DOI 10.1007/s11253-017-1429-0
Ukrainian Mathematical Journal, Vol. 69, No. 8, January, 2018 (Ukrainian Original Vol. 69, No. 8, August, 2017)

PROPERTY OF MIXING OF CONTINUOUS CLASSICAL SYSTEMS WITH STRONG SUPERSTABLE INTERACTIONS

O. L. Rebenko and M. V. Tertychnyi UDC 517.9 + 531.19

We consider an infinite system of point particles in $\mathbb{R}^d$ interacting via a strong superstable two-body potential ϕ of finite range with radius R. In the language of correlation functions, we obtain a simple proof of the decay of correlations between two clusters (groups of variables) in the case where the distance between these clusters is larger than the radius of interaction. The established result is true for sufficiently small values of the activity of particles.

1. Introduction

The property of mixing plays a key role in the investigation of dynamical systems because it guarantees their ergodicity and enables one to state that any nonequilibrium distribution in a system with mixing approaches the equilibrium distribution (see, e.g., [1]). Various aspects of the property of mixing are extensively discussed in the papers and surveys and, thus, it seems to be impossible even to mention all of these works (see, e.g., the surveys [2, 3]). The rigorous proof of this property for models of statistical mechanics was obtained only for lattice systems (spin systems of ferromagnets and lattice gases).

In the language of Gibbs distributions, the property of mixing means that the behavior of a system in some volumes separated by large distances is statistically independent (see, e.g., [4] or Remark 2.7 in [5]). From the technical point of view, it is most convenient to prove this property by estimating the correlations between the clusters of particles, i.e., the behavior of correlation functions in which one group of variables is separated by a large distance from the other group. These investigations were originated in [6–8] for spin systems. In [7], the behavior of continuous systems was discussed and some estimates were presented without rigorous proofs.

In the present paper, we establish the property of rapid decay of correlations in classical continuous systems for low values of the activity (or large nonnegative values of the chemical potential). We use the method proposed in [9] for a system of lattice gas and the approximation of a continuous system of classical gas by a continuous system of *cell gas* [10]. A cell gas is a continuous system of point interacting particles whose configuration space is constructed as follows: For any partition of the space $\mathbb{R}^d$ into elementary disjoint hypercubes with edge a, every cube in this partition may contain at most one particle. This can be realized by introducing a Gibbs measure guaranteeing the elimination of configurations in which at least one cube of the partition contains more than one particle. An approximation of this kind was proposed in [11–13] for systems of point particles interacting via a strong superstable potential [14].

2. Main Mathematical Notions and Quantities

2.1. Configuration Spaces of the Systems of Statistical Mechanics. By $\mathfrak{B}(\mathbb{R}^d)$ we denote a Borel σ-algebra of open sets in $\mathbb{R}^d$. Also let $\mathfrak{B}_c(\mathbb{R}^d)$ denote all subsets with compact closure. The *configuration space* $\Gamma := \Gamma_{\mathbb{R}^d}$

Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, Ukraine.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 69, No. 8, pp. 1084–1095, August, 2017. Original article submitted April 27, 2017.

 0041-5995/18/6908–1262 © 2018 Springer Science+Business Media New York

consists of all locally finite subsets of the space $\mathbb{R}^d$, i.e.,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \left\{ \gamma \subset \mathbb{R}^d \;\middle|\; |\gamma \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\},$$

where $|A|$ is the number of points in A. By Γ_0 we denote the set of all finite configurations of the space Γ. Actually, Γ_0 is a subspace of Γ but we consider it as an independent configuration space in which the topology can be introduced independently of Γ (see, e.g., [15]). We first define a configuration space with fixed number of points:

$$\Gamma^{(n)} := \{\gamma \in \Gamma \mid |\gamma| = n, \; n \in \mathbb{N}\},$$

$$\Gamma^{(0)} := \varnothing.$$

If all these configurations are located in a certain bounded set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, then the corresponding space is denoted by

$$\Gamma_\Lambda^{(n)} := \left\{ \gamma \in \Gamma^{(n)} \mid \gamma \subset \Lambda \right\}.$$

Thus, the spaces of finite configurations in $\mathbb{R}^d$ and $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ can be represented in the form of disjoint unions as follows:

$$\Gamma_0 := \coprod_{n=0}^{\infty} \Gamma^{(n)} \qquad \text{and} \qquad \Gamma_\Lambda := \coprod_{n=0}^{\infty} \Gamma_\Lambda^{(n)}.$$

The topological and measurable structures of the spaces Γ, Γ_0, and Γ_Λ are well studied (see, e.g., [15–18]).

In the investigation of various thermodynamic characteristics of infinite systems, an important role is played by the partition of the space $\mathbb{R}^d$ into elementary hypercubes with edges of length $a > 0$ and centers at the points $r \in a\mathbb{Z}^d \subset \mathbb{R}^d$:

$$\Delta_a(r) := \{x \in \mathbb{R}^d \mid (r^i - a/2) \le x^i < (r^i + a/2), \; i = 1, \ldots, d\}.$$

We write Δ instead of $\Delta_a(r)$ if it is not necessary to indicate the location of the center of a hypercube. We denote this partition by $\overline{\Delta}_a$. Without loss of generality, for the sake of convenience, we consider Λ, which are unions of finitely many hypercubes $\Delta \in \overline{\Delta}_a$.

For the construction of the above-mentions approximation, we introduce one more space, namely, the *configuration space of cell gas:*

$$\Gamma^{(a)} := \left\{ \gamma \in \Gamma \;\middle|\; |\gamma_\Delta| \le 1 \text{ for all } \Delta \in \overline{\Delta}_a \right\}.$$

We make the following remark in order to clarify that the configurations $\Gamma^{(a)}$ describe not only physical systems of rarefied gases:

Remark 2.1. For a system of interacting particles, the density, i.e., the number of particles in a unit volume, is one of the most important characteristics of the physical state of the system. An arbitrarily large value of this characteristic can be obtained within the framework of the description of a system in the configuration space $\Gamma^{(a)}$ if we choose sufficiently small values of the length a of edges of the hypercubes.

2.2. Interaction. In the present paper, we consider a system of point particles interacting via a two-body potential ϕ. The energy of any configuration $\gamma \in \Gamma_\Lambda$ or $\gamma \in \Gamma_0$ is specified as follows:

$$U_\phi(\gamma) = U(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(|x-y|),$$

where summation is carried out over all possible different pairs of points of the configuration γ. We also define the energy of interaction between the configurations $\eta,\ \gamma \in \Gamma_0\ (\eta \cap \gamma = \varnothing)$ as follows:

$$W(\eta;\gamma) := \sum_{\substack{x\in\eta\\ y\in\gamma}} \phi(|x-y|).$$

(A) Assumptions Imposed on the Interaction Potential

In the present paper, we consider two-body potentials ϕ of the general form continuous on $\mathbb{R}_+ \setminus \{0\}$ and such that there are constants

$$r_0 > 0, \quad B \geq 0, \quad R > 0, \quad \text{and} \quad \varphi_0 > 0,$$

for which

$$\phi(|x|) \equiv \phi^+(|x|) \geq \frac{\varphi_0}{|x|^s} \qquad \text{for} \quad |x| \leq r_0, \quad s \geq d, \tag{2.1}$$

$$\phi(|x|) \equiv 0 \qquad \text{for} \quad |x| \geq R, \tag{2.2}$$

where

$$\phi^+(|x|) := \max\big\{0, \phi(|x|)\big\},$$

$$\phi^-(|x|) := -\min\big\{0, \phi(|x|)\big\}.$$

These potentials are called strong superstable (for details, see [14]). This enables one to approximate the analyzed system by a system of cell gas [10]. As a consequence, we also use the fact that the energy of interaction of particles in the configuration γ satisfies the stability condition:

$$U(\gamma) \;=\; \sum_{x,y\in\gamma} \phi(|x-y|) \;\geq\; -B|\gamma|, \qquad \gamma \in \Gamma_0, \quad B \geq 0. \tag{2.3}$$

The Lennard-Jones potential

$$\phi(|x|) = \frac{C}{|x|^{12}} - \frac{D}{|x|^6},$$

where $C > 0$ and $D > 0$ are constants, is extensively used in the molecular physics. In the present paper, we consider potentials of the same kind and their behavior is shown in Fig. 1. However, the potential $\phi(|x|) \to 0$ as $|x| \to R$ from the left.

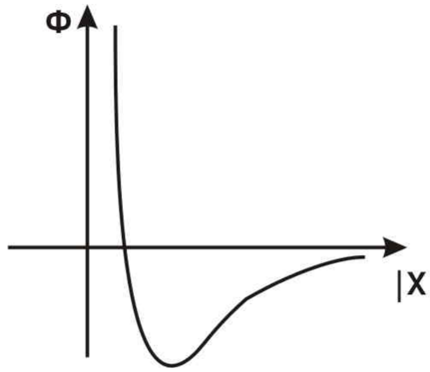

Fig. 1

2.3. Measures on Configuration Spaces of Continuous Systems. According to the Gibbs ideas, the physical state of a system is described by the probability measure which is first constructed in a certain bounded volume of the space $\mathbb{R}^d$ depending on the ensemble (microcanonical, canonical, or grand canonical) for the analyzed problem and then the thermodynamic limit transition is performed. We consider systems of statistical mechanics within the framework of grand canonical ensemble and, first, study a system of noninteracting point particles (*ideal gas*).

Let σ be a Lebesgue measure in $\mathbb{R}^d$. The state of an ideal gas in the equilibrium statistical mechanics is described by a *Poisson* measure $\pi_{z\sigma}$ on the configuration space Γ, where $z > 0$ is the activity (a physical parameter connected with the density of particles in the system). The measure $\pi_{z\sigma}$ with intensity $z\sigma$ is defined in what follows. To this end, we first introduce an analog of this measure in the spaces of finite configurations in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ (see [19]), which is sometimes called the *Lebesgue–Poisson* measure, by the formula

$$\int\limits_{\Gamma_\Lambda} F(\gamma)\lambda_{z\sigma}(d\gamma) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \int\limits_{\Lambda} \ldots \int\limits_{\Lambda} F(\{x_1, \ldots, x_n\})\sigma(dx_1)\ldots\sigma(dx_n)$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int\limits_{\Lambda} \ldots \int\limits_{\Lambda} F_n(x_1, \ldots, x_n)dx_1\ldots dx_n \tag{2.4}$$

for all measurable functions $F = \{F_n\}_{n\geq 0}$, $F_n \in L^\infty(\Lambda^n)$ $\big($or $F_n \in L^1(\mathbb{R}^{dn})\big)$. By using the measure $\lambda_{z\sigma}$, we construct a family of probability measures

$$\pi^\Lambda_{z\sigma} := e^{-z\sigma(\Lambda)}\lambda^\Lambda_{z\sigma}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{2.5}$$

With the help of definition (2.4), we can easily show that family (2.5) is pairwise consistent and, by the Kolmogorov theorem (see, e.g., [20]), there exists a unique probability measure $\pi_{z\sigma}$ on the configuration space Γ.

The Gibbs measure for the grand canonical ensemble on the space of configurations Γ_Λ is given by the formula

$$\mu_\Lambda(d\gamma) = \frac{1}{Z_\Lambda} e^{-\beta U(\gamma)}\lambda_{z\sigma}(d\gamma),$$

$$Z_\Lambda = \int\limits_{\Gamma_\Lambda} e^{-\beta U(\gamma)}\lambda_{z\sigma}(d\gamma),$$

where we have used the definition of the Lebesgue–Poisson measure (2.4). In the case of an infinite system in $\mathbb{R}^d$, according to the Dobrushin–Lanford–Ruelle approach (see [4, 21]), the Gibbs measure μ is defined on Γ with the help of a family of conditional probability distributions whose density is specified by the Radon–Nikodým derivative

$$\frac{d\mu_\Lambda}{d\lambda_{z\sigma}}(\eta \mid \bar{\gamma}_{\Lambda^c}) = \frac{\exp\{-\beta U(\eta \mid \bar{\gamma}_{\Lambda^c})\}}{Z_\Lambda(\bar{\gamma}_{\Lambda^c})},$$

where

$$U(\eta \mid \bar{\gamma}_{\Lambda^c}) := U(\eta) + W(\eta; \bar{\gamma}_{\Lambda^c})$$

and

$$\Lambda \in \mathcal{B}_c(\mathbb{R}^d), \qquad \Lambda^c = \mathbb{R}^d \setminus \Lambda, \qquad \eta \in \Gamma_\Lambda, \quad \bar{\gamma} \in \Gamma.$$

The existence and uniqueness of the Gibbs measure μ for the investigated interactions and the conditions are known results (see, e.g., the surveys [5, 22]) and the condition of mixing can be represented in the form

$$\mu(F_1 F_2) \ - \ \mu(F_1)\mu(F_2) \ \to \ 0$$

for two bounded functions $F_1, F_2 : \Gamma \mapsto \mathbb{R}$ such that $\operatorname{dist}(\operatorname{supp} F_1, \operatorname{supp} F_2) \to \infty$.

2.4. Correlation Functions and the Kirkwood–Salsburg Equations. In a certain sense, the correlation functions are moments of the Gibbs measure used to find the mean values of the observables, and the statistical sum plays an important role in the construction of thermodynamic functions, namely, the free energy and pressure of the system (see, e.g., [23–25]).

We write the expressions for the statistical sum Z_Λ and the corresponding collection of correlation functions ρ_Λ with the help of integrals over the measure $\lambda_{z\sigma}$. Thus, for the grand canonical ensemble, we obtain

$$Z_\Lambda(z,\beta) := \int\limits_{\Gamma_\Lambda} e^{-\beta U(\gamma)} \lambda_{z\sigma}(d\gamma),$$

$$\rho_\Lambda(\eta; z, \beta) := \frac{1}{Z_\Lambda(z,\beta)} \int\limits_{\Gamma_\Lambda} e^{-\beta U(\eta \cup \gamma)} \lambda_{z\sigma}(d\gamma), \quad \eta \in \Gamma_\Lambda.$$

For low values of the activity z, there exists a unique thermodynamic limit of ρ_Λ as $\Lambda \uparrow \mathbb{R}^d$ [23]. The limit functions $\rho(\eta; z, \beta)$ are the solutions of the infinite system of Kirkwood–Salsburg equations in the Banach space E_ξ (see, e.g., [23–25]) with the norm

$$\|\varphi\|_\xi := \sup_{\eta \in \Gamma_0 \setminus \varnothing} |\varphi(\eta)| \xi^{-|\eta|}, \quad \varphi \in E_\xi.$$

The system of Kirkwood–Salsburg equations can be represented in the form of a single operator equation as follows (see [23]):

$$\rho = z\widetilde{K}\rho + z\delta, \tag{2.6}$$

where the operator $\widetilde{K}$ acts upon an arbitrary function $\varphi \in E_\xi$ according to the rule

$$(\widetilde{K}\varphi)(\{x\}) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{i=1}^{k} \left(e^{-\beta\phi(|x-y_i|)} - 1\right) \varphi(\{y_1, \dots, y_k\})\, dy_1 \dots dy_k$$

$$:= \int\limits_{\Gamma_0} K(x;\gamma)\varphi(\gamma)\lambda_\sigma(d\gamma) \quad \text{for} \quad \eta = \{x\}, \quad K(x;\varnothing) = \varphi(\varnothing) = 1,$$

$$(\widetilde{K}\varphi)(\eta) = \sum_{x\in\eta} \widetilde{\pi}(x;\eta\setminus\{x\}) e^{-\beta W(x;\eta\setminus\{x\})}$$

$$\times \left[\varphi(\eta\setminus\{x\}) + \sum_{k=1}^{\infty} \frac{1}{k!} \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{i=1}^{k} \left(e^{-\beta\phi(|x-y_i|)} - 1\right) \varphi(\eta\setminus\{x\}\cup\{y_1,\dots,y_k\}) dy_1 \dots dy_k\right]$$

$$= \sum_{x\in\eta} \widetilde{\pi}(x;\eta\setminus\{x\}) e^{-\beta W(x;\eta\setminus\{x\})} \int\limits_{\Gamma_0} K(x;\gamma)\varphi(\eta\setminus\{x\}\cup\gamma)\lambda_\sigma(d\gamma) \quad \text{for} \quad |\eta| \geq 2,$$

where

$$\widetilde{\pi}(x;\eta\setminus\{x\}) = \frac{\pi_W(x;\eta\setminus\{x\})}{\sum_{y\in\eta} \pi_W(y;\eta\setminus\{y\})}, \tag{2.7}$$

$$\pi_W(x;\eta\setminus\{x\}) = \begin{cases} 1 & \text{for} \quad W(x;\eta\setminus\{x\}) \geq -2B, \\ 0, & \text{otherwise}, \end{cases} \tag{2.8}$$

$$\rho := \{\rho(\eta;z,\beta)\}_{\eta\in\Gamma_0\setminus\varnothing}, \quad \rho(\varnothing;z,\beta) = 1, \tag{2.9}$$

$$\delta(\eta) = \begin{cases} 1 & \text{for } |\eta| = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.10}$$

Remark 2.2. In Ruelle's notation [23], the operator $\widetilde{K} = \Pi K$, while (2.7), (2.8) is the realization of the operator Π:

$$(\Pi\varphi)(\eta) := \sum_{x\in\eta} \widetilde{\pi}(x;\eta\setminus\{x\})\varphi(x,\eta\setminus\{x\}).$$

Moreover, the operator $K\colon E_\xi \to E_{e^{2\beta B}\xi}$ and the operator $\widetilde{K}\colon E_\xi \to E_\xi$ (for details, see [23]).

The operator $\widetilde{K}$ is a bounded operator in the Banach space E_ξ.

The solution of Eq. (2.6) can be represented in the form of the series

$$\rho(\eta;z,\beta) = \sum_{n=0}^{\infty} z^{n+1} \left(\widetilde{K}^n\delta\right)(\eta;z,\beta)$$

convergent in the space E_ξ (and pointwise convergent for any fixed configuration $\eta \in \Gamma_0$) provided that the interaction satisfies Assumptions **(A)** and the values of the activity z lie inside the circle

$$|z| \leq e^{-2\beta B}\xi e^{-\xi C(\beta)}. \tag{2.11}$$

The optimal value of the parameter ξ is $\xi = C(\beta)^{-1}$, where

$$C(\beta) = \int\limits_{\mathbb{R}^d} \left| e^{-\beta\phi(|x|)} - 1 \right| dx.$$

2.5. Main Result.

Theorem 2.1. *Suppose that the potential $\phi(|x|)$ is continuous on $\mathbb{R}_+ \setminus \{0\}$ and satisfies Assumptions (2.1)–(2.3). Then, for any configurations $\eta, \eta_1 \in \Gamma_0$, $\eta \cap \eta_1 = \varnothing$, located at a distance*

$$\operatorname{dist}(\eta, \eta_1) := \min_{\substack{x \in \eta \\ y \in \eta_1}} |x - y| > R$$

form each other and sufficiently low values of the activity z, the following inequality is true:

$$\left|\rho(\eta \cup \eta_1) - \rho(\eta)\rho(\eta_1)\right| \leq C m_{\xi_0}^{\frac{\operatorname{dist}(\eta,\eta_1)}{R}} \xi_0^{|\eta| + |\eta_1|}, \tag{2.12}$$

where $m_{\xi_0} < 1$, $C > 0$, and $\xi_0 = \xi_0(\beta, z) > 0$ are independent of η and η_1.

3. Proof of Theorem 2.1

3.1. Correlation Functions in the Model of Cell Gas. As already indicated, in [11–13], it was established that, for the analyzed interactions, the correlation functions of the model of cell gas are pointwise convergent to the correlations functions $\rho(\eta; z, \beta)$ if the parameter of partition $a \to 0$. This implies that, in order to prove Theorem 2.1, it suffices to establish inequality (2.12) for the correlation functions of the model of cell gas with constants

$$m_{\xi_0} < 1, \quad C > 0, \quad \text{and} \quad \xi_0 = \xi_0(\beta, z) > 0$$

independent of the parameter of partition a.

To determine the correlation functions of the model of cell gas, we introduce the following function in the space Γ_0:

$$\chi_-^\Delta(\gamma) = \begin{cases} 1 & \text{for } \gamma \text{ from } |\gamma_\Delta| \in \{0, 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the statistical sum and the correlation functions of the model of cell gas are given by the formulas

$$Z_\Lambda^{(a)}(z, \beta) := \int\limits_{\Gamma_\Lambda} e^{-\beta U(\gamma)} \prod_{\Delta \in \overline{\Delta_a} \cap \Lambda} \chi_-^\Delta(\gamma) \lambda_{z\sigma}(d\gamma) := \int\limits_{\Gamma_\Lambda^{(a)}} e^{-\beta U(\gamma)} \lambda_{z\sigma}^a(d\gamma),$$

$$\rho_\Lambda^{(a)}(\eta; z, \beta) := \frac{1}{Z_\Lambda^{(a)}(z,\beta)} \int\limits_{\Gamma_\Lambda} e^{-\beta U(\eta\cup\gamma)} \lambda_{z\sigma}^a(\eta \cup d\gamma),$$

where

$$\lambda_{z\sigma}^a(\eta \cup d\gamma) := \prod_{\Delta \in \overline{\Delta_a} \cap \Lambda} \chi_-^\Delta(\eta\cup\gamma)\lambda_{z\sigma}(d\gamma).$$

We also define a family of *conditional* correlation functions $\rho_\Lambda^{(a)}(\cdot \mid \eta_1; z, \beta)$:

$$\rho_\Lambda^{(a)}(\eta \mid \eta_1; z, \beta) := \frac{z^{|\eta|}}{Z_\Lambda^{(a)}(\eta_1; z,\beta)} \int\limits_{\Gamma_\Lambda} e^{-\beta U(\eta\cup\gamma|\eta_1)} \lambda_{z\sigma}^a(\eta \cup \eta_1 \cup d\gamma), \tag{3.1}$$

where

$$U(\gamma \mid \eta) \ = U(\gamma) \ + \ W(\gamma; \eta),$$

and the corresponding statistical sum takes the form

$$Z_\Lambda^{(a)}(\eta_1; z, \beta) := \int\limits_{\Gamma_\Lambda^{(a)}} e^{-\beta U(\gamma|\eta_1)} \lambda_{z\sigma}(\eta_1 \cup d\gamma) = \int\limits_{\Gamma_\Lambda} e^{-\beta U(\gamma|\eta_1)} \lambda_{z\sigma}^a(\eta_1 \cup d\gamma).$$

3.2. Kirkwood–Salsburg Equations for the Correlation Functions of the Cell Gas. The Kirkwood–Salsburg equations for the correlation functions

$$\rho^{(a)}(\eta; z, \beta) \quad \text{and} \quad \rho^{(a)}(\cdot \mid \eta_1; z, \beta)$$

are the main instrument used to prove inequality (2.12). In [12], the equations for the functions $\rho^{(a)}(\eta; z, \beta)$ were used to prove the limit transition:

$$\lim_{a\to 0} \rho^{(a)}(\eta; z, \beta) \ = \rho(\eta; z, \beta).$$

We prove the existence of the thermodynamic limit $\Lambda \uparrow \mathbb{R}^d$ for the functions $\rho_\Lambda^{(a)}(\eta; z, \beta)$ in a similar way (see [12]). To deduce the equations for the functions $\rho_\Lambda^{(a)}(\cdot \mid \eta_1; z, \beta)$, we multiply the numerator and the denominator on the right-hand side of expression (3.1) by a factor

$$\frac{z^{|\eta_1|} e^{-\beta U(\eta_1)}}{Z_\Lambda^{(-)}(z, \beta, a)}.$$

This yields

$$\rho_\Lambda^{(a)}(\eta \mid \eta_1; z, \beta) := \frac{\dfrac{z^{|\eta\cup\eta_1|}}{Z_\Lambda^{(-)}(z,\beta,a)} \displaystyle\int_{\Gamma_\Lambda} e^{-\beta U(\eta\cup\eta_1\cup\gamma)} \lambda_{z\sigma}^a(\eta\cup\eta_1\cup d\gamma)}{\dfrac{z^{|\eta_1|}}{Z_\Lambda^{(-)}(z,\beta,a)} \displaystyle\int_{\Gamma_\Lambda} e^{-\beta U(\eta_1\cup\gamma)} \lambda_{z\sigma}^a(\eta_1\cup d\gamma)}. \tag{3.2}$$

In view of the definition of the correlation functions $\rho_\Lambda^{(a)}(\cdot; z, \beta)$, we rewrite relation (3.2) in the form

$$\rho_\Lambda^{(a)}(\eta \mid \eta_1; z, \beta) := \frac{\rho_\Lambda^{(a)}(\eta \cup \eta_1; z, \beta)}{\rho_\Lambda^{(a)}(\eta_1; z, \beta)}. \tag{3.3}$$

A similar relation can be also written for the limit functions $\rho^{(a)}(\eta \mid \eta_1; z, \beta)$. To obtain equations for the functions $\rho^{(a)}(\eta \mid \eta_1; z, \beta)$, we write the Kirkwood–Salsburg equation for the function $\rho^{(a)}(\eta \cup \eta_1; z, \beta)$, select a variable $x \in \eta$, and apply the operator $\widetilde{\pi}(x; \eta \setminus \{x\})$ [relations (2.7) and (2.8)]. As a result, we obtain

$$\rho^{(a)}(\eta \cup \eta_1; z, \beta) = z\widetilde{\pi}(x; \eta \setminus \{x\}) e^{-\beta W(x; \eta \setminus \{x\} \cup \eta_1)} \Bigg\{ \rho^{(a)}(\eta \setminus \{x\} \cup \eta_1; z, \beta)$$

$$+ \sum\!\!\!\!\!\!\int_{\substack{Q \subset \overline{\Delta}_a, Q \neq \varnothing \\ Q \cap \eta \setminus \{x\} \cup \eta_1 = \varnothing}} \prod_{y \in Q} \left(e^{-\beta \phi_{xy}} - 1\right) \rho^{(a)}(\eta \setminus \{x\} \cup \eta_1 \cup Q; z, \beta) \Bigg\}, \tag{3.4}$$

where

$$Q = \{\Delta_1, \ldots, \Delta_k\}, \quad k = 0, 1, \ldots,$$

are all possible subsets of cubes of the partition $\overline{\Delta}_a$ each of which contains a point of configuration. In relation (3.4), we use the notation

$$\sum\!\!\!\!\!\!\int_{Q \subset \overline{\Delta}_a} f(Q) = \sum_{k=0}^{\infty} \sum_{\{\Delta_1, \ldots, \Delta_k\} \subset \overline{\Delta}_a} \int_{\Delta_1} \ldots \int_{\Delta_k} f(y_1, \ldots, y_k) dy_1 \ldots dy_k.$$

In view of (3.3), we get

$$\rho^{(a)}(\eta \mid \eta_1; z, \beta) = z\widetilde{\pi}(x; \eta \setminus \{x\}) e^{-\beta W(x; \eta \setminus \{x\} \cup \eta_1)} \Bigg\{ \rho^{(a)}(\eta \setminus \{x\} \mid \eta_1; z, \beta)$$

$$+ \sum\!\!\!\!\!\!\int_{\substack{Q \subset \overline{\Delta}_a, Q \neq \varnothing \\ Q \cap \eta \setminus \{x\} \cup \eta_1 = \varnothing}} \prod_{y \in Q} \left(e^{-\beta \phi_{xy}} - 1\right) \rho^{(a)}(\eta \setminus \{x\} \cup Q \mid \eta_1; z, \beta) \Bigg\}. \tag{3.5}$$

We now rewrite the family of equations (3.5) in the form of a single operator equation

$$\rho^{(a)}(\eta \mid \eta_1) = z \left(\widetilde{K}_{\eta_1}^{(a)} \rho^{(a)} \right)(\eta \mid \eta_1) + z\delta_{\eta_1}(\eta), \tag{3.6}$$

where

$$\delta_{\eta_1}(\eta) = \begin{cases} e^{-\beta W(x; \eta_1)} & \text{for} \quad \eta = \{x\}, \\ 0, & \text{otherwise} \quad |\eta| \geq 2. \end{cases}$$

The operator $\widetilde{K}_{\eta_1}^{(a)}$ acts upon an arbitrary function $\varphi \in C_0(\Gamma_0^{(a)}$ according to the rule

$$\left(\widetilde{K}_{\eta_1}^{(a)}\varphi\right)(x) = e^{-\beta W(x;\eta_1)} \sum\!\!\!\!\!\!\!\int_{\substack{Q\subset\overline{\Delta}_a, Q\neq\varnothing \\ Q\cap\eta_1=\varnothing}} \prod_{y\in Q}\left(e^{-\beta\phi_{xy}}-1\right)\varphi(Q) \tag{3.7}$$

for $|\eta| = 1$ and according to the rule

$$\left(\widetilde{K}_{\eta_1}^{(a)}\varphi\right)(\eta) = \widetilde{\pi}(x;\eta\setminus\{x\})e^{-\beta W(x;\eta\setminus\{x\}\cup\eta_1)}$$
$$\times\left\{\varphi(\eta\setminus\{x\}) + \sum\!\!\!\!\!\!\!\int_{\substack{Q\subset\overline{\Delta}_a, Q\neq\varnothing \\ Q\cap\eta\setminus\{x\}\cup\eta_1=\varnothing}} \prod_{y\in Q}\left(e^{-\beta\phi_{xy}}-1\right)\varphi(\eta\setminus\{x\}\cup Q)\right\} \tag{3.8}$$

for $|\eta| \geq 2$.

It is worth noting that if $\operatorname{dist}(\eta,\eta_1) > R$, then $W(\eta;\eta_1) = 0$, $e^{-\beta W(x;\eta_1)} = 1$, and the summation in (3.7) and (3.8) is carried out over the subsets $Q \subset R(\Delta)$, where

$$R(\Delta) = \bigcup_{\Delta'\in\overline{\Delta}_a, \operatorname{dist}(\Delta',\Delta)\leq R} \Delta'.$$

The operator $\widetilde{K}_{\eta_1}^{(a)}$ is an operator in the space $E_\xi(\xi > 0)$ (see [23]). Its norm satisfies the inequality (see also the similar estimates obtained in [9] for the model of cell gas):

$$\left\|\widetilde{K}_{\eta_1}^{(-)}\right\|_\xi \leq e^{\beta D(a)} e^{\xi C(\beta)} \xi^{-1}.$$

The solution of Eq. (3.6) can be represented in the form of a series

$$\rho^{(a)}(\eta \mid \eta_1; z, \beta, a) = \sum_{n=0}^{\infty} z^{n+1}\left(\left(\widetilde{K}_{\eta_1}^{(a)}\right)^n \delta_{\eta_1}\right)(\eta)$$

convergent in E_ξ (and also pointwise convergent for every fixed configuration $\eta \in \Gamma_0$) and sufficiently low values of the activity z. Setting $\eta_1 = \varnothing$, we obtain similar equations for the functions $\rho^{(a)}(\eta; z, \beta)$, and the solutions of these equations are given by a similar series

$$\rho^{(a)}(\eta; z, \beta) = \sum_{n=0}^{\infty} z^{n+1}\left(\left(\widetilde{K}^{(a)}\right)^n \delta\right)(\eta).$$

The expression for the operator $\widetilde{K}^{(a)}$ $(|\eta| \geq 2)$ is similar to (3.8) but without the factor $e^{-\beta W(x;\eta_1)}$.

To prove inequality (2.12), we formulate the following lemma (see also [9], Lecture 6):

Lemma 3.1. *If functions f_1 and f_2 coincide $(f_1(\eta) = f_2(\eta))$ for all $\eta \in \Gamma_0$ such that*

$$\operatorname{dist}(\eta,\eta_1) \geq pR > 0, \quad p \in \mathbb{N},$$

then

$$\left(\widetilde{K}^{(a)} f_1\right)(\eta) = \left(\widetilde{K}^{(a)}_{\eta_1} f_2\right)(\eta)$$

for all η such that

$$\operatorname{dist}(\eta, \eta_1) \geq (p+1)R.$$

Proof. In view of Eqs. (3.7) and (3.8) and the fact that $Q \subset R(\Delta),$ we get

$$\operatorname{dist}(\eta \setminus \{x\} \cup Q, \eta_1) \geq pR$$

for $\operatorname{dist}(\eta, \eta_1) \geq (p+1)R.$ The summation in the expressions for the operators $\widetilde{K}^{(a)}$ and $\widetilde{K}^{(a)}_{\eta_1}$ is carried out over different domains since

$$Q \cap \eta \setminus \{x\} = \varnothing \quad \text{and} \quad Q \cap (\eta \setminus \{x\} \cup \eta_1) = \varnothing,$$

respectively. However, this does not lead to different operators because if

$$Q \cap \eta_1 \neq \varnothing \quad \text{and} \quad \operatorname{dist}(\eta, \eta_1) > R$$

under the conditions of the lemma, then

$$e^{-\beta\phi_{xy}} - 1 = 0 \quad \text{for any} \quad y \in Q.$$

If $\operatorname{dist}(\eta, \eta_1) \geq R,$ then $\delta(\eta) = \delta_{\eta_1}(\eta).$ Moreover, for any $\eta, \eta_1 \in \Gamma_0$ such that $\operatorname{dist}(\eta, \eta_1) \geq R,$ we have

$$\left(\widetilde{K}^{(a)} \delta\right)(\eta) = \left(\widetilde{K}^{(a)}_{\eta_1} \delta_{\eta_1}\right)(\eta).$$

Lemma 3.1 is proved.

In view of Lemma 3.1, we obtain

$$\left(\left(\widetilde{K}^{(a)}\right)^p \delta\right)(\eta) = \left(\left(\widetilde{K}^{(a)}_{\eta_1}\right)^p \delta_{\eta_1}\right)(\eta) \tag{3.9}$$

for all $p,\ 1 \leq p \leq \left[\dfrac{\operatorname{dist}(\eta, \eta_1)}{R}\right].$

For any $\eta, \eta_1 \in \Gamma_0,$ there exists $n \in \mathbb{N}_0$ such that

$$(n+1)R \geq \operatorname{dist}(\eta, \eta_1) \geq nR.$$

By using this fact and relation (3.9), we obtain the following estimate for any $\eta, \eta_1 \in \Gamma_0$ such that $\operatorname{dist}(\eta, \eta_1) \geq R$:

$$\left|\rho^{(a)}(\eta) - \rho^{(a)}(\eta \mid \eta_1)\right| \leq \left|\sum_{l=n+1}^{\infty} z^{l+1} \left(\left(\widetilde{K}^{(a)}_{\eta_1}\right)^l \delta\right)(\eta) - \sum_{l=n+1}^{\infty} z^{l+1} \left(\left(\widetilde{K}^{(a)}\right)^l \delta_{\eta_1}\right)(\eta)\right|. \tag{3.10}$$

By the definition of the norm in the space E_ξ and the fact that there exists ξ_0 such that the inequalities

$$z\big\|\widetilde{K}^{(a)}\big\|_{\xi_0} \le m_{\xi_0}(a), \quad z\big\|\widetilde{K}^{(a)}_{\eta_1}\big\|_{\xi_0} \le m_{\xi_0}(a), \quad 0 \le m_{\xi_0}(a) \le 1,$$

are true, we arrive at the inequalities

$$z^l\Big|\big(\big(\widetilde{K}^{(a)}\big)^l\delta\big)(\eta)\Big| \le z^l\big\|\big(\widetilde{K}^{(a)}\big)^l\delta\big\|_{\xi_0}\xi_0^{|\eta|} \le m^l_{\xi_0}\|\delta\|_{\xi_0}\xi_0^{|\eta|} \le m^l_{\xi_0}\xi_0^{|\eta|-1}, \tag{3.11}$$

$$z^l\Big|\big(\big(\widetilde{K}^{(a)}_{\eta_1}\big)^l\delta_{\eta_1}\big)(\eta)\Big| \le z^l\big\|\big(\widetilde{K}^{(a)}_{\eta_1}\big)^l\delta_{\eta_1}\big\|_{\xi_0}\xi_0^{|\eta|} \le m^l_{\xi_0}\|\delta_{\eta_1}\|_{\xi_0}\xi_0^{|\eta|} \le e^{\beta B}m^l_{\xi_0}\xi_0^{|\eta|-1}. \tag{3.12}$$

In view of estimates (3.11) and (3.12), we can rewrite estimate (3.10) in the form

$$\Big|\rho^{(a)}(\eta) - \rho^{(a)}(\eta \mid \eta_1)\Big| \le C_0 m^{n+1}_{\xi_0}\xi_0^{|\eta|} \le C_0 m_{\xi_0}^{\frac{\mathrm{dist}(\eta,\eta_1)}{R}}\xi_0^{|\eta|}. \tag{3.13}$$

In view of relation (3.3) and the boundedness of the correlation functions $\rho^{(a)}(\eta; z, \beta)$:

$$\rho^{(a)}(\eta_1; z, \beta) \le C_1\xi_1^{|\eta_1|},$$

where C_1 is independent of a (see [13]), we derive the following inequality from (3.13):

$$\Big|\rho^{(a)}(\eta \cup \eta_1) - \rho^{(a)}(\eta)\rho^{(a)}(\eta_1)\Big| \le C m_{\xi_0}^{\frac{\mathrm{dist}(\eta,\eta_1)}{R}}\xi^{|\eta|+|\eta_1|}. \tag{3.14}$$

The right-hand side of inequality (3.14) is independent of the parameter of partition a. Finally, passing to the limit as $a \to 0$, we complete the proof of Theorem 2.1.

REFERENCES

1. I. P. Kornfel'd, Ya. G. Sinai, and S. V. Fomin, *Ergodic Theory* [in Russian], Nauka, Moscow (1980).
2. K. S. Alexander, "Mixing properties and exponential decay for lattice systems in finite volumes," *Ann. Probab.*, **32**, No. 1A, 441–487 (2004).
3. R. C. Bradley, "Basic properties of strong mixing conditions. A survey and some open questions," *Probab. Surv.*, **2**, 107–144 (2005).
4. R. L. Dobrushin, "Description of a random field with the help of conditional probabilities and conditions of its regularity," *Teor. Veroyatn. Ee Primen.*, **13**, Issue 2, 201–229 (1968).
5. D. Conache, A. Daletskii, Yu. Kondratiev, and T. Pasurek, *Gibbs Measures on Marked Configuration Spaces: Existence and Uniqueness*, Preprint arXiv 1503.06349v2 (2015).
6. J. L. Lebowitz, "Bounds on the correlations and analyticity properties of ferromagnetic Ising spin systems," *Comm. Math. Phys.*, **28**, 313–321 (1972).
7. M. Duneau, D. Iagolnitzer, and B. Souillard, "Decrease properties of truncated correlation functions and analyticity properties for classical statistical mechanics," *Comm. Math. Phys.*, **31**, 191–208 (1973).
8. D. Iagolnitzer and B. Souillard, "On the analyticity in the potential in classical statistical mechanics," *Comm. Math. Phys.*, **60**, 131–152 (1978).
9. R. A. Minlos, *Introduction to Mathematical Statistical Physics*, American Mathematical Society, Providence, RI (1999).
10. A. L. Rebenko, "Cell gas model of classical statistical systems," *Rev. Math. Phys.*, **25**, No. 4, 1–28 (2013).
11. A. L. Rebenko and M. V. Tertychnyi, "Quasicontinuous approximation of statistical systems with strong superstable interactions," in: *Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences*, **4**, No. 3 (2007), pp. 172–182.
12. A. L. Rebenko and M. V. Tertychnyi, "Quasi-lattice approximation of statistical systems with strong superstable interactions. Correlation functions," *J. Math. Phys.*, **50**, No. 3, 1–16 (2009).

13. S. M. Petrenko, O. L. Rebenko, and M. V. Tertychnyi, "Quasicontinuous approximation in classical statistical mechanics," *Ukr. Mat. Zh.*, **63**, No. 3, 369–384 (2011); ***English translation:*** *Ukr. Math. J.*, **63**, No. 3, 425–442 (2011).
14. A. L. Rebenko and M. V. Tertychnyi, "On the stability, superstability and strong superstability of classical systems of statistical mechanics," *Meth. Funct. Anal. Topol.*, **14**, No. 3, 287–296 (2008).
15. Yu. G. Kondratiev and T. Kuna, "Harmonic analysis on configuration spaces. I. General theory," *Infin. Dimens. Anal. Quantum. Probal. Relat. Top.*, **5**, No. 2, 201–233 (2002).
16. A. Lenard, "States of classical statistical mechanical systems of infinitely many particles. I," *Arch. Ration. Mech. Anal.*, **59**, 219–239 (1975).
17. A. Lenard, "States of classical statistical mechanical systems of infinitely many particles. II," *Arch. Ration. Mech. Anal.*, **59**, 241–256 (1975).
18. R. L. Dobrushin, Ya. G. Sinai, and Yu. M. Sukhov, "Dynamical systems of statistical mechanics," in: *VINITI Series on Contemporary Problems in Mathematics. Fundamental Trends* [in Russian], Vol. 2, VINITI, Moscow (1985), pp. 235–284.
19. R. A. Minlos, "Lectures on statistical physics," *Usp. Mat. Nauk*, **23**, No. 1, 133–190 (1968).
20. K. R. Parthasarathy, *Probability Measure on Metric Spaces. Probability and Mathematical Statistics*, Academic Press, New York (1967).
21. O. E. Lanford and D. Ruelle, "Observables at infinity and states with short range correlations in statistical mechanics," *Comm. Math. Phys.*, **13**, No. 3, 194–215 (1969).
22. Yu. G. Kondratiev, T. Pasurek, and M. Röckner, "Gibbs measures of continuous systems: an analytic approach," *Rev. Math. Phys.*, **24**, No. 10, 1250026 (2012).
23. D. Ruelle, *Statistical Mechanics. Rigorous Results*, Benjamin, New York (1969).
24. D. Ya. Petrina, V. I. Gerasimenko, and P. V. Malyshev, *Mathematical Foundations of Classical Statistical Mechanics* [in Russian], Naukova Dumka, Kiev (1985).
25. D. Ya. Petrina, V. I. Gerasimenko, and P. V. Malyshev, *Mathematical Foundations of Classical Statistical Mechanics. Continuous Systems*, Gordon @ Breach Sci. Publ., New York, etc. (1989).