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ON THE SOLVABILITY OF ONE SYSTEM OF NONLINEAR HAMMERSTEIN-TYPE INTEGRAL EQUATIONS ON THE SEMIAXIS

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We study the problems of construction of positive summable and bounded solutions for the systems of nonlinear Hammerstein-type integral equations with difference kernels on the semiaxis. These systems have direct applications to the kinetic theory of gases, the theory of radiation transfer in spectral lines, and the theory of nonlinear Ricker competition models for running waves.

1. Introduction

Consider a system of nonlinear Hammerstein-type integral equations

$$\varphi_i(x)=\int_0^\infty K_i(x-t)H_i\big(t,\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)\big)dt,\qquad i=1,2,\ldots,n,\quad x\geq 0, \tag{1.1}$$

for a measurable vector function $\varphi(x)=\big(\varphi_1(x),\varphi_2(x),\ldots,\varphi_n(x)\big)^T$, where T is the sign of the operation of transposition. There are numerous nonlinear boundary-value problems for systems of differential equations of order n that can be reduced to nonlinear matrix Hammerstein-type integral equations of the form (1.1) (see [1] and the references therein). The indicated systems can also be used in the kinetic theory of gases, in the theory of nonlinear Ricker competition systems for running waves, and in the theory of radiation transfer in spectral lines (see [2–6]).

The kernels $\{K_i(x)\}_{i=1}^n$ are summable functions defined on the set $\mathbb{R}$ and satisfying the conditions

$$K_i(x)\geq 0,\quad x\in\mathbb{R},\qquad \int_{-\infty}^{+\infty}K_i(x)dx=1,\qquad \int_{-\infty}^{+\infty}|x|K_i(x)dx<+\infty, \tag{1.2}$$

$$K_i\in L_\infty(\mathbb{R}),\quad i=1,2,\ldots,n. \tag{1.3}$$

The functions $\{H_i(t,z_1,z_2,\ldots,z_n)\}_{i=1}^n$ are defined on the set $\mathbb{R}^+\times\mathbb{R}^n$, take real values, and satisfy both the condition

$$H_i(t,0,0,\ldots,0)=0,\qquad i=1,2,\ldots,n,\quad t\in\mathbb{R}^+,$$

and some additional conditions (see the formulation of the main result).

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The first results of investigations of the scalar Hammerstein integral equations were obtained in the 1920s in the pioneering works by Uryson and Hammerstein (see [7–8]). Later, in the 1950s, Krasnosel'skii and his colleagues originated systematic investigations of some classes of scalar nonlinear integral equations. Thus, in particular, various necessary and sufficient conditions guaranteeing the compactness of the Hammerstein integral operators were obtained by Krasnosel'skii, Zabreiko, Stetsenko, and Pustyl'nik (see [9–13]). Based on these results, under certain restrictions imposed on the nonlinearity, the theorems on existence and uniqueness were proved in the cited works for nonlinear integral equations with Hammerstein operators. Similar problems were also investigated by Browder's scientific school (see [14, 15] and the references therein). However, the compactness of the Hammerstein operator (and, in some cases, the boundedness of the domain of integration) played a significant role in these works.

Main difficulties in the investigation of nonlinear scalar or matrix integral equations of the form (1.1) are caused by the noncompactness of the corresponding nonlinear Hammerstein integral operator in the spaces $L_p(\mathbb{R}^+)$, $1 \le p \le +\infty$, its critical behavior, and the unboundedness of the domain of integration. For this reason, at present, there is no general operator theory of construction of fixed positive points for these equations. However, for some special cases, system (1.1) and its scalar analogs were investigated in [5, 16–20].

Thus, in [5], system (1.1) was investigated in the case where $n = 2$,

$$H_i(t, z_1, z_2) = z_i e^{u_i - z_i - v_i z_{3-i}},$$

and

$$K_i(\tau) = \frac{1}{\sqrt{4\pi d_i}} e^{-\frac{\tau^2}{4d_i}}, \qquad u_i, v_i, d_i > 0, \quad i = 1, 2.$$

In [16], system (1.1) was studied for

$$H_i(t, z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} c_{ij}(z_j - \omega_j(t, z_j)),$$

where

$$c_{ij} > 0, \qquad \sum_{j=1}^{n} c_{ij} \le 1, \qquad \omega_j(t, u) \downarrow \quad \text{with respect to} \quad u, \quad 0 \le \omega_j(t, u) \le \omega_j^0(t + u),$$

$$\omega_j^0 \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+), \qquad m_1(\omega_j^0) \equiv \int_0^{+\infty} x\omega_j^0(x)\, dx < +\infty, \quad i, j = 1, 2, \ldots, n,$$

and the kernel satisfies conditions (1.2) and (1.3) together with some technical conditions. Note that the Hammerstein scalar integral equation on the semiaxis with even conservative kernel and a nonlinearity of the form

$$z - \omega(z), \quad \omega \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+),$$

was studied in [17]. In [18], a nonlinear scalar integral equation with Hammerstein operator was investigated on the semiaxis. Moreover, in this case, the functions that describe nonlinearity, parallel with some technical conditions, also satisfy the Hölder–Lipschitz-type condition with respect to the second argument. In [19], the problems of solvability of Eq. (1.1) with $n = 1$ in the space $L_1(0, +\infty)$ were studied by using the method of investigation of the corresponding scalar integral equation similar to the method applied in the present paper. In [20], the authors considered nonlinear integral equations with noncompact operators and compact domains of integration. In [21], the problem of solvability of some systems of nonlinear Hammerstein–Nemytskii-type integral equations was analyzed on the entire real axis.

In the present paper, under certain restrictions imposed on the functions $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$, we establish the existence of a componentwise positive solution of system (1.1) in the space

$$\mathfrak{M} \equiv \Big\{\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T,\ \varphi_j \in L_1^0(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+),\ j = 1, 2, \ldots, n\Big\},$$

where $L_1^0(\mathbb{R}^+)$ is a space of summable functions on $\mathbb{R}^+$ vanishing as $+\infty$.

The obtained results are illustrated by examples of functions $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$ satisfying the conditions of the theorems presented in what follows.

2. Notation and Some Auxiliary Facts

2.1. Parameters $\{p_i\}_{i=1}^n$. We introduce the following functions defined on $\mathbb{R}^+ \equiv [0, +\infty)$:

$$\chi_i(p) = \int_0^\infty K_i(x) e^{-px}\, dx, \qquad p \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

In what follows, we assume that

$$\gamma_i \equiv \int_0^\infty K_i(x)\, dx > 0, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

By using (1.2) and (2.2), in view of (2.1), we obtain

$$\chi_i \in C(\mathbb{R}^+), \quad \chi_i(p) \downarrow \text{ with respect to } p \text{ on } \mathbb{R}^+, \quad \chi_i(0) = \gamma_j > 0, \quad \chi_i(+\infty) = 0, \quad i = 1, 2, \ldots, n.$$

Hence, by the Bolzano–Cauchy theorem (see [22]), for any $i \in \{1, 2, \ldots, n\}$, there exists (and, moreover, is unique) a number $p_i > 0$ such that

$$\chi_i(p_i) = \frac{\gamma_i}{2}, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

2.2. System of Linear Integral Wiener–Hopf Equations. Let $\{\beta_i(x)\}_{i=1}^n$ be positive measurable functions defined on the set $\mathbb{R}^+$ with the following properties:

$$\beta_i \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+), \qquad m_1(\beta_i) \equiv \int_0^\infty x\beta_i(x)\, dx < +\infty, \tag{2.4}$$

$$\lim_{x\to\infty} \beta_i(x) = 0, \quad \beta_i(x) \geq \frac{2}{\gamma_i} e^{-p_i x}, \qquad x \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n. \tag{2.5}$$

Further, let $A = (a_{ij})_{i,j=1}^{n\times n}$ be a primitive matrix with unit spectral radius $\mathbf{r}(A) = 1$; here $\mathbf{r}(A)$ is the modulus of the maximum absolute eigenvalue of the matrix A. Thus, by the Perron–Frobenius theorem, (see [23]), there exists a vector

$$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^T$$

with positive coordinates $\{\zeta_i\}_{i=1}^n,\ \zeta_i > 0$, such that

$$A\zeta = \zeta. \tag{2.6}$$

Consider a system of inhomogeneous integral Wiener–Hopf equations

$$f_i(x) = g_i(x) + \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t) f_j(t)\, dt, \qquad i = 1, 2, \dots, n, \quad x \in \mathbb{R}^+, \tag{2.7}$$

for a measurable vector function $f(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)^T$, where

$$g_i(x) = \int_0^\infty K_i(x-t)\beta_i(t)\, dt, \qquad i = 1, 2, \dots, n, \quad x \in \mathbb{R}^+, \tag{2.8}$$

$$\tilde{K}_{ij}(x) = a_{ij} K_i(x), \qquad i, j = 1, 2, \dots, n, \quad x \in \mathbb{R}. \tag{2.9}$$

In what follows, we need the following lemma:

Lemma 2.1. *Suppose that conditions (1.2), (1.3), (2.4), and (2.5) are satisfied. If*

$$K_i(-\tau) > K_i(\tau), \quad i = 1, 2, \dots, n, \quad \tau \in (0, +\infty),$$

then system (2.7) possesses a positive solution

$$f(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)^T, \qquad f_i(x) > 0, \quad i = 1, 2, \dots, n,$$

and, in addition,

(a) $f_i \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$;

(b) $\lim_{x\to+\infty} f_i(x) = 0,\ i = 1, 2, \dots, n$;

(c) $f_i(x) \geq e^{-p_i x},\ i = 1, 2, \dots n,\ x \in \mathbb{R}^+$.

Proof. We first note that:

(1) $g_i \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$;

(2) $m_1(g_i) < +\infty,\ i = 1, 2, \dots, n$;

(3) $\lim_{x\to\infty} g_i(x) = 0,\ i = 1, 2, \dots, n$.

Indeed, since $\lim_{x\to\infty}\beta_i(x)=0$ and the kernels $\{K_i(x)\}_{i=1}^n$ satisfy conditions (1.2) and (1.3), the inclusion from item 1 directly follows from representation (2.8). Moreover, the formula in item 3 is a consequence of the known limit relations for the operation of convolution (see [24, p. 61], Lemma 5). We now prove that

$$m_1(g_i)<+\infty,\quad i=1,2,\ldots,n.$$

To this end, by using (1.2), (1.3), (2.4), and (2.5), for any $\rho>0,$ we estimate the integral

$$\int_0^\rho xg_i(x)\,dx=\int_0^\rho x\int_0^\infty K_i(x-t)\beta_i(t)\,dt\,dx$$

$$=\int_0^\infty \beta_i(t)\int_0^\rho K_i(x-t)x\,dx\,dt$$

$$=\int_0^\infty \beta_i(t)\int_{-t}^{\rho-t} K_i(y)(t+y)\,dy\,dt$$

$$\le m_1(\beta_i)+\int_0^\infty \beta_i(t)\,dt\int_{-\infty}^{+\infty}|y|K_i(y)\,dy<+\infty,\quad i=1,2,\ldots,n.$$

Since $\rho>0$ is an arbitrary number, this enables us to conclude that $m_1(g_i)<+\infty,\ i=1,2,\ldots,n.$

Since $\mathbf{r}(A)=1,$ by using (1.2) and (2.9), we get

$$\mathbf{r}\left(\int_{-\infty}^{+\infty}\tilde{K}(x)dx\right)=\mathbf{r}(A)=1,\tag{2.10}$$

where

$$\tilde{K}(x)=\left(\tilde{K}_{ij}(x)\right)_{i,j=1}^n,\quad x\in\mathbb{R},\tag{2.11}$$

is the matrix kernel of system (2.7). On the other hand,

$$\nu(\tilde{K}_{ij})=a_{ij}\int_{-\infty}^{+\infty}xK_i(x)\,dx=a_{ij}\left(\int_{-\infty}^{0}xK_i(x)dx+\int_0^{+\infty}K_i(x)x\,dx\right)<0\tag{2.12}$$

because $K_i(-\tau)>K_i(\tau),\ \tau\in(0,+\infty),\ i=1,2,\ldots,n.$

Therefore, system (2.7) has a componentwise positive solution in the space $L_1(\mathbb{R}^+)$ (see [25, p. 216], Theorem 8.3 with $n=1$). Since $K_i\in L_\infty(\mathbb{R})$ and the free terms g_i have properties (1)–(3), relation (2.7) also implies that $f_i\in L_\infty(\mathbb{R}^+)$. On the other hand, by Lemma 5 in [24], we conclude that

$$f_i\in L_1^0(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+),\quad i=1,2,\ldots,n.$$

To complete the proof of the lemma, it remains to show that estimate (c) holds. Indeed, in view of (2.5), it follows from (2.7) that

$$f_i(x) \geq g_i(x) \geq \frac{2}{\gamma_i}\int_0^\infty K_i(x-t)e^{-p_i t}dt = \frac{2}{\gamma_i}e^{-p_i x}\int_{-\infty}^{x} K_i(t)e^{p_i t}dt$$

$$\geq \frac{2}{\gamma_i}e^{-p_i x}\int_{-\infty}^{0} K_i(t)e^{p_i t}dt = \frac{2}{\gamma_i}e^{-p_i x}\int_0^\infty K_i(-\tau)e^{-p_i\tau}d\tau$$

$$\geq \frac{2}{\gamma_i}e^{-p_i x}\int_0^\infty K_i(\tau)e^{-p_i\tau}d\tau = \frac{2}{\gamma_i}e^{-p_i x}\chi_i(p_i) = e^{-p_i x}, \quad i = 1, 2, \ldots, n.$$

Lemma 2.1 is proved.

2.3. System of Homogeneous Wiener–Hopf Integral Equations. Consider a system of homogeneous Wiener–Hopf integral equations

$$S_i(x) = \sum_{j=1}^{n}\int_0^\infty \tilde{K}_{ij}(x-t)S_j(t)\,dt, \qquad i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+, \tag{2.13}$$

with the normalization condition

$$S_i(0) = \sup_{x\in\mathbb{R}^+} f_i(x), \quad i = 1, 2, \ldots, n. \tag{2.14}$$

In (2.13), it is assumed that the kernel functions $\{K_i(x)\}_{i=1}^n$ satisfy conditions (1.2) and (1.3) and, in addition,

$$K_i(-\tau) > K_i(\tau), \qquad \tau \in (0, +\infty), \quad i = 1, 2, \ldots, n. \tag{2.15}$$

Thus, it is known (see [26, p. 235], Theorem 14.3) that problem (2.13), (2.14) has a componentwise monotonically increasing and essentially bounded solution $S(x) = \big(S_1(x), S_2(x), \ldots, S_n(x)\big)^T$. It is clear that

$$\eta = \delta\zeta = (\delta\zeta_1, \delta\zeta_2, \ldots, \delta\zeta_n)^T, \qquad \delta \equiv \max_{1\leq i\leq n}\frac{\sup_{x\in\mathbb{R}^+} S_i(x)}{\zeta_i}, \tag{2.16}$$

is also an eigenvector of the matrix A corresponding to the eigenvalue $\lambda = \mathbf{r}(A) = 1$:

$$A\eta = \eta. \tag{2.17}$$

We now show that

$$\eta_j \geq e^{-p_j t}, \qquad j = 1, 2, \ldots, n, \quad t \in \mathbb{R}^+.$$

Indeed, by using (2.14) and Lemma 2.1, in view of (2.16), we obtain

$$\eta_j = \delta\zeta_j \geq \sup_{x\in\mathbb{R}^+} S_j(x) \geq S_j(x) \geq f_j(x) \geq g_j(x) \geq e^{-p_j x}, \qquad x \in \mathbb{R}^+, \quad j = 1, 2, \ldots, n.$$

2.4. Construction of Majorizing Functions for $H_i(t, z_1, z_2, \ldots, z_n)$. Consider a sequence of functions

$$\{Q_j(z)\}_{j=1}^n$$

given on $\mathbb{R}$ with the following properties:

(A_1) $Q_j(z) \uparrow$ with respect to z on $[0, \eta_j]$, $j = 1, 2, \ldots, n$,

(A_2) $Q_j(0) = 0$, $Q_j(\eta_j) = \eta_j$, $j = 1, 2, \ldots, n$,

(A_3) the functions $Q_j(z)$ satisfy the Lipschitz conditions on the segment $[0, \eta_j]$, i.e., for each $j \in \{1, 2, \ldots, n\}$, there exists a number $L_j > 0$ such that, for any z^j, $\tilde{z}^j \in [0, \eta_j]$, the inequality

$$\left|Q_j(z^j) - Q_j(\tilde{z}^j)\right| \leq L_j |z^j - \tilde{z}^j|$$

is true.

The following lemma plays the key role in subsequent reasoning:

Lemma 2.2. *For any*

$$\alpha \in \left(0, \min\left(1, \frac{1}{\max_{1\leq j\leq n} L_j}\right)\right) \equiv I,$$

the functions

$$\tilde{Q}_j(z) = \eta_j - \alpha Q_j(\eta_j - z), \quad j = 1, 2, \ldots, n,$$

have the following properties:

(B_1) $\tilde{Q}_j(z) \uparrow$ *with respect to* z *on* $[0, \eta_j]$, $j = 1, 2, \ldots, n$,

(B_2) $\tilde{Q}_j(0) > 0$ *and* $\tilde{Q}_j(\eta_j) = \eta_j$, $j = 1, 2, \ldots, n$,

(B_3) *for any* z^j, $\tilde{z}^j \in [0, \eta_j]$, *the following inequalities are true:*

$$\left|\tilde{Q}_j(z^j) - \tilde{Q}_j(\tilde{z}^j)\right| \leq \alpha^* |z^j - \tilde{z}^j|, \quad j = 1, 2, \ldots, n,$$

where $\alpha^* \equiv \alpha \max_{1\leq j\leq n} L_j \in (0, 1)$.

(B_4) *the lower bounds*

$$\tilde{Q}_j(z) \geq z, \qquad z \in [0, \eta_j], \quad j = 1, 2, \ldots, n,$$

are true.

Proof. The properties (B_1)–(B_3) can be easily verified. We prove the property (B_4). Consider the functions

$$W_j(z) \equiv \tilde{Q}_j(z) - z, \qquad z \in [0, \eta_j], \quad j = 1, 2, \ldots, n.$$

We have

$$W_j(0) = \tilde{Q}_j(0) = (1-\alpha)\eta_j > 0 \quad \text{because} \quad \alpha \in I, \quad j = 1, 2, \ldots, n,$$

$$W_j(\eta_j) = \tilde{Q}_j(\eta_j) - \eta_j = 0, \quad j = 1, 2, \ldots, n,$$

$$W_j \in C[0, \eta_j], \quad j = 1, 2, \ldots, n.$$

We verify that $W_j(z) \downarrow$ with respect to z on $[0, \eta_j]$. Let $u_1^j, u_2^j \in [0, \eta_j],\ u_1^j > u_2^j$, be arbitrary numbers. Then

$$W_j(u_1^j) - W_j(u_2^j) = u_2^j - u_1^j + \alpha(Q_j(\eta_j - u_2^j) - Q_j(\eta_j - u_1^j))$$

$$\leq u_2^j - u_1^j + \alpha L_j(u_1^j - u_2^j) = (\alpha L_j - 1)(u_1^j - u_2^j) < 0$$

because $\alpha \in I,\ j = 1, 2, \ldots, n$. Therefore,

$$W_j(z) \geq 0, \quad z \in [0, \eta_j], \quad \text{i.e.,} \quad \tilde{Q}_j(z) \geq z, \quad z \in [0, \eta_j], \quad j = 1, 2, \ldots, n.$$

Lemma 2.2 is proved.

We now formulate the main result of the present paper.

3. Main Result

3.1. Formulation of the Theorem. The following theorem is the main result of the present paper:

Theorem 3.1. *Suppose that real-valued functions* $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$ *satisfy the following conditions:*

(a_1) for any fixed $t \in \mathbb{R}^+$, *the functions* $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n \uparrow$ *with respect to* z_j *on the segment*

$$[e^{-p_j t}, \eta_j], \quad j = 1, 2, \ldots, n,$$

where the numbers $\{p_i\}_{i=1}^n$ *are given by equality (2.3);*

(a_2) the functions $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$ *satisfy the multidimensional Carathéodory condition on the set*

$$\Omega_\eta \equiv \mathbb{R}^+ \times [0, \eta_1] \times [0, \eta_2] \times \ldots \times [0, \eta_n]$$

in the collection of arguments

$$(z_1, z_2, \ldots, z_n) \in [0, \eta_1] \times [0, \eta_2] \times \ldots \times [0, \eta_n],$$

i.e., for each $(z_1, z_2, \ldots, z_n) \in [0, \eta_1] \times [0, \eta_2] \times \ldots \times [0, \eta_n]$, *the functions* $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$ *are measurable with respect to the argument* $t \in \mathbb{R}^+$ *and, for almost all* $t \in \mathbb{R}^+$, *these functions are continuous in the collection of arguments* $(z_1, z_2, \ldots, z_n)$ *on the set* $[0, \eta_1] \times [0, \eta_2] \times [0, \eta_n]$;

(a_3) the inequalities

$$H_i\big(t, e^{-p_1 t}, e^{-p_2 t}, \ldots, e^{-p_n t}\big) \geq \frac{2}{\gamma_i} e^{-p_i t}, \qquad t \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n,$$

are true;

(a_4) there exist a number $\alpha \in I$, functions $\big\{\beta_i(t)\big\}_{i=1}^n$ with properties (2.4) and (2.5), and a primitive matrix $A = (a_{ij})_{i,j=1}^{n\times n}$ with $r(A) = 1$ such that

$$H_i(t, z_1, z_2, \ldots, z_n) \leq \alpha \sum_{j=1}^n a_{ij} Q_j(z_j) + \beta_i(t),$$

$$i = 1, 2, \ldots, n, \quad t \in \mathbb{R}^+, \qquad z_j \in [e^{-p_j t}, \eta_j], \quad j = 1, 2, \ldots, n.$$

Then, under conditions (1.2), (1.3), and (2.15), system (1.1) has a componentwise positive solution in $\mathfrak{M}$.

3.2. Proof of the Theorem. We split the proof into several steps.

Step 1 (auxiliary system of nonlinear Hammerstein equations). Consider an auxiliary system of nonlinear Hammerstein equations

$$\psi_i(x) = \sum_{j=1}^n \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(\psi_j(t) - f_j(t))dt + \phi_i(x), \qquad x \geq 0, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

for the required vector function $\psi(x) = \big(\psi_1(x), \psi_2(x), \ldots, \psi_n(x)\big)^T$, where $\{f_i(x)\}_{i=1}^n$ is a positive bounded and summable solution of system (2.7) (see Lemma 2.1) and

$$\phi_i(x) = \sum_{j=1}^n \int_0^\infty \tilde{K}_{ij}(x-t) f_j(t)dt, \qquad i = 1, 2, \ldots, n, \quad x \geq 0. \tag{3.2}$$

We introduce the following iterations:

$$\psi_i^{(m+1)}(x) = \sum_{j=1}^n \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(\psi_j^{(m)}(t) - f_j(t))dt + \phi_i(x),$$
$$i = 1, 2, \ldots, n, \quad x \geq 0, \qquad \psi_i^{(0)}(x) = S_i(x), \quad m = 0, 1, 2, \ldots. \tag{3.3}$$

We now prove that

$$\psi_i^{(m)}(x) \uparrow \quad \text{with respect to} \quad m,$$
$$\psi_i^{(m)}(x) \leq \eta_i + f_i(x), \tag{3.4}$$
$$i = 1, 2, \ldots, n, \quad x \geq 0, \quad m = 0, 1, 2, \ldots.$$

We first prove the property of monotonicity with respect to $m.$ By virtue of Lemma 2.2, relations (2.7) and (2.13), and the inequalities $0 \le S_i(x) - f_i(x) \le \eta_i,\ x \in \mathbb{R}^+,\ i = 1, 2, \ldots, n,$ we get

$$\psi_i^{(1)}(x) = \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(S_j(t) - f_j(t))dt + \phi_i(x)$$

$$\ge \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)(S_j(t) - f_j(t))dt + \phi_i(x) = S_i(x) = \psi_i^{(0)}(x).$$

On the other hand,

$$\psi_i^{(0)}(x) \le \sup_{x\ge 0} S_i(x) \le \delta\zeta_i \le \eta_i + f_i(x), \quad i = 1, 2, \ldots, n.$$

We also note that $0 \le \psi_i^{(1)}(x) - f_i(x) \le \eta_i$. Indeed, by virtue of (2.17), we can write

$$\psi_i^{(1)}(x) - f_i(x) \ge S_i(x) - f_i(x) \ge 0,$$

$$\psi_i^{(1)}(x) - f_i(x) \le \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(\eta_j)dt + \phi_i(x) - f_i(x)$$

$$\le \sum_{j=1}^{n} a_{ij}\eta_j + \phi_i(x) - f_i(x)$$

$$\le \eta_i, \qquad i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+.$$

Assume that

$$\psi_i^{(m)}(x) \ge \psi_i^{(m-1)}(x) \quad \text{and} \quad \psi_i^{(m)}(x) \le \eta_i + f_i(x), \quad i = 1, 2, \ldots, n.$$

Thus, for some $m \in \mathbb{N},$ by using Lemma 2.2 and relations (2.7) and (2.13), we get

$$\psi_i^{(m+1)}(x) \ge \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j\big(\psi_j^{(m-1)}(t) - f_j(t)\big)\, dt + \phi_i(x) = \psi_i^{(m)}(x)$$

and

$$\psi_i^{(m+1)}(x) \le \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(\eta_j)dt + \phi_i(x)$$

$$\le \sum_{j=1}^{n} a_{ij}\eta_j + \phi_i(x) = \eta_i + \phi_i(x)$$

$$\le \eta_i + f_i(x).$$

Therefore, the sequence of vector functions

$$\psi^{(m)}(x) = \big(\psi_1^{(m)}(x), \psi_2^{(m)}(x), \ldots, \psi_n^{(m)}(x)\big)^T, \quad m = 0, 1, 2, \ldots,$$

has a pointwise limit as $m \to \infty$. Moreover, by virtue of Lemma 2.2 and the B. Levi limit theorem, the limit vector function

$$\psi(x) = \big(\psi_1(x), \psi_2(x), \ldots, \psi_n(x)\big)^T,$$

$$\psi_i(x) = \lim_{m\to\infty} \psi_i^{(m)}(x),$$

satisfies system (3.1). By using (3.3) and (3.2), we also get the following two-sided estimate for $\{\psi_i(x)\}_{i=1}^n$:

$$S_i(x) \le \psi_i(x) \le \eta_i + f_i(x), \qquad x \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

In what follows, we show that

$$\eta_i + f_i - \psi_i \in L_1^0(\mathbb{R}^+), \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Step 2 [proof of inclusion (3.6)]. To this end, we consider the following auxiliary inhomogeneous Hammerstein system:

$$F_i(x) = \tilde{g}_i(x) + \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\big(\eta_j - \tilde{Q}_j(\eta_j - F_j(t))\big)\, dt, \qquad i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+, \tag{3.7}$$

where

$$\tilde{g}_i(x) = \eta_j \int_x^\infty K_i(t)\, dt + g_i(x), \qquad i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+. \tag{3.8}$$

We introduce the following successive approximations:

$$F_i^{(m+1)}(x) = \tilde{g}_i(x) + \sum_{j=1}^{n} \int_0^\infty \tilde{K}_{ij}(x-t)\big(\eta_j - \tilde{Q}_j(\eta_j - F_j^{(m)}(t))\big)\, dt, \qquad F_i^{(0)}(x) \equiv 0, \quad i = 1, 2, \ldots, n, \quad m = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+. \tag{3.9}$$

By induction on $m,$ it is possible to prove that

(C$_1$) $F_i^{(m)} \in L_1(\mathbb{R}^+),\ m = 0, 1, 2, \ldots,\ i = 1, 2, \ldots, n,$

(C$_2$) $F_i^{(m)}(x) \uparrow$ with respect to $m,$

(C$_3$) $$\int_0^\infty F_i^{(m)}(x)\,dx \le \eta_i \max_{1\le i\le n}\|\tilde{g}_i\|_{L_1(\mathbb{R}^+)}(1-\alpha^*)^{-1},$$

$$m=0,1,2,\ldots,\quad i=1,2,\ldots,n,\quad \text{and}\quad \alpha^*=\alpha\max_{1\le j\le n}L_j,$$

(C$_4$) $F_i^{(m)}(x)\le \eta_i - S_i(x)+f_i(x),\ x\in\mathbb{R}^+,\ m=0,1,2,\ldots,\ i=1,2,\ldots,n.$

The assertions (C$_1$)–(C$_3$) are checked by the standard methods. We prove the assertion (C$_4$). For $m=0$, the assertion (C$_4$) is a corollary of inequality (3.5). Assume that

$$F_i^{(m)}(x)\le \eta_i - S_i(x)+f_i(x),\quad i=1,2,\ldots,n,\quad x\in\mathbb{R}^+,$$

for some $m\in\mathbb{N}$. Thus, by virtue of Lemma 2.2 and relation (2.17), it follows from (3.9) that

$$\begin{aligned}F_i^{(m+1)}(x) &\le \eta_i\int_x^\infty K_i(t)dt+g_i(x)+\sum_{j=1}^n\int_0^\infty \tilde{K}_{ij}(x-t)(\eta_j-\tilde{Q}_j(S_j(t)-f_j(t)))dt\\ &=\eta_i\int_x^\infty K_i(t)dt+\sum_{j=1}^n a_{ij}\eta_j\int_{-\infty}^x K_i(t)dt+g_i(x)-\sum_{j=1}^n\int_0^\infty \tilde{K}_{ij}(x-t)\tilde{Q}_j(S_j(t)-f_j(t))dt\\ &\le \eta_i+g_i(x)-\sum_{j=1}^n\int_0^\infty \tilde{K}_{ij}(x-t)(S_j(t)-f_j(t))dt\\ &=\eta_i+g_i(x)-S_i(x)+\phi_i(x)=\eta_i-S_i(x)+f_i(x),\qquad i=1,2,\ldots,n.\end{aligned}$$

It follows from the assertions (C$_1$)–(C$_4$) that system (3.7) has a componentwise positive, summable, and essentially bounded solution, which is a pointwise limit of the sequence

$$\{F^{(m)}(x)\}_{m=0}^\infty,\quad F^{(m)}(x)=\left(F_1^{(m)}(x),F_2^{(m)}(x),\ldots,F_n^{(m)}(x)\right)^T$$

as $m\to\infty$. Since

$$F_i(x)=\lim_{m\to\infty}F_i^{(m)}(x)\in L_1(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+),$$

by using the inequalities

$$0\le F_i(x)\le \tilde{g}_i(x)+\sum_{j=1}^n a_{ij}\alpha^*\int_0^\infty K_i(x-t)F_j(t)\,dt,\quad i=1,2,\ldots,n,$$

and the limits

$$\lim_{x\to\infty}\tilde{g}_i(x)=0,\qquad \lim_{x\to\infty}\int_0^\infty K_i(x-t)F_j(t)\,dt=0$$

{this follows from the well-known properties of the convolution (see [24, p. 61], Lemma 5}, we obtain

$$\lim_{x\to+\infty}F_i(x)=0,\quad i=1,2,\ldots,n.$$

Thus, we have proved that system (3.7) possesses a positive summable and essentially bounded solution $F(x) = \big(F_1(x), F_2(x), \dots, F_n(x)\big)^T$. Moreover, $\lim_{x\to\infty} F_i(x) = 0,\ i = 1, 2, \dots, n,$ and

$$0 \le F_i(x) \le \eta_i - S_i(x) + f_i(x) \le \eta_i, \qquad i = 1, 2, \dots, n, \quad x \in \mathbb{R}^+.$$

We introduce the following class of measurable vector functions:

$$\mathcal{P}_\eta \equiv \Big\{\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T,\ 0 \le \varphi_i(t) \le \eta_i,\ t \in \mathbb{R}^+,\ i = 1, 2, \dots, n\Big\}.$$

It is clear that $F \in \mathcal{P}_\eta$. In what follows, we prove that system (3.7) possesses a unique solution from the class $\mathcal{P}_\eta$. Indeed, assume the contrary, i.e., that there exist two different solutions $F, F^* \in \mathcal{P}_\eta$ of system (3.7). In view of Lemma 2.2, it follows from (3.7) that

$$\begin{aligned}\frac{|F_i(x) - F_i^*(x)|}{\eta_i} &\le \frac{\alpha^*}{\eta_i} \sum_{j=1}^n a_{ij}\eta_j \int_0^\infty \frac{K_i(x-t)}{\eta_j} |F_j(t) - F_j^*(t)| dt \\ &\le \frac{\alpha^*}{\eta_i} \max_{1\le j\le n} \sup_{t\in\mathbb{R}^+} \frac{|F_j(t) - F_j^*(t)|}{\eta_j} \sum_{j=1}^n a_{ij}\eta_j \\ &= \alpha^* \max_{1\le j\le n} \sup_{t\in\mathbb{R}^+} \frac{|F_j(t) - F_j^*(t)|}{\eta_j}, \quad i = 1, 2, \dots, n.\end{aligned}$$

The obtained inequality implies that

$$(1-\alpha^*) \max_{1\le j\le n} \sup_{t\in\mathbb{R}^+} \frac{|F_j(t) - F_j^*(t)|}{\eta_j} \le 0. \tag{3.10}$$

Since $\alpha^* \in (0,1)$, by using (3.10), we conclude that $F_i(x) = F_i^*(x)$ almost everywhere on $\mathbb{R}^+,\ i = 1, 2, \dots, n$. Hence, system (3.7) possesses a unique solution in $\mathcal{P}_\eta$. On the other hand, we can directly show that the vector function

$$\big(\eta_1 - \psi_1 + f_1, \eta_2 - \psi_2 + f_2, \dots, \eta_n - \psi_n + f_n\big)^T \in \mathcal{P}_\eta$$

satisfies system (3.7). Indeed, in view of (3.1), we get

$$\begin{aligned}&\tilde{g}_i(x) + \sum_{j=1}^n \int_0^\infty \tilde{K}_{ij}(x-t)(\eta_j - \tilde{Q}_j(\psi_j(t) - f_j(t)))dt \\ &= \eta_i \int_x^\infty K_i(t)dt + g_i(x) + \eta_i \int_{-\infty}^x K_i(t)dt - (\psi_i(x) - \phi_i(x)) \\ &= \eta_i + g_i(x) - \psi_i(x) + \phi_i(x) = \eta_i - \psi_i(x) + f_i(x), \quad i = 1, 2, \dots, n.\end{aligned}$$

Since $\eta - \psi + f \in \mathcal{P}_\eta$, by using the above-mentioned result, we conclude that

$$\eta - \psi + f \in \mathfrak{M},$$

i.e., inclusion (3.6) is proved.

Step 3 [convergence of successive approximations for the main system (1.1)]. We now consider special successive approximations

$$\varphi_i^{(m+1)}(x) = \int_0^\infty K_i(x-t) H_i\big(t, \varphi_1^{(m)}(t), \varphi_2^{(m)}(t), \ldots, \varphi_n^{(m)}(t)\big)\, dt,$$
$$\varphi_i^{(0)}(x) = e^{-p_i x}, \qquad i = 1, 2, \ldots, n, \quad m = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+, \tag{3.11}$$

where the numbers $\{p_i\}_{i=1}^n$ are determined from equality (2.3).

By induction on m, we can prove that:

(D$_1$) $\varphi_i^{(m)}(x) \uparrow$ with respect to m, $i = 1, 2, \ldots, n$,

(D$_2$) $\varphi_i^{(m)}(x) \le \eta_i - \psi_i(x) + f_i(x)$, $i = 1, 2, \ldots, n$, $m = 0, 1, 2, \ldots,$ $x \in \mathbb{R}^+$.

Indeed, for $m = 0$, the inequalities in the condition (D$_2$) directly follow from the chain of inequalities

$$\eta_i - \psi_i(x) + f_i(x) \ge \tilde{g}_i(x) \ge g_i(x) \ge \frac{2}{\gamma_i} \int_0^\infty K_i(x-t) e^{-p_i t}\, dt \ge e^{-p_i x},$$
$$i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+.$$

In what follows, we prove that $\varphi_i^{(1)}(x) \ge \varphi_i^{(0)}(x)$. By virtue of the condition (a$_3$) of Theorem 1, it follows from (3.11) that

$$\varphi_i^{(1)}(x) = \int_0^\infty K_i(x-t) H_i(t, e^{-p_1 t}, e^{-p_2 t}, \ldots, e^{-p_n t}) dt$$
$$\ge \frac{2}{\gamma_i} \int_0^\infty K_i(x-t) e^{-p_i t} dt \ge e^{-p_i x} = \varphi_i^{(0)}(x).$$

Assume that

$$\varphi_i^{(m)}(x) \ge \varphi_i^{(m-1)}(x) \quad \text{and} \quad \varphi_i^{(m)}(x) \le \eta_i - \psi_i(x) + f_i(x)$$

for some $m \in \mathbb{N}$, $i = 1, 2, \ldots, n$. Thus, in view of the conditions (a$_1$) and (a$_4$) of Theorem 1 and relations (3.11), we obtain

$$\varphi_i^{(m+1)}(x) \ge \int_0^\infty K_i(x-t) H_i\big(t, \varphi_1^{(m-1)}(t), \varphi_2^{(m-1)}(t), \ldots, \varphi_n^{(m-1)}(t)\big) dt = \varphi_i^{(m)}(x)$$

and

$$\varphi_i^{(m+1)}(x) \le \alpha \sum_{j=1}^{n} a_{ij} \int_0^\infty K_i(x-t) Q_j(\eta_j - \psi_j(t) + f_j(t))dt + g_i(x)$$

$$= \sum_{j=1}^{n} a_{ij} \int_0^\infty K_i(x-t)(\eta_j - \tilde{Q}_j(\psi_j(t) - f_j(t)))dt + g_i(x)$$

$$\le \sum_{j=1}^{n} a_{ij}\eta_j - (\psi_i(x) - \phi_i(x)) + g_i(x) = \eta_i + f_i(x) - \psi_i(x).$$

Thus, the assertions (D_1) and (D_2) are completely proved. Hence, the sequence of vector functions

$$\varphi^{(m)}(x) = \left(\varphi_1^{(m)}(x), \varphi_2^{(m)}(x), \ldots, \varphi_n^{(m)}(x)\right)^T, \quad m = 0, 1, 2, \ldots,$$

possesses a pointwise limit as $m \to \infty$:

$$\lim_{m\to\infty} \varphi_i^{(m)}(x) = \varphi_i(x), \quad i = 1, 2, \ldots, n.$$

Moreover,

$$e^{-p_i x} \le \varphi_i(x) \le \eta_i - \psi_i(x) + f_i(x), \qquad i = 1, 2, \ldots, n, \quad x \in \mathbb{R}^+. \tag{3.12}$$

By the B. Levi theorem and the condition (a_2), the limit vector function $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T$ satisfies system (1.1). Since

$$\eta - \psi + f \in \mathfrak{M},$$

it follows from (3.12) that $\varphi \in \mathfrak{M}$.

Theorem 3.1 is proved.

Remark. In the course of the proof of Theorem 3.1, we have also established the two-sided estimate (3.12) for the solution $\varphi(x) = \left(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)\right)^T$.

4. Examples of the Functions $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$. Uniqueness Theorem for a Special Case of System (1.1)

To illustrate the obtained result, we present examples of the functions $\left\{H_i(t, z_1, z_2, \ldots, z_n)\right\}_{i=1}^n$ satisfying the conditions (a_1)–(a_4). All conditions of Theorem 3.1 are satisfied for the class of functions

$$H_i(t, z_1, z_2, \ldots, z_n) = \alpha \sum_{j=1}^{n} a_{ij} Q_j(z_j) + \frac{2(1+\varepsilon) z_i e^{-p_i t}}{\gamma_i \left(z_i + \varepsilon e^{-p_i t}\right)}, \quad i = 1, 2, \ldots, n,$$

where $\varepsilon > 0$ is an arbitrary number.

In this case, as functions $\left\{\beta_i(t)\right\}_{i=1}^n$, we can take a family

$$\beta_i(t) = \frac{2(1+\varepsilon)}{\gamma_i} e^{-p_i t}, \qquad i = 1, 2, \ldots, n, \quad t \in \mathbb{R}^+.$$

We can also consider a more general example of functions $\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$:

$$H_i(t, z_1, z_2, \ldots, z_n) = \frac{\alpha}{lt^2+1}\sum_{j=1}^n a_{ij}Q_j(z_j) + \frac{2(1+\varepsilon)^q z_i^q e^{-p_i t}}{\gamma_i(z_i + \varepsilon e^{-p_i t})^q}, \tag{4.1}$$

where $q \geq 1,\ l \geq 0$ are arbitrary numbers. Here, as functions $\{\beta_i(t)\}_{i=1}^n$, we can use

$$\beta_i(t) = \frac{2(1+\varepsilon)^q}{\gamma_i}e^{-p_i t}, \qquad i = 1, 2, \ldots, n, \quad t \in \mathbb{R}^+.$$

Note that system (1.1) with nonlinearity (4.1) and a kernel of the form

$$K_i(\tau) = \frac{1}{\sqrt{4\pi d_i}}e^{-\frac{(\tau+c)^2}{4d_i}}, \qquad \tau \in \mathbb{R}, \quad c \geq 0, \quad d_i > 0, \quad i = 1, 2, \ldots, n,$$

is encountered in the theory of nonlinear Ricker competition systems (see [5]).

As functions $\{Q_j(z)\}_{j=1}^n$, we can use the following functions:

(a) $Q_j(z) = \dfrac{z^p}{\eta_j^{p-1}},\ p > 1,$

(b) $Q_j(z) = z + \dfrac{\eta_j}{\pi}\sin^2\dfrac{\pi z}{\eta_j},$

(c) $Q_j(z) = 2z - \dfrac{z^2}{\eta_j},\ j = 1, 2, \ldots, n.$

In what follows, we prove that, for sufficiently small $\varepsilon > 0,$ in the case where the functions

$$\{H_i(t, z_1, z_2, \ldots, z_n)\}_{i=1}^n$$

admit representation (4.1), the solution of system (1.1) is unique in the following class of measurable and essentially bounded vector functions:

$$\mathcal{L} = \Big\{\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T,$$
$$\eta_i \geq \varphi_i(x) \geq e^{-p_i x},\ \ x \in \mathbb{R}^+,\ \ \varphi_i \in L_\infty(\mathbb{R}^+),\ \ i = 1, 2, \ldots, n\Big\}.$$

Consider a nonlinear system of integral equations

$$\varphi_i(x) = \alpha\sum_{j=1}^n a_{ij}\int_0^\infty \frac{K_i(x-t)}{(lt^2+1)}Q_j(\varphi_j(t))dt$$
$$+ \frac{2(1+\varepsilon)^q}{\gamma_i}\int_0^\infty K_i(x-t)\frac{e^{-p_i t}\varphi_i^q(t)}{(\varphi_i(t)+\varepsilon e^{-p_i t})^q}dt, \tag{4.2}$$

where $x \in \mathbb{R}^+,\ i = 1, 2, \ldots, n,$ for $\varphi(x) = \big(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)\big)^T.$

The following theorem is true:

Theorem 4.1. *Suppose that conditions (1.2), (1.3), and (2.15) are satisfied and* ε^* *is a positive solution of the characteristic equation*

$$(1+x)^{q-2}x + \frac{(\alpha^*-1)\gamma}{2q} = 0 \tag{4.3}$$

for x, *where*

$$\gamma \equiv \min_{1\le i\le n} \gamma_i, \qquad q \ge 1, \qquad \alpha^* = \alpha \max_{1\le j\le n} L_j, \quad \alpha \in \left(0, \min\left(1, \frac{1}{\max_{1\le j\le n} L_j}\right)\right).$$

If $\varepsilon \in (0, \varepsilon^*)$, *then the solution of system (4.2) is unique in the class of vector functions* $\mathcal{L}$.

Proof. Assume the contrary, i.e., that system (4.2) has two different solutions $\varphi, \tilde{\varphi} \in \mathcal{L}$. Thus, it follows from (4.2) that

$$\begin{aligned}\frac{|\varphi_i(x)-\tilde{\varphi}_i(x)|}{\eta_i} &\le \frac{\alpha}{\eta_i}\sum_{j=1}^n a_{ij}\int_0^\infty K_i(x-t)|Q_j(\varphi_j(t)) - Q_j(\tilde{\varphi}_j(t))|dt \\ &\quad + \frac{2(1+\varepsilon)^q}{\eta_i\gamma_i}\int_0^\infty K_i(x-t)e^{-p_i t}\left|\frac{\varphi_i^q(t)}{(\varphi_i(t)+\varepsilon e^{-p_i t})^q} - \frac{\tilde{\varphi}_i^q(t)}{(\tilde{\varphi}_i(t)+\varepsilon e^{-p_i t})^q}\right|dt \\ &\le \frac{\alpha}{\eta_i}\sum_{j=1}^n a_{ij}L_j\int_0^\infty K_i(x-t)|\varphi_j(t)-\tilde{\varphi}_j(t)|dt \\ &\quad + \frac{2(1+\varepsilon)^q}{\eta_i\gamma_i}\int_0^\infty K_i(x-t)e^{-p_i t}\left|\frac{\varphi_i^q(t)}{(\varphi_i(t)+\varepsilon e^{-p_i t})^q} - \frac{\tilde{\varphi}_i^q(t)}{(\tilde{\varphi}_i(t)+\varepsilon e^{-p_i t})^q}\right|dt \\ &\equiv I_i(x) + J_i(x), \qquad i=1,2,\ldots,n, \quad x\in\mathbb{R}^+.\end{aligned} \tag{4.4}$$

By virtue of the Lagrange formula of finite increments, we get the following estimate for the term $J_i(x)$:

$$J_i(x) \le \frac{2(1+\varepsilon)^q\varepsilon q}{\eta_i\gamma}\int_0^\infty K_i(x-t)e^{-2p_i t}\frac{\Theta_i^{q-1}(t)}{(\Theta_i(t)+\varepsilon e^{-p_i t})^{q+1}}|\varphi_i(t)-\tilde{\varphi}_i(t)|dt,$$

$$i = 1,2,\ldots,n,$$

where

$$\eta_i \ge \Theta_i(t) \ge e^{-p_i t}, \quad t\in\mathbb{R}^+, \qquad \Theta_i \in L_\infty(\mathbb{R}^+), \quad i=1,2,\ldots,n. \tag{4.5}$$

It follows from (4.5) that

$$\frac{\Theta_i^{q-1}(t)}{(\Theta_i(t)+\varepsilon e^{-p_i t})^{q+1}} \le \frac{1}{(1+\varepsilon)^2 e^{-2p_i t}}, \qquad t\in\mathbb{R}^+, \quad i=1,2,\ldots,n. \tag{4.6}$$

Hence, in view of (1.2) and (4.6), we get

$$J_i(x) \leq \frac{2(1+\varepsilon)^{q-2}\varepsilon q}{\eta_i \gamma} \int_0^\infty K_i(x-t)|\varphi_i(t) - \tilde{\varphi}_i(t)|dt$$

$$\leq \frac{2(1+\varepsilon)^{q-2}\varepsilon q}{\gamma} \max_{1\leq i\leq n} \sup_{t\geq 0} \frac{|\varphi_i(t) - \tilde{\varphi}_i(t)|}{\eta_i}. \tag{4.7}$$

By virtue of (2.17), we get the following relation for the term $I_i(x)$:

$$I_i(x) \leq \frac{\alpha^*}{\eta_i} \sum_{j=1}^n a_{ij}\eta_j \int_0^\infty K_i(x-t) \frac{|\varphi_j(t) - \tilde{\varphi}_j(t)|}{\eta_j} dt$$

$$\leq \frac{\alpha^*}{\eta_i} \max_{1\leq j\leq n} \sup_{t\geq 0} \frac{|\varphi_j(t) - \tilde{\varphi}_j(t)|}{\eta_j} \sum_{j=1}^n a_{ij}\eta_j$$

$$= \alpha^* \max_{1\leq i\leq n} \sup_{t\geq 0} \frac{|\varphi_i(t) - \tilde{\varphi}_i(t)|}{\eta_i}. \tag{4.8}$$

By using estimates (4.7) and (4.8), we derive the following inequality from (4.4):

$$\frac{|\varphi_i(x) - \tilde{\varphi}_i(x)|}{\eta_i} \leq \left(\alpha^* + \frac{2(1+\varepsilon)^{q-2}\varepsilon q}{\gamma}\right) \max_{1\leq i\leq n} \sup_{x\geq 0} \frac{|\varphi_i(x) - \tilde{\varphi}_i(x)|}{\eta_i}.$$

This inequality implies that

$$\left(1 - \alpha^* - \frac{2(1+\varepsilon)^{q-2}\varepsilon q}{\gamma}\right) \max_{1\leq i\leq n} \sup_{x\geq 0} \frac{|\varphi_i(x) - \tilde{\varphi}_i(x)|}{\eta_i} \leq 0. \tag{4.9}$$

Since $\varepsilon \in (0, \varepsilon^*)$, $q \geq 1$, and the function $(1+\varepsilon)^{q-2}\varepsilon \uparrow$ with respect to ε, we obtain the following inequality from (4.9) and (4.3):

$$1 - \alpha^* - \frac{2(1+\varepsilon)^{q-2}\varepsilon q}{\gamma} > 1 - \alpha^* - \frac{2(1+\varepsilon^*)^{q-2}\varepsilon^* q}{\gamma} = 0. \tag{4.10}$$

Thus, it follows from (4.9) and (4.10) that

$$\max_{1\leq i\leq n} \sup_{x\geq 0} \frac{|\varphi_i(x) - \tilde{\varphi}_i(x)|}{\eta_i} \leq 0,$$

which is possible only in the case where $\varphi_i(t) = \tilde{\varphi}_i(t)$, $i = 1, 2, \ldots, n$, almost everywhere on $\mathbb{R}^+$. The obtained contradiction proves the theorem.

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