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NEW FRACTIONAL INTEGRAL INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS AND THEIR APPLICATIONS

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We establish some new fractional integral inequalities for differentiable convex functions and give several applications for the Beta-function.

1. Introduction

Throughout this paper, we assume that $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f\colon [a,b] \to \mathbb{R}$, is known in the literature as the Hermite–Hadamard inequality [7].

For some results generalizing, improving, and extending the inequality (1.1), see [1–6] and [8–17].

In [14], Tseng, et al. established the following Hermite–Hadamard-type inequality refining inequality (1.1).

Theorem A. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]$$

$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The third inequality in (1.2) is known in the literature as the Bullen inequality.

In [4], Dragomir and Agarwal established the following results connected with the second inequality in (1.1).

Theorem B. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

which is the trapezoid inequality provided that $|f'|$ is convex on $[a,b]$.

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In [11], Kirmaci and Özdemir established the following results connected with the first inequality in (1.1):

Theorem C. *Under the assumptions of Theorem B, the following inequality is true:*

$$\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right),$$

which is the midpoint inequality provided that $|f'|$ *is convex on* $[a,b]$.

In [12], Pearce and Pečarić established the following Hermite–Hadamard-type inequalities for differentiable functions:

Theorem D. *If* $f : I^o \subseteq R \to R$ *is a differentiable mapping on* $I^o,\ a,b \in I^o$ *with*

$$a < b, \quad f' \in L_1[a,b], \quad q \geq 1,$$

and $|f'|^q$ *is convex on* $[a,b]$, *then the following inequalities are true:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q},$$

$$\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

We now recall the following definition [13]:

Definition 1.1. *Let* $f \in L_1[a,b]$. *The Riemann–Liouville integrals*

$$J^{\alpha}_{a+}f \qquad \text{and} \qquad J^{\alpha}_{b-}f$$

of order $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma-function and*

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

In [13], Sarikaya, et al. established the following Hermite–Hadamard-type inequalities for the fractional integrals:

Theorem E. *Let $f\colon [a,b] \to \mathbb{R}$ be positive with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2}$$

for $\alpha > 0$.

Theorem F. *Under the assumptions of Theorem B, the following inequality is true:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \le \frac{2^\alpha - 1}{2^{\alpha+1}(\alpha+1)}(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

for $\alpha > 0$.

In [9], Hwang, et al. established the following fractional integral inequalities:

Theorem G. *Under the assumptions of Theorem B, the following Hermite–Hadamard-type inequality is true for fractional integrals:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{b-a}{4(\alpha+1)}\left(\alpha - 1 + \frac{1}{2^{\alpha-1}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

for $\alpha > 0$.

Theorem H. *Under the assumptions of Theorem B, the following inequality for fractional integrals is true with* $\dfrac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}$:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\le \left(\frac{1}{8} + \frac{3^{\alpha+1} - 2^{\alpha+1} + 1}{4^{\alpha+1}(\alpha+1)} - \frac{1}{2(\alpha+1)}\right)(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \tag{1.3}$$

for $\alpha > 0$.

Theorem I. *Under the assumptions of Theorem B, the following Bullen-type inequality for fractional integrals is true:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \left[\frac{3^\alpha - 1}{4^\alpha} f\left(\frac{a+b}{2}\right) + \frac{4^\alpha - 3^\alpha + 1}{4^\alpha}\frac{f(a)+f(b)}{2}\right]\right|$$

$$\leq \frac{1}{\alpha+1}\left(\frac{2^\alpha+1}{2^{\alpha+1}}-\frac{3^{\alpha+1}+1}{4^{\alpha+1}}\right)(b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big) \tag{1.4}$$

for $\alpha > 0$.

Theorem J. *Under the assumptions of Theorem B, the following Simpson-type inequality for fractional integrals is true:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] - \left[\frac{5^\alpha-1}{6^\alpha}f\left(\frac{a+b}{2}\right)+\frac{6^\alpha-5^\alpha+1}{6^\alpha}\frac{f(a)+f(b)}{2}\right]\right|$$

$$\leq \left[\frac{1}{\alpha+1}\left(\frac{2^\alpha+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^\alpha-1}{12\cdot 6^\alpha}\right)\right](b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big) \tag{1.5}$$

for $\alpha > 0$.

Remark 1.1.

(1) The assumption that $f\colon [a,b]\to\mathbb{R}$ is positive with $0\leq a<b$ in Theorem E can be weakened as $f\colon[a,b]\to\mathbb{R}$ with $a<b$.

(2) In Theorem D, let $q=1$. Then Theorem D reduces to Theorems B and C.

(3) In Theorems F and G, let $\alpha=1$. Then Theorems F and G reduce to Theorems B and C, respectively.

(4) In Theorem H, let $\alpha=1$. Then inequality (1.3) is connected with the second inequality in (1.2).

(5) In Theorem I, let $\alpha=1$. Then inequality (1.4) is a Bullen-type inequality.

(6) In Theorem J, let $\alpha=1$. Then inequality (1.5) is a Simpson-type inequality.

In the present paper, we establish some new Hermite–Hadamard-type inequalities for the fractional integrals generalizing Theorems D and G–J. Some applications for the Beta-function are given.

2. Main Results

Theorem 2.1. *Under the assumptions of Theorem D, the following Hermite–Hadamard-type inequality for fractional integrals is true:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{2(\alpha+1)}\left(\alpha-1+\frac{1}{2^{\alpha-1}}\right)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \tag{2.1}$$

for $\alpha > 0$.

Proof. In [9], we assume that

$$h_1(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - (b-a)^\alpha, & x \in \left[a, \dfrac{a+b}{2}\right), \\ (b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Then the following identities are true:

$$\begin{aligned} &\frac{1}{2(b-a)^\alpha} \int_a^b h_1(x) f'(x)\, dx \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\, dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha (a)\right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha (a)\right] - f\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.2}$$

As a result of simple computation, we arrive at the following identities:

$$x = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b, \quad x \in [a,b], \tag{2.3}$$

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx \\ &\quad + \int_{\frac{a+b}{2}}^b \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx \\ &= \int_a^{\frac{a+b}{2}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx \\ &\quad + \int_a^{\frac{a+b}{2}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx \\ &= \left|f'(a)\right|^q \int_a^{\frac{a+b}{2}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx := M_1, \end{aligned} \tag{2.4}$$

$$\int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \frac{x-a}{b-a} |f'(b)|^q \, dx$$

$$+ \int_{\frac{a+b}{2}}^b [(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha] \frac{x-a}{b-a} |f'(b)|^q \, dx$$

$$= \int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \frac{x-a}{b-a} |f'(b)|^q \, dx$$

$$+ \int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \frac{b-x}{b-a} |f'(b)|^q \, dx$$

$$= |f'(b)|^q \int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \, dx := M_2, \tag{2.5}$$

$$\int_a^b |h_1(x)| \, dx = 2 \int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \, dx. \tag{2.6}$$

Further, by using the power mean inequality, identities (2.3)–(2.6) and the convexity of $|f'|^q$, we obtain the inequality

$$\left| \int_a^b h_1(x) f'(x) \, dx \right| \le \int_a^b |h_1(x)| \, |f'(x)| \, dx$$

$$\le \left[\int_a^b |h_1(x)| \, dx \right]^{\frac{q-1}{q}} \left[\int_a^b |h_1(x)| \, |f'(x)|^q \, dx \right]^{1/q}$$

$$= \left[\int_a^b |h_1(x)| \, dx \right]^{\frac{q-1}{q}} \left[\int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \, |f'(x)|^q \, dx \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b [(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha] \, |f'(x)|^q \, dx \right]^{1/q}$$

$$\leq \left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}} (M_1+M_2)^{1/q}$$

$$= 2\int_a^{\frac{a+b}{2}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

$$= \frac{(b-a)^{\alpha+1}}{\alpha+1}\left(\alpha - 1 + \frac{1}{2^{\alpha-1}}\right)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{2.7}$$

Inequality (2.1) follows from identity (2.2) and inequality (2.7).
Theorem 2.1 is proved.

Remark 2.1. In Theorem 2.1, let $q = 1$. Then Theorem 2.1 reduces to Theorem G.

Theorem 2.2. *Under the assumptions of Theorem D, the following inequality for fractional integrals is true with* $\dfrac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}$:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right|$$

$$\leq \left(\frac{1}{4} + \frac{3^{\alpha+1} - 2^{\alpha+1} + 1}{2\cdot 4^\alpha(\alpha+1)} - \frac{1}{\alpha+1}\right)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \tag{2.8}$$

for $\alpha > 0$.

Proof. In [9], let

$$h_2(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - (b-a)^\alpha, & x \in \left[a, \frac{3a+b}{4}\right), \\ (b-x)^\alpha - (x-a)^\alpha, & x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right), \\ (b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha, & x \in \left[\frac{a+3b}{4}, b\right]. \end{cases}$$

Then the following identities hold:

$$\frac{1}{2(b-a)^\alpha}\int_a^b h_2(x) f'(x)\,dx = \frac{\alpha}{2(b-a)^\alpha}\int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\,dx$$

$$- \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}$$

$$= \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha(a)\right] - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}$$

$$= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]. \tag{2.9}$$

As a result of simple computation, we arrive at the identities

$$\int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$
$$+ \int_{\frac{a+3b}{4}}^{b} \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$
$$= \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$
$$+ \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx$$
$$= \left|f'(a)\right|^q \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx := N_1, \tag{2.10}$$

$$\int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$
$$+ \int_{\frac{a+3b}{4}}^{b} \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$
$$= \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$
$$+ \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx := N_2, \tag{2.11}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(x-a)^\alpha - (b-x)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx$$

$$= \left|f'(a)\right|^q \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] dx := N_3, \tag{2.12}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(x-a)^\alpha - (b-x)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [(b-x)^\alpha - (x-a)^\alpha]\, dx := N_4, \tag{2.13}$$

$$\int\limits_a^b |h_2(x)|\, dx = 2\left[\int\limits_a^{\frac{3a+b}{4}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha]\, dx + \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [(b-x)^\alpha - (x-a)^\alpha]\, dx\right]. \tag{2.14}$$

Further, by using the power mean inequality, identities (2.3), (2.10)–(2.14), and the convexity of $|f'|^q,$ we get the inequality

$$\begin{aligned}
\left|\int\limits_a^b h_2(x) f'(x)\, dx\right| &\le \int\limits_a^b |h_2(x)| \left|f'(x)\right| dx \\
&\le \left[\int\limits_a^b |h_2(x)|\, dx\right]^{\frac{q-1}{q}} \left[\int\limits_a^b |h_2(x)| \left|f'(x)\right|^q dx\right]^{1/q} \\
&= \left[\int\limits_a^b |h_2(x)|\, dx\right]^{\frac{q-1}{q}} \left[\int\limits_a^{\frac{3a+b}{4}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha] \left|f'(x)\right|^q dx\right. \\
&\quad + \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [(b-x)^\alpha - (x-a)^\alpha] \left|f'(x)\right|^q dx \\
&\quad + \int\limits_{\frac{a+b}{2}}^{\frac{a+3b}{4}} [(x-a)^\alpha - (b-x)^\alpha] \left|f'(x)\right|^q dx \\
&\quad \left. + \int\limits_{\frac{a+3b}{4}}^{b} [(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha] \left|f'(x)\right|^q dx\right]^{1/q}
\end{aligned}$$

$$\leq \left[\int_a^b |h_2(x)|\, dx \right]^{\frac{q-1}{q}} (N_1 + N_2 + N_3 + N_4)$$

$$= 2 \left[\int_a^{\frac{3a+b}{4}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha]\, dx \right.$$

$$\left. + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [(b-x)^\alpha - (x-a)^\alpha]\, dx \right] \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}$$

$$= \left(\frac{1}{2} + \frac{3^{\alpha+1} - 2^{\alpha+1} + 1}{4^\alpha (\alpha+1)} - \frac{2}{\alpha+1} \right) (b-a)^{\alpha+1} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}. \tag{2.15}$$

Inequality (2.8) follows from identity (2.9) and inequality (2.15).
Theorem 2.2 is proved.

Remark 2.2.

(1) In Theorem 2.2, let $q = 1$. Then Theorem 2.2 reduces to Theorem H.

(2) In Theorems 2.1 and 2.2, let $\alpha = 1$. Then Theorems 2.1 and 2.2 reduce to Theorem D.

Theorem 2.3. *Under the assumptions of Theorem D, the following Bullen-type inequality for fractional integrals is true:*

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right.$$

$$\left. - \left[\frac{3^\alpha - 1}{4^\alpha} f\left(\frac{a+b}{2} \right) + \frac{4^\alpha - 3^\alpha + 1}{4^\alpha} \frac{f(a) + f(b)}{2} \right] \right|$$

$$\leq \frac{1}{\alpha+1} \left(\frac{2^\alpha + 1}{2^\alpha} - \frac{3^{\alpha+1} + 1}{2 \cdot 4^\alpha} \right) (b-a) \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q} \tag{2.16}$$

for $\alpha > 0$.

Proof. Let

$$h_3(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - \dfrac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha, & x \in \left[a, \dfrac{a+b}{2} \right), \\[2ex] (b-x)^\alpha - (x-a)^\alpha + \dfrac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha, & x \in \left[\dfrac{a+b}{2}, b \right]. \end{cases}$$

Then the following identities are true:

$$\frac{1}{2(b-a)^{\alpha}}\int_{a}^{b} h_3(x)f'(x)\,dx$$

$$=\frac{\alpha}{2(b-a)^{\alpha}}\int_{a}^{b}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,dx$$

$$-\left[\frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}}\frac{f(a)+f(b)}{2}\right]$$

$$=\frac{\alpha\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}(a)\right]$$

$$-\left[\frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}}\frac{f(a)+f(b)}{2}\right]$$

$$=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}(a)\right]$$

$$-\left[\frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}}\frac{f(a)+f(b)}{2}\right]. \tag{2.17}$$

As a result of simple computations, we arrive at the following identities:

$$\int_{a}^{\frac{3a+b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx$$

$$+\int_{\frac{a+3b}{4}}^{b}\left[(x-a)^{\alpha}-(b-x)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx$$

$$=\int_{a}^{\frac{3a+b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx$$

$$+\int_{a}^{\frac{3a+b}{4}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|^{q}dx$$

$$= \left|f'(a)\right|^q \int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_1, \tag{2.18}$$

$$\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{a+3b}{4}}^{b} \left[(x-a)^\alpha - (b-x)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$= \int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_2, \tag{2.19}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx$$

$$= \left|f'(a)\right|^q \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_3, \tag{2.20}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_4, \tag{2.21}$$

$$\int_a^b |h_3(x)| dx = 2\left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx\right.$$

$$\left. + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx\right]. \tag{2.22}$$

Thus, by using the power mean inequality, identities (2.3) and (2.18)–(2.22), and the convexity of $|f'|^q$, we establish the inequality

$$\left|\int_a^b h_3(x) f'(x) dx\right| \le \int_a^b |h_3(x)| \left|f'(x)\right| dx$$

$$\le \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_3(x)|\left|f'(x)\right|^q dx\right]^{1/q}$$

$$= \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}\right](b-a)^\alpha \left|f'(x)\right|^q dx\right.$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx$$

$$\left. + \int_{\frac{a+3b}{4}}^{b} \left[(x-a)^\alpha - (b-x)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx\right]^{1/q}$$

$$\le \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} (P_1 + P_2 + P_3 + P_4)$$

$$= 2\left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx\right.$$

$$\left. + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right]\right] dx \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}$$

$$= \frac{1}{\alpha+1}\left(\frac{2^\alpha + 1}{2^{\alpha-1}} - \frac{3^{\alpha+1} + 1}{4^\alpha}\right)(b-a)^{\alpha+1}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{2.23}$$

Inequality (2.16) follows from identity (2.17) and inequality (2.23).
Theorem 2.3 is proved.

Remark 2.3. In Theorem 2.3, let $q = 1$. Then Theorem 2.3 reduces to Theorem I.

Theorem 2.4. *Under the assumptions of Theorem D, the following Simpson-type inequality for fractional integrals is true:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] - \left[\frac{5^\alpha-1}{6^\alpha}f\left(\frac{a+b}{2}\right)+\frac{6^\alpha-5^\alpha+1}{6^\alpha}\frac{f(a)+f(b)}{2}\right]\right|$$

$$\leq \left[\frac{1}{\alpha+1}\left(\frac{2^\alpha+1}{2^\alpha}-\frac{5^{\alpha+1}+1}{3\cdot 6^\alpha}\right)+\left(\frac{5^\alpha-1}{6^{\alpha+1}}\right)\right](b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \tag{2.24}$$

for $\alpha>0$.

Proof. In [9], let

$$h_4(x)=\begin{cases}(b-x)^\alpha-(x-a)^\alpha-\dfrac{5^\alpha-1}{6^\alpha}(b-a)^\alpha, & x\in\left[a,\dfrac{a+b}{2}\right),\\[2ex] (b-x)^\alpha-(x-a)^\alpha+\dfrac{5^\alpha-1}{6^\alpha}(b-a)^\alpha, & x\in\left[\dfrac{a+b}{2},b\right].\end{cases}$$

Then, the following identities hold:

$$\frac{1}{2(b-a)^\alpha}\int_a^b h_4(x)f'(x)\,dx$$

$$=\frac{\alpha}{2(b-a)^\alpha}\int_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,dx - \left[\frac{5^\alpha-1}{6^\alpha}f\left(\frac{a+b}{2}\right)+\frac{6^\alpha-5^\alpha+1}{6^\alpha}\frac{f(a)+f(b)}{2}\right]$$

$$=\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}(a)\right] - \left[\frac{5^\alpha-1}{6^\alpha}f\left(\frac{a+b}{2}\right)+\frac{6^\alpha-5^\alpha+1}{6^\alpha}\frac{f(a)+f(b)}{2}\right]$$

$$=\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}(a)\right] - \left[\frac{5^\alpha-1}{6^\alpha}f\left(\frac{a+b}{2}\right)+\frac{6^\alpha-5^\alpha+1}{6^\alpha}\frac{f(a)+f(b)}{2}\right]. \tag{2.25}$$

As a result of simple computations, we arrive at the following identities:

$$\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx$$

$$+\int_{\frac{a+5b}{6}}^{b} \left[(x-a)^\alpha-(b-x)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx$$

$$=\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx$$

$$+\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(a)\right|^q dx$$

$$=\left|f'(a)\right|^q\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx:=Q_1, \tag{2.26}$$

$$\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx$$

$$+\int_{\frac{a+5b}{6}}^{b} \left[(x-a)^\alpha-(b-x)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx$$

$$=\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx$$

$$+\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(b)\right|^q dx$$

$$=\left|f'(b)\right|^q\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx:=Q_2, \tag{2.27}$$

$$\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx$$

$$= \left|f'(a)\right|^q \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] dx := Q_3, \tag{2.28}$$

$$\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx$$

$$= \left|f'(b)\right|^q \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right] dx := Q_4, \tag{2.29}$$

$$\int_a^b |h_4(x)|\,dx = 2\left[\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]dx + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]dx\right]. \tag{2.30}$$

Further, by using the power mean inequality, identities (2.3) and (2.26)–(2.30), and the convexity of $|f'|^q$, we get the inequality

$$\begin{aligned}
\left|\int_a^b h_4(x) f'(x)dx\right| &\le \int_a^b |h_4(x)|\left|f'(x)\right|dx \\
&\le \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_4(x)|\left|f'(x)\right|^q dx\right]^{1/q} \\
&= \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx\right. \\
&\quad + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx \\
&\quad + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx \\
&\quad \left.+ \int_{\frac{a+5b}{6}}^{b} \left[(x-a)^\alpha - (b-x)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]\left|f'(x)\right|^q dx\right] \\
&\le \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} (Q_1 + Q_2 + Q_3 + Q_4) \\
&= 2\left[\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]dx\right.
\end{aligned}$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^{\alpha} - (b-x)^{\alpha} + \frac{5^{\alpha}-1}{6^{\alpha}} (b-a)^{\alpha} \right] dx \Bigg]$$

$$\times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}$$

$$= \left[\frac{1}{\alpha+1} \left(\frac{2^{\alpha}+1}{2^{\alpha-1}} - \frac{5^{\alpha+1}+1}{9 \cdot 6^{\alpha-1}} \right) + \left(\frac{5^{\alpha}-1}{3 \cdot 6^{\alpha}} \right) \right]$$

$$\times (b-a)^{\alpha+1} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}. \tag{2.31}$$

Inequality (2.24) now follows from identity (2.25) and inequality (2.31).
Theorem 2.4 is proved.

Remark 2.4. In Theorem 2.4, let $q = 1$. Then Theorem 2.4 reduces to Theorem J.

3. Applications for the Beta-functions

Throughout this section, let

$$\alpha > 0, \quad \rho \geq 1, \quad q \geq 1, \quad a = 0, \quad b = 1,$$

let $\Gamma(\alpha)$ be the Gamma-function, and let

$$f(x) = x^{\rho-1} \qquad (x \in [0,1]).$$

Then $|f'|$ is convex on $[0,1]$.

We now recall the definition of the *Beta-function*

$$B(p,r) = \int_0^1 x^{p-1} (1-x)^{r-1} \, dx \quad (p, r > 0).$$

Remark 3.1. By using Theorems 2.1–2.4, we get

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{a+}^{\alpha} f(b) = \frac{\alpha}{2} \int_0^1 (1-x)^{\alpha-1} x^{\rho-1} dx = \frac{\alpha}{2} B(\rho, \alpha)$$

and

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) = \frac{\alpha}{2} \int_0^1 x^{\alpha+\rho-2} dx = \frac{\alpha}{2(\alpha+\rho-1)}.$$

By virtue of Theorems 2.1–2.4 and Remark 3.1, we arrive at the following propositions:

Proposition 3.1. *In Theorem 2.1, the following inequality is true:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\frac{1}{2^{\rho-1}}\right|\le\left(\frac{1}{2}-\frac{2^{\alpha}-1}{2^{\alpha}(\alpha+1)}\right)\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.2. *In Theorem 2.2, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\frac{3^{\rho-1}+1}{2\cdot 4^{\rho-1}}\right|$$
$$\le\left[\frac{3^{\alpha+1}+1}{2\cdot 4^{\alpha}(\alpha+1)}+\frac{1}{4}-\frac{2^{\alpha}+1}{2^{\alpha}(\alpha+1)}\right]\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.3. *In Theorem 2.3, the following inequality is true:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\left(\frac{3^{\alpha}-1}{2^{\rho-1}4^{\alpha}}+\frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\right)\right|$$
$$\le\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha}}-\frac{3^{\alpha+1}+1}{2\cdot 4^{\alpha}}\right)\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.4. *In Theorem 2.4, the following inequality holds:*

$$\left|\frac{\alpha}{2}B(\rho,\alpha)+\frac{\alpha}{2(\alpha+\rho-1)}-\left(\frac{5^{\alpha}-1}{2^{\rho-1}6^{\alpha}}+\frac{6^{\alpha}-5^{\alpha}+1}{2\cdot 6^{\alpha}}\right)\right|$$
$$\le\left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha}}-\frac{5^{\alpha+1}+1}{3\cdot 6^{\alpha}}\right)+\left(\frac{5^{\alpha}-1}{6^{\alpha+1}}\right)\right]\frac{\rho-1}{2^{1/q}}.$$

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