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BOUNDEDNESS OF RIESZ-TYPE POTENTIAL OPERATORS ON VARIABLE EXPONENT HERZ–MORREY SPACES

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We show the boundedness of the Riesz-type potential operator of variable order $\beta(x)$ from the variable exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ into the weighted space $M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, where $\alpha(x)\in L^\infty(\mathbb{R}^n)$ is log-Hölder continuous both at the origin and at infinity, $\omega=(1+|x|)^{-\gamma(x)}$ with some $\gamma(x)>0$, and $1/q_1(x)-1/q_2(x)=\beta(x)/n$ when $q_1(x)$ is not necessarily constant at infinity. It is assumed that the exponent $q_1(x)$ satisfies the logarithmic continuity condition both locally and at infinity and, moreover, $1<(q_1)_\infty\le q_1(x)\le(q_1)_+<\infty,\ x\in\mathbb{R}^n$.

1. Introduction

In the last decade, there is an evident intensification of investigations related both to the theory of variable exponent function spaces and to the operator theory in these spaces. This is explained by the keen interest not only in real analysis but also in partial differential equations and in applied mathematics because they are applicable to the modeling for electrorheological fluids, continuum mechanics, image restoration (see, e.g., [1–7] and the references therein), etc.

The theory of function spaces with variable exponent has rapidly made progress during the past twenty years since some elementary properties were established by Kováčik and Rákosník [8]. One of the main problems of the theory is the problem of boundedness of the Hardy–Littlewood maximal operator in variable Lebesgue spaces.

In 2012, Almeida and Drihem [9] discussed the boundedness of a wide class of sublinear operators on the Herz spaces

$$K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n) \qquad \text{and} \qquad \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$$

with variable exponents $\alpha(\cdot)$ and $q(\cdot)$. Meanwhile, they also established the Hardy–Littlewood–Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko [10, 11] introduced new Herz-type function spaces with variable exponent, where all three parameters are variable, and proved the property of boundedness for some sublinear operators. Later, in 2015, Rafeiro and Samko [12] studied the validity of a Sobolev-type theorem for the Riesz potential operator in continual variable exponent Herz spaces. In recent papers, Wu [13, 14] also considered the problem of boundedness of a fractional Hardy-type operator and a Riesz-type potential operator on the Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha\in\mathbb{R}$ and $p\in(0,\infty)$.

Motivated by the results mentioned above and based on some facts taken from [9, 15], we investigate the mapping properties of the operator $I_{\beta(\cdot)}$ within the framework of variable exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, where the Riesz-type potential operator of variable order is given by the formula

$$I_{\beta(\cdot)}(f)(x)=\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\beta(x)}}dy, \quad 0<\beta(x)<n.$$

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2. Preliminaries

In this section, we define some function spaces with variable exponent and present basic properties and useful lemmas. Throughout this paper, we use the following notation:

By $|S|$ we denote the Lebesgue measure and by χ_S we denote the characteristic function of a measurable set $S \subset \mathbb{R}^n$;

f_S denotes the mean value of f on a measurable set S, namely,

$$f_S := \frac{1}{|S|} \int\limits_S f(x)\, dx;$$

$B(x, r)$ is a ball of radius r centered at x; $B_0 = B(0, 1)$;

C is a constant whose value may differ from line to line independent of the main parameters involved;

For any exponent $1 < q(x) < \infty$, by $q'(x)$ we denote its conjugate exponent, namely,

$$1/q(x) + 1/q'(x) = 1.$$

2.1. Function Spaces with Variable Exponent. Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first define Lebesgue spaces with variable exponent.

Definition 2.1. *Let $q(\cdot)\colon \Omega \to (1, \infty)$ be a measurable function.*

(i) The variable Lebesgue spaces $L^{q(\cdot)}(\Omega)$ are defined by

$$L^{q(\cdot)}(\Omega) = \big\{ f \text{ is measurable function: } F_q(f/\eta) < \infty \text{ for some constant } \eta > 0 \big\},$$

where

$$F_q(f) := \int\limits_\Omega |f(x)|^{q(x)}\, dx.$$

The Lebesgue space $L^{q(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \eta > 0\colon F_q(f/\eta) = \int\limits_\Omega \Big(\frac{|f(x)|}{\eta}\Big)^{q(x)} dx \le 1 \right\}$$

is a Banach space.

(ii) The space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined as follows:

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \big\{ f \text{ is a measurable function: } f \in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0 \subset \Omega \big\}.$$

(iii) The weighted Lebesgue space $L^{q(\cdot)}_\omega(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{q(\cdot)}_\omega(\Omega)} = \|\omega^{1/q(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Further, we define some classes of variable exponent functions. Given a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined as

$$Mf(x) = \sup_{r>0} r^{-n} \int\limits_{B(x,r)} |f(y)|\, dy \quad \forall\, x \in \mathbb{R}^n;$$

here and what follows, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Definition 2.2. *Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$, we write*

$$q_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q(x) \qquad \text{and} \qquad q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x).$$

(i) $q'_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q'(x) = \dfrac{q_+}{q_+ - 1}$ *and* $q'_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q'(x) = \dfrac{q_-}{q_- - 1}$.

(ii) Denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ such that $0 < q_- \le q(x) \le q_+ < \infty$.

(iii) Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot)\colon \mathbb{R}^n \to (1,\infty)$ such that $1 < q_- \le q(x) \le q_+ < \infty$.

(iv) $\mathcal{B}(\Omega) = \big\{q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\colon$ *the maximal operator M is bounded on $L^{q(\cdot)}(\Omega)$*$\big\}$.

Definition 2.3. *Let $q(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ be a real-valued function.*

(1) Denote by $\mathcal{C}^{\log}_{\rm loc}(\mathbb{R}^n)$ the set of all local $\log$-Hölder continuous functions $q(\cdot)$ such that

$$|q(x) - q(y)| \le \frac{-C}{\ln(|x-y|)}, \qquad |x-y| \le 1/2, \quad x, y \in \mathbb{R}^n.$$

(2) Denote by $\mathcal{C}^{\log}_0(\mathbb{R}^n)$ the set of all $\log$-Hölder functions $q(\cdot)$ continuous at the origin and such that

$$|q(x) - q(0)| \le \frac{C}{\ln\Big(e + \dfrac{1}{|x|}\Big)}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

(3) Denote by $\mathcal{C}^{\log}_\infty(\mathbb{R}^n)$ the set of all $\log$-Hölder functions $q(\cdot)$ continuous at infinity and such that

$$\big|q(x) - q_\infty\big| \le \frac{C_\infty}{\ln(e + |x|)}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $q_\infty = \lim_{|x|\to\infty} q(x)$.

(4) Denote by

$$\mathcal{C}^{\log}(\mathbb{R}^n) := \mathcal{C}^{\log}_{\rm loc}(\mathbb{R}^n) \cap \mathcal{C}^{\log}_\infty(\mathbb{R}^n)$$

the set of all global $\log$-Hölder continuous functions $q(\cdot)$.

Remark 2.1. The condition $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$ is equivalent to the condition of uniform continuity

$$|q(x)-q(y)| \le \frac{C}{\ln(e+|x|)}, \qquad |y|\ge |x|, \quad x,y\in\mathbb{R}^n.$$

The condition $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$ was originally introduced in this form in [16].

We now define the variable exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$. Let

$$B_k=\{x\in\mathbb{R}^n:\ |x|\le 2^k\}, \quad A_k = B_k\setminus B_{k-1}, \quad \text{and} \quad \chi_k=\chi_{A_k} \quad \text{for} \quad k\in\mathbb{Z}.$$

Definition 2.4. *Suppose that* $0\le\lambda<\infty,\ 0<p<\infty,$

$$q(\cdot)\in\mathcal{P}(\mathbb{R}^n), \qquad \textit{and} \qquad \alpha(\cdot)\colon \mathbb{R}^n\to\mathbb{R} \quad \textit{with} \quad \alpha(\cdot)\in L^\infty(\mathbb{R}^n).$$

The variable exponent Herz–Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ *is defined as follows:*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)=\Big\{f\in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\})\colon \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}<\infty\Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}=\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\|2^{k\alpha(\cdot)}f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}.$$

We now compare the variable Herz–Morrey space

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$$

with the variable Herz space [9]

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n),$$

where

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\})\colon \sum_{k=-\infty}^{\infty}\|2^{k\alpha(\cdot)}f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}<\infty\right\}.$$

Obviously,

$$M\dot{K}^{\alpha(\cdot),0}_{p,q(\cdot)}(\mathbb{R}^n)=\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n).$$

If $\alpha(\cdot)$ is constant, then we get

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)=M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$$

(see [13]). If both $\alpha(\cdot)$ and $q(\cdot)$ are constants and $\lambda=0$, then

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)=\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$$

are classical Herz spaces.

2.2. Recent Results for the Riesz-Type Potential $I_{\beta(\cdot)}$. In this section, we recall some recent results obtained for the Riesz-type potential operator $I_{\beta(\cdot)}$. The order of potential $\beta(x)$ is not assumed to be continuous. We assume that it is a measurable function on Ω satisfying the following assumptions:

$$\beta_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \beta(x) > 0,$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)\beta(x) < n, \tag{2.3}$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p_\infty \beta(x) < n.$$

The open problem of boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ from the variable exponent space $L^{p(\cdot)}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$ with the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n},$$

was first solved in the case of bounded domains $\Omega \subset \mathbb{R}^n$ (see [17]). After Diening [18] proved the boundedness of the maximal operator over bounded domains, the validity of the Sobolev theorem for bounded domains became an unconditional statement.

In 2008, in the case of bounded sets, Almeida, Hasanov, and Samko [19] proved the boundedness of the maximal operator in variable exponent Morrey spaces. In 2009, Hästö [20] used his new "local-to-global" approach to extend the result obtained in [19] concerning the maximal operator to the entire space $\mathbb{R}^n$. In 2010, in the case of bounded sets, Guliyev, Hasanov, and Samko [21] considered the problem of boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable exponent Morrey-type spaces.

For the entire space $\mathbb{R}^n$, under the condition that the exponent $p(x)$ is constant outside some ball of large radius, the Sobolev theorem was proved by Diening [22].

Another version of the Sobolev theorem for the space $\mathbb{R}^n$ was proved in [23] for the exponent $p(x)$ not necessarily constant in a neighborhood of infinity but with some extra power weight fixed at infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

Theorem A. *Let $\beta(x)$ meet conditions (2.3) with $q_1(\cdot)$ instead of $p(\cdot)$. Suppose that*

$$q_1(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$$

and

$$1 < (q_1)_\infty \le q_1(x) \le (q_1)_+ < \infty. \tag{2.4}$$

Then the following weighted Sobolev-type estimate is valid for the operator $I_{\beta(\cdot)}$:

$$\big\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)},$$

where $q_2(x)$ is given by the formula

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}, \tag{2.5}$$

and

$$\gamma(x) = C_\infty \beta(x)\left(1 - \frac{\beta(x)}{n}\right) \le \frac{n}{4} C_\infty, \tag{2.6}$$

C_∞ is the Dini–Lipschitz constant from (2.2) with $q(\cdot)$ replaced by $q_1(\cdot)$.

In 2013, in the case of unbounded sets, Guliyev and Samko [24] studied the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable-exponent Morrey-type spaces. Recently, the author [14] has obtained the results similar to Theorem A for the variable-exponent Herz–Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$.

Remark 2.2. The fractional maximal operator is defined as follows:

$$M_{\beta(\cdot)}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{n-\beta(x)}} \int\limits_{B(x,r)} |f(y)|\, dy. \tag{2.7}$$

The pointwise estimate for (2.7) is also valid. This yields Theorem A.

2.3. Auxiliary Propositions and Lemmas. In this section, we formulate some auxiliary propositions and lemmas required to prove our main theorems. Here, we present only partial results used in our subsequent presentation.

Proposition 2.1. *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

(i) If $q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$, then $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

(ii) $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if and only if $q'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

The first part of Proposition 2.1 was independently proved by Cruz–Uribe, et al. [16] and Nekvinda [25]. The second part of Proposition 2.1 belongs to Diening [26] (see Theorem 8.1 or Theorem 1.2 in [27]).

Remark 2.3. Since

$$\left|q'(x) - q'(y)\right| \le \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

we immediately conclude that if $q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$, then the same is true for $q'(\cdot)$, i.e., if the condition is satisfied, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ and $L^{q'(\cdot)}(\mathbb{R}^n)$. Furthermore, Diening also proved general results for the Musielak–Orlicz spaces.

The next proposition is a generalization of variable-exponent Herz spaces from [9]. It was proved in [15].

Proposition 2.2. *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p \in (0,\infty)$, and $\lambda \in [0,\infty)$. If a real-valued function*

$$\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^{\log}_0(\mathbb{R}^n) \cap \mathcal{C}^{\log}_\infty(\mathbb{R}^n),$$

then

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}$$

$$\approx \max\left\{ \sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{\tilde{k}_1} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}, \right.$$

$$\sup_{\substack{k_0\geq 0\\ k_0\in\mathbb{Z}}}\left(2^{-k_0\lambda}\left(\sum_{k=-\infty}^{\tilde{k}_2} 2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}\right.$$

$$\left.\left. + 2^{-k_0\lambda}\left(\sum_{k=0}^{\tilde{k}_3} 2^{k\alpha_\infty p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}\right)\right\},$$

where $\tilde{k}_1 = k_0,\ \tilde{k}_2 = -1,\ \tilde{k}_3 = k_0$.

The next lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [8].

Lemma 2.1 (generalized Hölder's inequality)**.** *Suppose that* $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. *Then, for any* $f \in L^{q(\cdot)}(\mathbb{R}^n)$ *and any* $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, *the following inequality is true:*

$$\int\limits_{\mathbb{R}^n} \big|f(x)g(x)\big|dx \leq C_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $C_q = 1 + 1/q_- - 1/q_+$.

The following lemma can be found in [28]:

Lemma 2.2.

(I) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exist positive constants* $\delta \in (0,1)$ *and* $C > 0$ *such that*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^\delta$$

for all balls B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$.

(II) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a positive constant* $C > 0$ *such that*

$$C^{-1} \leq \frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C$$

for all balls B *in* $\mathbb{R}^n$.

Remark 2.4.

(i) If

$$q_1(\cdot),\ q_2(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n),$$

then we can see that

$$q_1'(\cdot),\, q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n).$$

Hence, it is possible take positive constants

$$0 < \delta_1 < 1/(q_1')_+, \qquad 0 < \delta_2 < 1/(q_2)_+$$

such that

$$\frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \qquad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{2.8}$$

hold for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$ (see [28, 29]).

(ii) On the other hand, Kopaliani [30] proved the following fact: If the exponent $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is equal to a constant outside a certain large ball, then $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if and only if $q(\cdot)$ satisfies the Muckenhoupt-type condition

$$\sup_{Q:\ \text{cube}} \frac{1}{|Q|} \|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} < \infty.$$

3. Main Result and Its Proof

Our main result can be formulated as follows:

Theorem 3.1. *Suppose that $q_1(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $\beta(x)$ satisfy conditions (2.3) with $q_1(\cdot)$ instead of $p(\cdot)$. Consider a variable exponent $q_2(\cdot)$ defined by (2.5). Let $q_1(\cdot)$ and $q_2'(\cdot)$ satisfy condition (2.4), let $0 < p_1 \le p_2 < \infty$, $\lambda \ge 0$, and let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity with*

$$\lambda - n\delta_2 < \alpha(0) \le \alpha_\infty < \lambda + n\delta_1,$$

where $\delta_1 \in (0, 1/(q_1')_+)$ and $\delta_2 \in (0, 1/(q_2)_+)$ are the constants appearing in (2.8). Then

$$\big\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ is defined as in (2.6), and the Dini–Lipschitz constant has the form

$$\max\left\{C_\infty, \frac{2C_\infty}{\big((q_1)_- - 1\big)^2}\right\}$$

if $q(\cdot)$ in (2.2) is replaced by $q_1(\cdot)$.

Remark 3.1.

(i) Under the assumptions of Theorem 3.1, the result similar to Theorem 3.1 is also valid for the fractional maximal operator $M_{\beta(\cdot)}(f)$ defined by (2.7) (for some details, see [14]).

(ii) If $\alpha(\cdot)$ is a constant exponent, then the above result can be found in [14].

(iii) The result presented is also valid for $\lambda = 0$.

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$, if we denote

$$f_j := f\chi_j = f\chi_{A_j} \qquad \text{for all} \quad j \in \mathbb{Z},$$

then we get

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Since $0 < p_1/p_2 \le 1$, applying the inequality

$$\left(\sum_{i=-\infty}^{\infty} |a_i|\right)^{p_1/p_2} \le \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2}, \tag{3.1}$$

and Proposition 2.2, we obtain

$$\begin{aligned}
&\big\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\big\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \\
&\approx \max\Bigg\{ \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_2} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1/p_2}, \\
&\qquad \sup_{\substack{k_0\ge0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \Bigg[\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_2} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1/p_2} \\
&\qquad\qquad + \left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_2} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1/p_2} \Bigg] \Bigg\} \\
&\le \max\Bigg\{ \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right), \\
&\qquad \sup_{\substack{k_0\ge0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \Bigg[\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \\
&\qquad\qquad + \left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1} \|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \Bigg] \Bigg\} \\
&\equiv: \max\{E_1, E_2 + E_3\},
\end{aligned}$$

where

$$E_1 = \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1}\right),$$

$$E_2 = \sup_{\substack{k_0\geq 0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1}\right),$$

$$E_3 = \sup_{\substack{k_0\geq 0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1}\right).$$

It is not difficult to show that the estimate of E_1 is similar to the estimate of E_2. Therefore, we now consider only the estimates for E_1 and E_3.

Thus, for E_1, we obtain

$$\begin{aligned} E_1 &\leq C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=-\infty}^{k-2}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\\ &\quad + C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k-1}^{k+1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\\ &\quad + C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k+2}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\\ &\equiv: C(E_{11}+E_{12}+E_{13}). \end{aligned}$$

First, we estimate E_{12}. By using Theorem A and Proposition 2.2, we get

$$\begin{aligned} E_{12} &\leq C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k-1}^{k+1}\|f_j\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\\ &\leq C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|f\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\right)\\ &\leq C\|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}. \end{aligned}$$

We now estimate E_{11}. Note that if

$$x\in A_k, \qquad j\leq k-2, \qquad \text{and} \qquad y\in A_j,$$

then

$$|x-y|\backsim|x| \qquad \text{and} \qquad 2|y|\leq|x|.$$

Therefore, by using the generalized Hölder's inequality, we find

$$\begin{aligned} \left|I_{\beta(\cdot)}(f_j)(x)\chi_k(x)\right| &\leq \int\limits_{A_j} \frac{|f(y)|}{|x-y|^{n-\beta(x)}} dy \chi_k(x) \\ &\leq C \cdot 2^{-kn} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |x|^{\beta(x)} \chi_k(x). \end{aligned} \tag{3.2}$$

Note that

$$I_{\beta(\cdot)}(\chi_{B_k})(x) \geq I_{\beta(\cdot)}(\chi_{B_k})(x)\chi_{B_k}(x) \geq C|x|^{\beta(x)}\chi_k(x). \tag{3.3}$$

Using Theorem A, Lemma 2.2, (2.8), (3.2), and (3.3), we find

$$\begin{aligned} &\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\quad \leq C \cdot 2^{-kn} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(\chi_{B_k})\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\quad \leq C \cdot 2^{-kn} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\quad \leq C \cdot 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.4}$$

On the other hand, we mention the following fact:

Case I ($\tilde{k}_i < 0,\ i = 1, 2, 3$)**:**

$$\begin{aligned} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &\leq 2^{-j\alpha(0)} \left(\sum_{i=-\infty}^{j} 2^{i\alpha(0)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{1/p_1} \\ &\leq 2^{j(\lambda-\alpha(0))} \left(2^{-j\lambda} \left(\sum_{i=-\infty}^{j} \|2^{i\alpha(\cdot)} f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{1/p_1} \right) \\ &\leq C \cdot 2^{j(\lambda-\alpha(0))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.5}$$

Case II ($\tilde{k}_i \geq 0,\ i = 1, 2, 3$)**:**

$$\begin{aligned} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &\leq 2^{-j\alpha_\infty} \left(\sum_{i=0}^{j} 2^{i\alpha_\infty p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{1/p_1} \\ &\leq C \cdot 2^{j(\lambda-\alpha_\infty)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.6}$$

Here, we have used Definition 2.4, Proposition 2.2, and the condition for $\alpha(\cdot)$.

Thus, combining (3.4) and (3.5) and using $\alpha(0) \le \alpha_\infty < \lambda + n\delta_1,$ we conclude that

$$E_{11} \le C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)$$

$$\le C\,\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\right)$$

$$\le C\,\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

We now estimate E_{13}. Note that, if

$$x \in A_k, \quad j \ge k+2, \quad \text{and} \quad y \in A_j,$$

then

$$|x-y| \backsim |y| \qquad \text{and} \qquad 2|x| \le |y|.$$

Hence, by using the generalized Hölder's inequality, we find

$$\left|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)(x)\chi_k(x)\right|$$

$$\le (1+|x|)^{-\gamma(x)} \int\limits_{A_j} \frac{|f(y)|}{|x-y|^{n-\beta(x)}}\,dy\chi_k(x)$$

$$\le C\int_{A_j} |f(y)|(1+|x|)^{-\gamma(x)}|y|^{\beta(x)-n}\,dy\chi_k(x)$$

$$\le C\cdot 2^{-jn}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\chi_k(x). \tag{3.7}$$

By analogy with (3.3), we get

$$I_{\beta(\cdot)}(\chi_{B_j})(x) \ge I_{\beta(\cdot)}(\chi_{B_j})(x)\chi_{B_j}(x) \ge C|x|^{\beta(x)}\chi_j(x). \tag{3.8}$$

Using Theorem A, Lemma 2.2, (2.8), (3.7), and (3.8), we obtain

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\cdot 2^{-jn}\|\chi_{B_j}\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

$$\le C\cdot 2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.9}$$

Therefore, combining (3.5) with (3.9) and using

$$\lambda - n\delta_2 < \alpha(0) \le \alpha_\infty,$$

we conclude that

$$E_{13} \le C \sup_{\substack{k_0<0 \\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right)$$

$$\le C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0<0 \\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \right) \le C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_{11}, E_{12}, and E_{13}, we find

$$E_1 \le C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

For E_3, by analogy with the estimate of E_1, in view of Theorem A, Proposition 2.2, and relations (2.8), (3.1)–(3.4), and (3.6)–(3.9), we get

$$E_3 \le C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

The joint estimate for E_1, E_2 and E_3 implies that

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Theorem 3.1 is proved.

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