

is given by the formula

$$\mathbf{L}_0^{\varepsilon}(x)\varphi^{\varepsilon}(w,x) = \mathbf{L}\varphi(w) + \varepsilon^2\theta_w^{\varepsilon}(x)\varphi(w), \tag{14}$$

where the remainder $\theta_w^{\varepsilon}(x)$ *is uniformly bounded in* x.

The limit operator $\mathbf{L}$ *is given by the formula*

$$\mathbf{L} = \Pi\big[\mathbf{C}(x) + \Gamma_1(x)R_0\Gamma_1(x) + \Gamma_2(x)\big]\Pi. \tag{15}$$

Proof. In order that equality (14) be true, it is necessary that the coefficients of the same powers of ε on the left- and right-hand sides be equal. To this end, we determine

$$\begin{aligned}\mathbf{L}_0^{\varepsilon}(x)\varphi^{\varepsilon}(w,x) = \varepsilon^{-2}\mathbf{Q}(x)\varphi(w) &+ \varepsilon^{-1}\big[\mathbf{Q}\varphi_1(w,x) + \Gamma_1(x)\varphi(w)\big] \\ &+ \big[\mathbf{Q}\varphi_2(w,x) + \Gamma_1(x)\varphi_1(w,x) + \Gamma_2(x)\varphi(w) + \mathbf{C}(x)\varphi(w)\big] \\ &+ \varepsilon\big[\Gamma_1(x)\varphi_2(w,x) + \Gamma_2(x)\varphi_1(w,x)) + \mathbf{C}(x)\varphi_1(w,x)\big] \\ &+ \varepsilon^2\big[\Gamma_2(x)\varphi_2(w,x) + \mathbf{C}(x)\varphi_2(w,x)\big].\end{aligned}$$

Since

$$\mathbf{Q}\varphi(w) = 0 \Leftrightarrow \varphi(w) \in N_Q,$$

it is clear that $\varphi(w)$ is independent of x.

The balance condition (4) is the condition of solvability of the equation

$$\mathbf{Q}\varphi_1(w,x) + \Gamma_1(x)\varphi(w) = 0.$$

Therefore,

$$\varphi_1(w,x) = R_0\Gamma_1(x)\varphi(w).$$

The last equation takes the form

$$\mathbf{Q}\varphi_2(w,x) + \Gamma_1(x)\varphi_1(w,x) + \Gamma_2(x)\varphi(w) + \mathbf{C}(x)\varphi(w) = \mathbf{L}\varphi(w).$$

We rewrite it as follows:

$$\mathbf{Q}\varphi_2(w,x) = \big[\mathbf{L} - \Gamma_1(x)R_0\Gamma_1(x) - \Gamma_2(x) - \mathbf{C}(x)\big]\varphi(w).$$

The condition of solvability of this equation gives the limit operator $\mathbf{L}$ in the form (15).

We complete the proof of the theorem by using the same scheme as in the proof of Theorem 4.2 in [3].

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