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ESTIMATION OF THE GENERALIZED BESSEL–STRUVE TRANSFORM IN A SPACE OF GENERALIZED FUNCTIONS

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We study the so-called Bessel–Struve transform in a certain class of generalized functions called Boehmians. By using different convolution products, we generate the Boehmian spaces in which the extended transform is well defined. We also show that the Bessel–Struve transform of a Boehmian is an isomorphism continuous with respect to a certain type of convergence.

1. Introduction

Although special types of what would later be known as Bessel functions were studied by Euler, Lagrange, and the Bernoullis, the Bessel functions were first used by Bessel to describe the three-body motion with Bessel functions appearing in the series expansion of planetary perturbations and series solutions to second-order differential equations encountered in diverse situations. On the other hand, the Struve functions occur in numerous problems of physics and applied mathematics, e.g., in optics, as the normalized line spread function, in fluid dynamics, and also (quite prominently) in acoustics for the impedance calculations.

The normalized Bessel and Struve functions of index α were, respectively, given by Watson [3] as follows:

$$\jmath_\alpha(z)2^{\alpha}\Gamma(\alpha+1)\,z^{-\alpha}\jmath_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\dfrac{z}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}$$

and

$$k_\alpha(z)=2^{\alpha}\Gamma(\alpha+1)\,z^{-\alpha}H_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\dfrac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\dfrac{3}{2}\right)\Gamma\left(n+\alpha+\dfrac{3}{2}\right)}.$$

A kind of Fourier transforms called the Bessel–Struve transform was considered by Hamem, et al. [2]:

$$\boldsymbol{f}^{\alpha}_{\beta,s}\left(f(x)\right)(\lambda)=\int_{-\infty}^{\infty}f(x)\sigma_\alpha\left(-i\lambda x\right)d\mu_\alpha(x),$$

where $\alpha>-\dfrac{1}{2}$ and σ_α is the Bessel–Struve kernel given by the equation

$$\sigma_\alpha(x)=\jmath_\alpha(ix)-ik_\alpha(ix).$$

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The Bessel–Struve kernel is the solution of the initial-value problem

$$\ell_\alpha u(x) = \lambda^2 u(x),$$

where

$$u\left(0\right) = 1 \qquad \text{and} \qquad u'\left(0\right) = \frac{\lambda\Gamma\left(\alpha+1\right)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{3}{2}\right)}.$$

It additionally satisfies the integral representation

$$\sigma_\alpha\left(\lambda x\right) = \frac{\lambda\Gamma\left(\alpha+1\right)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int\limits_0^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} e^{\lambda x t}\, dt,$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

Moreover, the Bessel–Struve transform is related to the Weyl integral transform [2]

$$\boldsymbol{w}_\alpha\left(f\right)\left(y\right) = \frac{2\Gamma\left(\alpha+1\right)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int\limits_{|y|}^1 \left(x^2-y^2\right)^{\alpha-\frac{1}{2}} xf\left(\operatorname{sgn}\left(y\right)x\right) dx$$

and satisfies the relation

$$\boldsymbol{f}^\alpha_{\beta,s}\left(f\right) = \mathcal{F}f \circ \boldsymbol{w}_\alpha\left(f\right), \tag{1}$$

where $f \in \boldsymbol{l}^1_\alpha(\mathbb{R})$ and $\mathcal{F}f$ is the Fourier transform of f,

$$\mathcal{F}\left(f(x)\right)\left(y\right) = \int\limits_{-\infty}^{\infty} f(x)e^{-ixy}\, dx.$$

The Mellin-type convolution product of the first kind is given by the integral equation [10]

$$f \times g(y) = \int\limits_0^\infty f\left(yx^{-1}\right) x^{-1} g(x)\, dx. \tag{2}$$

The space $\boldsymbol{l}^p_\alpha(\mathbb{R})$ consists of the real-valued measurable functions f defined on $\mathbb{R}$ and such that

$$\|f\|^p_\alpha := \begin{cases} \left(\int\limits_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha\left(x\right)\right)^{1/p} < \infty, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty, \end{cases}$$

where

$$d\mu_\alpha(x) = A(x)dx \qquad \text{and} \qquad A(x) = |x|^{2\alpha+1}.$$

By $\boldsymbol{\kappa}(\mathbb{R})$ we denote the space of test functions with bounded supports over $\mathbb{R}$. Thus, $\boldsymbol{\kappa}(\mathbb{R})$ is, indeed, a dense subspace of $\boldsymbol{l}^p(\mathbb{R})$ for every choice of p. Here, $\boldsymbol{l}^1(0,\infty)$ denotes the Lebesgue space of complex-valued integrable functions defined on $(0,\infty)$ and $\boldsymbol{l}^p_\alpha(0,\infty)$ denotes the restriction of $\boldsymbol{l}^p_\alpha(\mathbb{R})$ to the open interval $(0,\infty)$.

The following definition is very useful for our subsequent investigation:

Definition 1. *Let* $\alpha > -\dfrac{1}{2}$, *let* $\mathtt{A}(t) = |t|^{2\alpha+1}$, *and let* $f,\ g$ *in* $\boldsymbol{l}^1(0,\infty)$. *Then we define the product* $\otimes$ *of* f *and* g *by the integral*

$$f \otimes g(y) = \int_0^\infty f(yt)\, g(t) d\mu(t), \tag{3}$$

where $d\mu(t) = \mathtt{A}(t)dt$.

By using (2) and (3), we get the following proposition:

Proposition 1. *Let* $f,\ g$, *and* h *be integrable functions in* $\boldsymbol{l}^1(0,\infty)$ *and let* $y > 0$. *Then*

$$f \otimes (g \times h)(y) = (f \otimes g) \otimes h(y).$$

Proof. Assume that the hypothesis of the theorem are satisfied for $f,\ g$ and h in $\boldsymbol{l}^1(0,\infty)$. Then, in view of (2) and (3), we get

$$f \otimes (g \times h)(y) = \int_0^\infty f(yt) \int_0^\infty x^{-1} g\left(tx^{-1}\right) h(x)\, dx\, d\mu(t).$$

By Fubini's theorem, we obtain

$$f \otimes (g \times h)(y) = \int_0^\infty h(x)x^{-1} \int_0^\infty g\left(tx^{-1}\right) f(yt)\, d\mu(t)\, dx.$$

Changing the variables, we get

$$f \otimes (g \times h)(y) = \int_0^\infty h(x) \int_0^\infty f(yxz) g(z)\, d\mu(z)\, d\mu(x).$$

Proposition 1 is proved.

By virtue of Proposition 2.1 of [2], we conclude that $\boldsymbol{w}_\alpha$ is a bounded operator from $\boldsymbol{l}^1_\alpha(\mathbb{R})$ into $\boldsymbol{l}^1(\mathbb{R})$. Hence, we get the following remark:

Remark 1. Let $f \in \boldsymbol{l}^1_\alpha(\mathbb{R})$. Then $\boldsymbol{f}^\alpha_{\beta,s}(f) \in \boldsymbol{l}^1_\alpha(\mathbb{R})$.

The proof of this remark follows from equation (1) and the injectivity of $\mathcal{F}$. We therefore omit the details.

2. Generated Spaces of Boehmians

Boehmians were used for all objects defined by an algebraic construction similar to the construction of the field of quotients and, in some cases, it gives just the field of quotients. The appearance of Boehmians has recently brought drastic changes in the concept of applied functional analysis. The idea of construction of Boehmians was inspired by the concept of Mikusinski regular operators.

The minimal structure necessary for the construction of Boehmians consists of the following axioms:

A(i) A nonempty set $\mathfrak{a}$.

A(ii) A commutative semigroup $(\mathfrak{b}, \bullet)$.

A(iii) An operation $\star\colon \mathfrak{a} \times \mathfrak{b} \to \mathfrak{a}$ such that, for each $x \in \mathfrak{a}$ and $s_1, s_2 \in \mathfrak{b}$,

$$x \star (s_1 \bullet s_2) = (x \star s_1) \star s_2.$$

A(iv) A collection $\Delta \subset \mathfrak{b}^{\mathbb{N}}$ such that:

(a) if $x, y \in \mathfrak{a}$, $(s_n) \in \Delta$, $x \bullet s_n = y \bullet s_n$ for all n, then $x = y$;

(b) if $(s_n), (t_n) \in \Delta$, then $(s_n \bullet t_n) \in \Delta$.

The elements of Δ are called delta sequences. Consider

$$Q = \big\{(x_n, s_n)\colon x_n \in \mathfrak{a},\ \ (s_n) \in \Delta,\ \ x_n \star s_m = x_m \star s_n\ \ \forall m, n \in \mathbb{N}\big\}.$$

If $(x_n, s_n), (y_n, t_n) \in Q$, $x_n \star t_m = y_m \star s_n$ $\forall m, n \in \mathbb{N}$, then we say that $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence classes in Q is denoted by $\boldsymbol{b}$. The elements of $\boldsymbol{b}$ are called Boehmians.

Between $\mathfrak{a}$ and $\boldsymbol{b}$, there is a canonical embedding expressed as

$$x \to \frac{x \star s_n}{s_n}.$$

The operation $\star$ can be extended to $\boldsymbol{b} \times \mathfrak{a}$ by

$$\frac{x_n}{s_n} \star t = \frac{x_n \star t}{s_n}.$$

The relationship between the notion of convergence and the product $\star$ is specified as follows:

(i) if $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and $\phi \in \mathfrak{b}$ is any fixed element, then $f_n \star \phi \to f \star \phi$ in $\mathfrak{a}$ (as $n \to \infty$);

(ii) if $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \to f$ in $\mathfrak{a}$ (as $n \to \infty$).

The operation $\star$ is extended to $\boldsymbol{b} \times \mathfrak{b}$ as follows:

$$\text{If} \quad \left[\frac{(f_n)}{(s_n)}\right] \in \boldsymbol{b} \quad \text{and} \quad \phi \in \mathfrak{b}, \quad \text{then} \quad \left[\frac{(f_n)}{(s_n)}\right] \star \phi = \left[\frac{(f_n) \star \phi}{s_n}\right].$$

The convergence in $\boldsymbol{b}$ is defined as follows:

A sequence (h_n) in $\boldsymbol{b}$ is said to be δ **convergent** to h in $\boldsymbol{b}$, $h_n \overset{\delta}{\to} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n \star s_n)$, $(h \star s_n) \in \mathfrak{a}\ \forall k, n \in \mathbb{N}$ and $(h_n \star s_k) \to (h \star s_k)$ as $n \to \infty$, in $\mathfrak{a}$ for every $k \in \mathbb{N}$.

A sequence (h_n) in $\boldsymbol{b}$ is said to be Δ **convergent** to h in $\boldsymbol{b}$, $h_n \overset{\Delta}{\to} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in \mathfrak{a}\ \forall n \in \mathbb{N}$ and $(h_n - h) \star s_n \to 0$ as $n \to \infty$ in $\mathfrak{a}$.

Several integral transforms were extended to various spaces of Boehmians by numerous authors, namely, by Al-Omari and Kilicman [9, 15, 20], Al-Omari [13], Mikusinski, and Zayed [16], Karunakaran and Roopkumar [17], Karunakaran and Vembu [18], Roopkumar [19], Nemzer [21], Al-Omari, Loonker, Banerji, and Kalla [11], and many others. However, the readers are assumed to be acquainted with the abstract construction of Boehmian spaces. Otherwise, we refer the readers to [4–9, 11, 13] and [15–21]. In what follows, we need the following lemma:

Lemma 1. *Let $f \in \boldsymbol{l}^1_\alpha(0,\infty)$ and $\psi \in \boldsymbol{\kappa}(0,\infty)$. Then*

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \big(\boldsymbol{f}^{\alpha}_{\beta,s} f \otimes \psi(x)\big)(\lambda).$$

Proof. Under the condition of the theorem, we can write

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \int_0^\infty \int_0^\infty f\left(xt^{-1}\right) t^{-1}\psi(t) dt \sigma_\alpha\left(-i\lambda x\right) d\mu_\alpha(x).$$

By Fubini's theorem, this can be rewritten as

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \int_0^\infty \psi(t) \int_0^\infty f\left(xt^{-1}\right) \sigma_\alpha\left(-i\lambda x\right) d\mu_\alpha(x)\, dt.$$

Changing the variables, we get

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \int_0^\infty \big(\boldsymbol{f}^{\alpha}_{\beta,s} f(z);\lambda(t)\big)\, \psi(t)\, d\mu(t).$$

Hence, equation (3) implies that

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \big(\boldsymbol{f}^{\alpha}_{\beta,s} f \otimes \psi\big)(\lambda).$$

Lemma 1 is proved.

The spaces generated here are the space $\beta_1\big(\boldsymbol{l}^1_\alpha, \boldsymbol{\kappa}, \times), \times, \Delta\big)$ and the space $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$. By Δ (wherever it appears) we denote the set of delta sequences (δ_n) from $\boldsymbol{\kappa}(0,\infty)$, where

$$\int_0^\infty \delta_n(x)\, dx = 1, \tag{4}$$

$$\int_0^\infty \left|\delta_n(x)\right| dx < m, \qquad m \text{ is a positive real number,} \tag{5}$$

$$\operatorname{supp} \delta_n \subseteq [-\varepsilon, \varepsilon], \qquad \varepsilon \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

It is also necessary to recall some properties of the product $\times$, namely (see [12, 10]):

$$f \times g = g \times f, \tag{7}$$

$$f \times (g + h) = f \times g + f \times h, \tag{8}$$

$$f \times (g \times h) = (f \times g) \times h, \tag{9}$$

$$(\alpha f) \times g = \alpha\,(f \times g) = f \times (\alpha g)\,, \quad \alpha \in \mathbb{C}. \tag{10}$$

We merely generate the space $\beta_1\big(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$, while the space $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$ can be generated similarly.

Theorem 1. *Let $f \in \boldsymbol{l}_\alpha^1(0,\infty)$ and $\psi \in \boldsymbol{\kappa}(0,\infty)$, $\alpha > -\dfrac{1}{2}$. Then $f \times \psi \in \boldsymbol{l}_\alpha^1(0,\infty)$.*

Proof. Let $f \in \boldsymbol{l}_\alpha^1(0,\infty)$ and $\psi \in \boldsymbol{\kappa}(0,\infty)$ be given. Assume that $K = [a, b]$, $0 < a < b$, is a compact subset of $(0,\infty)$ such that $\operatorname{supp} \psi \subseteq K$. Then, for $\alpha > -\dfrac{1}{2}$, we obtain

$$\int_0^\infty |f \times \psi(y)|\, d\mu(y) = \int_0^\infty \left| \int_0^\infty f\left(yt^{-1}\right) t^{-1} \psi(t)\, dt \right| d\mu(y)$$

$$\leq \int_a^b |\psi(t)|\, t^{-1} \int_0^\infty \left| f\left(yt^{-1}\right)\right| d\mu(y)\, dt.$$

By the change of variables $z = yt^{-1}$, we get

$$\int_0^\infty |f \times \psi(y)|\, d\mu(y) \leq \int_a^b |\psi(t)|\, t^{2\alpha}\, dt \int_0^\infty |f(z)|\, d\mu(z).$$

This can be interpreted as follows:

$$\left|f \times \psi(y)\right|_\alpha^1 \leq m^* \, \|f\|_\alpha^1\,, \tag{11}$$

where

$$m^* = \int_a^b |\psi(t)|\, t^{2\alpha} dt.$$

Theorem 1 is proved.

Theorem 2. *Let* $f \in \boldsymbol{l}_\alpha^1(0,\infty)$ *and* $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty),\ \alpha > -\dfrac{1}{2}$. *Then*

(i) $f \times (\psi_1 + \psi_2) = f \times \psi_1 + f \times \psi_2,$

(ii) $f \times (\psi_1 \times \psi_2) = (f \times \psi_1) \times (\psi_2),$

(iii) $(\alpha f) \times \psi 1 = \alpha\,(f \times \psi_1) = f \times (\alpha \psi_1),\ \alpha \in \mathbb{C}.$

The proofs of identities (i) and (iii) follow from simple integral calculus. Identity (ii) directly follows from (9). This proves the theorem.

Theorem 3. *Let* $f_n \to f \in \boldsymbol{l}_\alpha^1(0,\infty)$ *as* $n \to \infty$ *and let* $\psi \in \boldsymbol{\kappa}(0,\infty),\ \alpha > -\dfrac{1}{2}$. *Then*

$$f_n \times \psi \to f \times \psi \qquad \text{as} \quad n \to \infty$$

in $\boldsymbol{l}_\alpha^1(0,\infty)$.

This theorem is proved by simple integration. Hence, we omit the details.

Theorem 4. *Let* $f \in \boldsymbol{l}_\alpha^1(0,\infty)$ *and let* $(\delta_n) \in \Delta, \alpha > -\dfrac{1}{2}$. *Then*

$$f \times \delta_n \to f \qquad \text{as} \quad n \to \infty$$

in $\boldsymbol{l}_\alpha^1(0,\infty)$.

Proof. Assume that $f \in \boldsymbol{l}_\alpha^1(0,\infty)$ and $(\delta_n) \in \Delta$ are given. Since the space $\boldsymbol{\kappa}(0,\infty)$ is dense in $\boldsymbol{l}_\alpha^1(0,\infty)$ we can find $\psi \in \boldsymbol{\kappa}(0,\infty)$ such that

$$\|f - \psi\|_\alpha^1 < \varepsilon \tag{12}$$

for $\varepsilon > 0$.

Moreover, by virtue of (11) and the fact that $(\delta_n) \in \boldsymbol{\kappa}(0,\infty),$ we obtain

$$\big\|(f-\psi) \times \delta_n\big\|_\alpha^1 \le m^* \,\|f - \psi\|_\alpha^1$$

for a real number m^*.

Hence, inserting (12) in above equation we get

$$\big\|(f-\psi) \times \delta_n\big\|_\alpha^1 \le \varepsilon m^*. \tag{13}$$

Thus, we get

$$\big\|\psi \times \delta_n - \psi\big\|_\alpha^1 = \int\limits_0^\infty |(\psi \times \delta_n - \psi)(y)|\, d\mu(y)$$

$$= \int\limits_0^\infty \left| \int\limits_0^\infty \psi\left(yt^{-1}\right) t^{-1} \delta_n(t) dt - \psi(y) \int\limits_0^\infty \delta_n(t) dt \right| d\mu(y)$$

$$\leq \int_0^\infty \int_0^\infty \left|\psi\left(yt^{-1}\right)t^{-1} - \psi(y)\right| |\delta_n(t)|\, dt\, d\mu(y). \tag{14}$$

Now, let $g_y(t) = \psi\left(yt^{-1}\right)t^{-1}$. Then $g_y(t)$ is a uniformly continuous function in $\boldsymbol{\kappa}(0,\infty)$. Therefore, we can find $\delta > 0$ such that

$$\left|g_y(t) - g(1)\right| < \varepsilon \qquad \text{whenever} \quad |y-1| < \delta.$$

Thus, using (4) in (14), we conclude that

$$\left\|\psi \times \delta_n - \psi\right\|_\alpha^1 \leq \int_0^\infty \int_0^\infty \left|g_y(t) - g_y(1)\right| |\delta_n(t)|\, dt\, d\mu(y) \leq \varepsilon \int_c^d d\mu(y), \tag{15}$$

where $[a,b]$ is an interval containing the support of g_y.

Therefore, inequality (15) implies that

$$\left\|\psi \times \delta_n - \psi\right\|_\alpha^1 \leq A\varepsilon, \tag{16}$$

where

$$A = \int_c^d d\mu(y).$$

In view of (13), (16) and (12), we arrive at the inequalities

$$\left\|f \times \delta_n - f\right\|_\alpha^1 \leq \left\|(f-\psi) \times \delta_n\right\|_\alpha^1 + \left\|\psi \times \delta_n - \psi\right\|_\alpha^1 t \left\|f - \psi\right\|_\alpha^1 \leq \varepsilon m^* + A\varepsilon + \varepsilon.$$

Hence, the equation presented above gives

$$\left\|f \times \delta_n - f\right\|_\alpha^1 \leq B\varepsilon,$$

where $B = m^* + A + 1$.

Theorem 4 is proved.

Thus, the space $\beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ is generated.

The sum of two Boehmians in $\beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ and the procedure of multiplication by a scalar can be defined as follows:

$$\left[\frac{(f_n)}{(\delta_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(f_n) \times \psi_n + (g_n) \times (\delta_n)}{(\delta_n) \times (\psi_n)}\right]$$

and

$$\alpha\left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\alpha(f_n)}{(\delta_n)}\right],$$

where $\alpha \in \mathbb{C}$ and $\mathbb{C}$ is the space of complex numbers.

The operation $\times$ and the operation of differentiation are defined as follows:

$$\left[\frac{(f_n)}{(\delta_n)}\right] \times \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(f_n) \times (g_n)}{(\delta_n) \times (\psi_n)}\right]$$

and

$$\mathcal{D}^\alpha \left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\mathcal{D}^\alpha (f_n)}{(\delta_n)}\right],$$

respectively.

A sequence of Boehmians (β_n) in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ is said to be δ convergent to a Boehmian β in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta-sequence (δ_n) such that

$$(\beta_n \times \delta_k), (\beta \times \delta_k) \in \boldsymbol{l}^1_\alpha \quad \forall k, n \in \mathbb{N},$$

and

$$(\beta_n \times \delta_k) \to (\beta \times \delta_k) \quad \text{as} \quad n \to \infty, \quad \text{in} \quad \boldsymbol{l}^1_\alpha, \quad \text{for every} \quad k \in \mathbb{N}.$$

The equivalent statement for the δ convergence has the following form:

$\beta_n \xrightarrow{\delta} \beta \ (n \to \infty)$ in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ if and only if there exist $(\varphi_{n,k}), (\varphi_k) \in \boldsymbol{l}^1_\alpha$ and $(\delta_k) \in \Delta$ such that

$$\beta_n = \left[\frac{(\varphi_{n,k})}{(\delta_k)}\right], \qquad \beta = \left[\frac{(\varphi_k)}{(\delta_k)}\right],$$

and

$$\varphi_{n,k} \to \varphi_k \qquad \text{as} \quad n \to \infty$$

in $\boldsymbol{l}^1_\alpha$ for each $k \in \mathbb{N}$.

A sequence of Boehmians (β_n) in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ is said to be Δ convergent to a Boehmian β in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists $(\delta_n) \in \Delta$ such that $(\beta_n - \beta) \times \delta_n \in \boldsymbol{l}^1_\alpha \ \forall n \in \mathbb{N}$ and $(\beta_n - \beta) \times \delta_n \to 0$ as $n \to \infty$ in $\boldsymbol{l}^1_\alpha$.

Similarly, the following theorems generate the Boehmian space $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$.

Theorem 5. *Let $f \in \boldsymbol{l}^1(0,\infty)$ and $\psi \in \boldsymbol{\kappa}(0,\infty)$. Then $f \otimes \psi \in \boldsymbol{l}^1(0,\infty)$.*

Theorem 6. *Let $f \in \boldsymbol{l}^1(0,\infty)$ and $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty)$. Then*

(i) $f \otimes (\psi_1 + \psi_2) = f \otimes \psi_1 + f \otimes \psi_2$,

(ii) $(\alpha f) \otimes \psi_1 = \alpha\,(f \otimes \psi_1) = f \otimes (\alpha \psi_1),\ \alpha \in \mathbb{C}$.

Theorem 7. *For $f \in \boldsymbol{l}^1(0,\infty)$ and $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty)$, the following relation is true:*

$$f \otimes (\psi_1 \times \psi_2) = (f \otimes \psi_1) \otimes \psi_2.$$

The proofs of Theorems 5 and 6 are similar to the proofs of Theorems 1 and 2, respectively. The proof of Theorem 7 follows from Proposition 1.

Theorem 8.

(i) *Let* $f_n \to f$ *in* $\boldsymbol{l}^1(0,\infty)$ *as* $n \to \infty$ *and let* $\psi \in \boldsymbol{\kappa}(0,\infty)$. *Then* $f_n \otimes \psi \to f \otimes \psi$ *as* $n \to \infty$.

(ii) *Let* $f_n \in \boldsymbol{l}^1(0,\infty)$ *and let* $(\delta_n) \in \Delta$. *Then* $f_n \otimes \delta_n \to f$ *as* $n \to \infty$.

The proof of Part (i) of the theorem is obtained by simple integration, whereas proof of the second part is analogous to the proof of Theorem 3. Hence, we prefer to omit the details.

The sum of two Boehmians in $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ and the operation of multiplication by a scalar can be also defined as follows:

$$\left[\frac{(f_n)}{(\delta_n)}\right] + \left[\frac{(g_n)}{(\varepsilon_n)}\right] = \left[\frac{(f_n) \otimes \varepsilon_n + (g_n) \otimes (\delta_n)}{(\delta_n) \times (\varepsilon_n)}\right]$$

and

$$\alpha \left[\frac{(f_n)}{(\delta_n)}\right] = \left[\alpha \frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\alpha (f_n)}{(\delta_n)}\right],$$

where $\alpha \in \mathbb{C}$ and $\mathbb{C}$ is the space of complex numbers.

The operation $\otimes$ and the operation of differentiation are, respectively, defined as

$$\left[\frac{(f_n)}{(\delta_n)}\right] \otimes \left[\frac{(g_n)}{(\varepsilon_n)}\right] = \left[\frac{(f_n) \otimes (g_n)}{(\delta_n) \times (\varepsilon_n)}\right]$$

and

$$\mathcal{D}^k \left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\mathcal{D}^k (f_n)}{(\delta_n)}\right].$$

The notions of δ- and Δ-convergence in $\beta_1\left(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ and $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ can be defined in a natural way as above.

3. The Bessel–Struve Transform of a Boehmian

Let $\beta \in \beta_1\left(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ and $\beta = \left[(f_n)(\delta_n)\right]$. Then, for every $\alpha > -\dfrac{1}{2}$, we define the Bessel–Struve transform of β as follows:

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\delta_n)}\right]\right) = \left[\frac{\left(\boldsymbol{f}^\alpha_{\beta,s} f_n\right)}{(\delta_n)}\right]. \tag{17}$$

The right-hand side of (17) belongs to $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ by virtue of Remark 1. The definition presented above is indeed well defined. Let

$$\left[\frac{(f_n)}{(\omega_n)}\right] = \left[\frac{(g_n)}{(\varepsilon_n)}\right] \in \beta_1\left(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\right).$$

Then, by the notion of equivalence classes in $\beta_1\left(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$, we find

$$f_n \times \varepsilon_m = g_m \times \omega_n.$$

Relation (17) and the notion of equivalence classes in $\beta_2\big(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\big)$ yield

$$\boldsymbol{f}^{\alpha}_{\beta,s}f_n\otimes\varepsilon_m=\boldsymbol{f}^{\alpha}_{\beta,s}g_m\otimes\omega_n.$$

Hence, we conclude that

$$\frac{\big(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\big)}{(\omega_n)}\sim\frac{\big(\boldsymbol{f}^{\alpha}_{\beta,s}g_n\big)}{(\varepsilon_n)}\qquad\text{in}\quad\beta_2\big(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\big).$$

Thus, we get

$$\left[\frac{\big(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\big)}{(\omega_n)}\right]=\left[\frac{\big(\boldsymbol{f}^{\alpha}_{\beta,s}g_n\big)}{(\varepsilon_n)}\right].$$

This proves the claim.

Theorem 9. $\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}$ *is an isomorphism from* $\beta_1\big(\boldsymbol{l}^1_\alpha,\boldsymbol{\kappa},\times),\times,\Delta\big)$ *into* $\beta_2\big(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\big)$.

Proof. We first establish that $\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}$ is injective. Given

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)=\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(g_n)}{(\varepsilon_n)}\right]\right),$$

by virtue of Lemma 1 and the notion of equivalent classes of $\beta_2\big(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\big)$, we conclude that

$$\boldsymbol{f}^{\alpha}_{\beta,s}f_n\otimes\varepsilon_m=\boldsymbol{f}^{\alpha}_{\beta,s}g_m\otimes\omega_n.$$

Therefore, Lemma 1 implies that $\boldsymbol{f}^{\alpha}_{\beta,s}(f_n\times\varepsilon_m)=\boldsymbol{f}^{\alpha}_{\beta,s}(g_m\times\omega_n)$. Employing $\boldsymbol{f}^{\alpha}_{\beta,s}$ gives

$$f_n\times\varepsilon_m=g_m\times\omega_n.$$

On the other hand, the notion of equivalent classes in $\beta_1\big(\boldsymbol{l}^1_\alpha,(\boldsymbol{\kappa},\times),\times,\Delta\big)$ yields

$$\left[\frac{(f_n)}{(\omega_n)}\right]=\left[\frac{(g_n)}{(\varepsilon_n)}\right].$$

We now show that $\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}$ is a surjective mapping. Let

$$\left[\frac{\big(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\big)}{(\omega_n)}\right]\in\beta_2\big(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\big)$$

be arbitrary. Then

$$\boldsymbol{f}^{\alpha}_{\beta,s}f_n\otimes\omega_m=\boldsymbol{f}^{\alpha}_{\beta,s}f_m\otimes\omega_n$$

for any choice of $m,n\in\mathbb{N}$. Hence, for every $m,n\in\mathbb{N}$, $f_n,f_m\in\boldsymbol{l}^1_\alpha(0,\infty)$, satisfy the relation

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f_n\times\omega_m\big)=\boldsymbol{f}^{\alpha}_{\beta,s}\big(f_m\times\omega_n\big).$$

This means that

$$\left[\frac{(f_n)}{(\omega_n)}\right] \in \beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$$

is such that

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right) = \left[\frac{\big(\boldsymbol{f}^\alpha_{\beta,s} f_n\big)}{(\omega_n)}\right].$$

Theorem 9 is proved.

In addition, we now deduce the formula of extension of $\times$ to $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ as follows:

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right) \otimes \phi.$$

It can be proved as follows: By virtue of (17) we can write

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \left[\frac{\big(\boldsymbol{f}^\alpha_{\beta,s}(f_n \times \phi)\big)}{(\omega_n)}\right].$$

Hence, Lemma 1 gives

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \left[\frac{\big(\boldsymbol{f}^\alpha_{\beta,s} f_n \otimes \phi\big)}{(\omega_n)}\right].$$

The definition of the product $\times$ implies that

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \left[\frac{\big(\boldsymbol{f}^\alpha_{\beta,s} f_n\big)}{(\omega_n)}\right] \times \phi.$$

Thus, it follows from relation (17) that

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right) \otimes \phi.$$

Hence, it is now possible to conclude that

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right] \times \phi\right) = \breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right) \otimes \phi.$$

Theorem 10. $\breve{\boldsymbol{f}}^\alpha_{\beta,s}: \ \beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big) \ \to \ \beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$ *is continuous with respect to the* δ*-and* Δ*-convergence.*

This theorem is proved by using a technique similar to the technique presented below in the citations.

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