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ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF DIFFERENCE EQUATIONS

Sh.-T. Lan[1] **and Z.-X. Chen**[2]

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We estimate the order of growth of meromorphic solutions of some linear difference equations and study the relationship between the exponent of convergence of zeros and the order of growth of the entire solutions of linear difference equations.

1. Introduction and Results

In the present paper, we use the main notions of Nevanlinna's theory (see [8, 12, 13]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of the roots of $f(z)$.

In recent years, numerous results are rapidly obtained for complex differences and difference equations (see [1, 10, 9, 3, 2, 5, 7]). Chiang and Feng [7] studied the growth of meromorphic solutions of homogeneous linear difference equations. In the case where there exists only one coefficient with the maximal order, they obtained the following result:

Theorem A. *Let $A_0(z), \ldots, A_n(z)$ be entire functions for which there exists an integer $l,\ 0 \le l \le n$, such that*

$$\sigma(A_l) > \max_{\substack{1 \le j \le n \\ j \ne l}} \{\sigma(A_j)\}.$$

If $f(z)$ is a meromorphic solution to

$$A_n(z)y(z+n) + \ldots + A_1(z)y(z+1) + A_0(z)y(z) = 0,$$

then $\sigma(f) \ge \sigma(A_l) + 1$.

Laine and Yang [11] showed that if the leading coefficient depends on the type but not on the order, Theorem A remains true. Their result can be formulated as follows:

Theorem B. *Let $A_0(z), \ldots, A_n(z)$ be entire functions of finite order such that, among the coefficients of the maximal order $\sigma = \max\{\sigma(A_k), 0 \le k \le n\}$, the type of exactly one coefficient is strictly greater than the other types. If $f(z) \not\equiv 0$ is a meromorphic solution of the equation*

$$A_n(z)f(z+\omega_n) + \ldots + A_1(z)f(z+\omega_1) + A_0(z)f(z) = 0, \tag{1.1}$$

then $\sigma(f) \ge \sigma + 1$.

Laine and Yang [11] raised the following question.

[1] Department of Mathematics, Guangzhou Civil Aviation College, Guangzhou, China.
[2] School of Mathematical Sciences, South China Normal University, Guangzhou, China.

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Question: Is it true that all meromorphic solutions $f(z)(\not\equiv 0)$ of equation (1.1) satisfy the inequality

$$\sigma(f) \geq 1 + \max_{0 \leq j \leq n} \sigma(A_j)$$

if there is no leading coefficient?

We impose certain restrictions on the coefficients of the difference equation to give an answer to the posed question and obtain the following results:

Theorem 1.1. *Let $c_j,\ j = 1, \ldots, n$, be different constants and let*

$$A_j(z) = P_j(z)e^{h_j(z)} + Q_j(z), \quad j = 1, \ldots, n,$$

where $h_j(z)$ are polynomials of degree $k \geq 1$ and $P_j(z)(\not\equiv 0)$ and $Q_j(z)$ are entire functions whose order is lower than k. Among the leading coefficients of $h_j(z),\ j \in \{1, \ldots, n\}$, with the maximal modulus, there exists a term unequal to the other terms. If $f(z)(\not\equiv 0)$ is a meromorphic solution of equation

$$A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) = 0, \tag{1.2}$$

then $\sigma(f) \geq k + 1$.

Corollary 1.1. *Let k and $A_j(z),\ j = 1, \ldots, n$, be defined as in Theorem 1.1, let $B_i(z),\ i = 1, \ldots, m$, be entire functions whose order is lower than k, and let $c_j,\ j = 1, \ldots, n + m$, be different constants. If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of the equation*

$$B_m(z)f(z + c_{n+m}) + \ldots + B_1(z)f(z + c_{n+1}) + A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) = 0, \tag{1.3}$$

then $\sigma(f) \geq k + 1$.

Example 1.1. The function

$$f(z) = e^{z^2}$$

satisfies the difference equation

$$e^{-2iz}f(z + i) + e^{2iz}f(z - i) - 2e^{-1}f(z) = 0.$$

Clearly,

$$\sigma(f) = 2 = \deg h_1 + 1 = \deg h_2 + 1.$$

This example shows that the equality in Corollary 1.1 can be attained. Hence, the estimate in Corollary 1.1 is sharp.

By using Theorems A, B, and 1.1, we deduce the following corollary:

Corollary 1.2. *Let $c_j,\ j = 1, 2$, be different nonzero constants, let $h_j(z),\ j = 1, 2$, be polynomials, and let $A_j(z)$ $(\not\equiv 0),\ j = 0, 1, 2$, be entire functions such that*

$$\max\{\sigma(A_j), 0 \leq j \leq 2\} < \max\{\deg h_1, \deg h_2\}.$$

If $f(z)$ $(\not\equiv 0)$ *is a meromorphic solution of the equation*

$$A_2(z)e^{h_2(z)}f(z+c_2)+A_1(z)e^{h_1(z)}f(z+c_1)+A_0(z)f(z)=0, \tag{1.4}$$

then

$$\sigma(f)\geq \max\{\deg h_1, \deg h_2\}+1.$$

Chen [6] studied complex oscillation problems for the entire solutions $f(z)$ of homogeneous and inhomogeneous linear difference equations respectively, and obtained certain relations between $\lambda(f)$ and $\sigma(f)$. These results can be formulated as follows:

Theorem C. *Let* $A_j(z)$, $j=1,\dots,n$, *be entire functions such that there exists at least one transcendental* A_j *and let* c_j, $j=1,\dots,n$, *be constants unequal to each other. Suppose that* $f(z)$ *is a finite-order transcendental entire solution of the homogeneous linear difference equation (1.2) satisfying the inequality*

$$\sigma(f)>\max\{\sigma(A_j)\colon 1\leq j\leq n\}+1.$$

Then $\lambda(f)\geq\sigma(f)-1$. *Moreover, if* $n=2$, *then* $\lambda(f)=\sigma(f)$.

Theorem D. *Let* $F(z)$, $A_j(z)$, $j=1,\dots,n$, *be entire functions such that* $F(z)A_n(z)\not\equiv 0$ *and let* c_k, $k=1,\dots,n$, *be constants unequal to each other. Suppose that* $f(z)$ *is a finite-order entire solution of the nonhomogeneous linear difference equation*

$$A_n(z)f(z+c_n)+\ldots+A_1(z)f(z+c_1)=F(z).$$

If

$$\sigma(f)>\max\{\sigma(F),\sigma(A_j)\colon 1\leq j\leq n\},$$

then $\lambda(f)=\sigma(f)$.

In what follows, we continue to study complex oscillation problems for the entire solutions of linear difference equations (1.2) and (1.4), and obtain the following results, extending Theorems C and D:

Theorem 1.2. *Let* c_j, $j=1,\dots,n$, *be different constants and let* $A_j(z)$ $(\not\equiv 0)$, $j=1,\dots,n$, *be entire functions of finite order. Suppose that* $f(z)$ *is a finite-order entire solution of equation (1.2) such that*

$$\sigma(f)>\max\{\sigma(A_j)\colon 1\leq j\leq n\}+1.$$

Then $f(z)$ *takes every finite value* d *infinitely often and* $\lambda(f-d)=\sigma(f)$.

Example 1.2. The entire function $f(z)=e^{z^2}$ satisfies the linear difference equation

$$f(z+1)-e^{2z+1}f(z)=0.$$

Obviously, $A_2(z)\equiv 1$ and $A_1(z)=-e^{2z+1}$. We see that

$$\sigma(f)=2=\max\{\sigma(A_1),\sigma(A_2)\}+1$$

but $\lambda(f) = 0 < \sigma(f)$. This example shows that the condition in Theorem 1.2, i.e.,

$$\sigma(f) > \max\{\sigma(A_j)\colon 1 \le j \le n\} + 1,$$

cannot be weakened.

By Theorems D and 1.2, we get the following corollary.

Corollary 1.3. *Under the conditions of Theorem 1.2, for any small entire function* $\varphi(z)$ $(\not\equiv 0)$ *satisfying* $\sigma(\varphi) < \sigma(f)$, *we have* $\lambda(f - \varphi) = \sigma(f)$.

Corollary 1.4. *Let* $h_1(z)$ *and* $h_2(z)$ *be polynomials such that*

$$h_1(z) = a_n z^n + \ldots + a_0 \qquad \text{and} \qquad h_2(z) = b_m z^m + \ldots + b_0,$$

where $a_n b_m \neq 0$, *let* $A_j(z)$ $(\not\equiv 0)$, $j = 0, 1, 2$, *be entire functions whose order is lower than* $\max\{n, m\}$, *and let* c_k, $k = 1, 2$, *be different nonzero constants such that* $c_2 a_n - c_1 b_m \neq 0$, *while* $n = m$. *If* $f(z)$ $(\not\equiv 0)$ *is a finite-order entire solution of (1.4), then*

$$\lambda(f) = \sigma(f) \ge \max\{n, m\} + 1.$$

Example 1.1 shows that the condition $c_2 a_n - c_1 b_m \neq 0$ for $n = m$ in Corollary 1.4 cannot be weakened.

2. Proofs of the Theorems and Corollaries

We need the following lemmas to prove the formulated theorems and corollaries:

Lemma 2.1 [4]. *Suppose that* $f(z)$ *is a meromorphic function with* $\sigma(f) = \sigma < \infty$. *Then, for any given* $\varepsilon > 0$, *one can find a set* $E \subset (1, \infty)$ *of finite linear measure or finite logarithmic measure such that*

$$|f(z)| \le \exp\{r^{\sigma+\varepsilon}\}$$

for all z *satisfying the relation* $|z| = r \notin [0, 1] \cup E$ *as* $r \to \infty$.

Lemma 2.2 [7]. *Let* η_1 *and* η_2 *be two arbitrary complex numbers and let* $f(z)$ *be a meromorphic function of finite order* σ. *For given* $\varepsilon > 0$, *there exists a subset* $E \subset (0, \infty)$ *of finite logarithmic measure such that, for all* z *satisfying the relation* $|z| = r \notin E \cup [0, 1]$, *the following inequality is true:*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \le \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}.$$

Proof of Theorem 1.1. Contrary to our assertion, we assume that $\sigma(f) < k + 1$. Let

$$h_j(z) = a_{jk} z^k + h_j^*(z), \tag{2.1}$$

where $a_{jk} \neq 0$ are constants and $h_j^*(z)$ are polynomials with $\deg h_j^* \le k - 1,\ j = 1, \ldots, n$.

We set

$$I=\left\{i\colon\ |a_{ik}|=\max_{1\le j\le n}|a_{jk}|\right\},\qquad \theta_j=\arg a_{jk}\in[0,2\pi),\quad j\in I.$$

There exists $l\in I$ such that $a_{lk}\neq a_{jk}, j\in I\setminus\{l\}$. This fact and the definitions of I and θ_j, enable us to conclude that

$$|a_{jk}|=|a_{lk}|,\qquad \theta_j\neq\theta_l,\quad j\in I\setminus\{l\}.$$

We now choose θ such that

$$\cos(k\theta+\theta_l)=1. \tag{2.2}$$

Thus, by $\theta_j\neq\theta_l,\ j\in I\setminus\{l\}$, we find

$$\cos(k\theta+\theta_j)<1,\quad j\in I\setminus\{l\}. \tag{2.3}$$

Denote

$$a=\max_{1\le j\le n}\{|a_{jk}|\},\quad b=\max_{j\notin I}\{|a_{jk}|\}\quad c=\max\big\{b,a\cos(k\theta+\theta_j),\ j\in I\setminus\{l\}\big\}<a, \tag{2.4}$$

and

$$\sigma=\sigma(f)<k+1,\qquad \beta=\max_{1\le j\le n}\{\sigma(P_j),\sigma(Q_j)\}<k. \tag{2.5}$$

Clearly,

$$\sigma\left(\frac{P_j}{P_l}\right)\le\max\{\sigma(P_j),\sigma(P_l)\}\le\beta,\qquad 1\le j\le n,\quad j\neq l,$$

$$\sigma\left(\frac{Q_j}{P_l}\right)\le\max\{\sigma(Q_j),\sigma(P_l)\}\le\beta,\quad 1\le j\le n.$$

By Lemma 2.1, for any given ε,

$$0<2\varepsilon<\min\{1,k+1-\sigma,k-\beta,a-c\},$$

there is a set $E_1\subset(1,\infty)$ with finite logarithmic measure such that, for all z satisfying $|z|=r\notin E_1\cup[0,1]$, we obtain

$$\left|\frac{P_j(z)}{P_l(z)}\right|\le\exp\{r^{\beta+\varepsilon}\},\quad 1\le j\le n, j\neq l;\qquad \left|\frac{Q_j(z)}{P_l(z)}\right|\le\exp\{r^{\beta+\varepsilon}\},\quad 1\le j\le n. \tag{2.6}$$

It is clear that $\exp\{-h_l^*(z)\}$ is of regular order $\deg h_l^*$ and $\exp\{h_j^*(z)\},\ 1\le j\le n,\ j\neq l$, is of regular order $\deg h_j^*$. Note that $\deg h_j^*\le k-1,\ 1\le j\le n$. Thus, for all large $z,\ |z|=r$, we get

$$|\exp\{-h_l^*(z)\}|\le\exp\{r^{k-1+\varepsilon}\},\qquad \big|\exp\{h_j^*(z)\}\big|\le\exp\{r^{k-1+\varepsilon}\},\quad 1\le j\le n, j\neq l. \tag{2.7}$$

Applying Lemma 2.2 to $f(z)$, we conclude that there is a set $E_2 \subset (1,\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_2 \cup [0,1]$, we can write

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}, \qquad 1 \le j \le n, \quad j \ne l. \tag{2.8}$$

By using (1.2) and (2.1), we obtain

$$\begin{aligned}-\exp\{a_{lk}z^k\} = &\sum_{j\in I\setminus\{l\}} \exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\} + \frac{Q_j(z)}{P_l(z)}\right)\\ &+ \sum_{j\notin I} \exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\} + \frac{Q_j(z)}{P_l(z)}\right)\\ &+ \exp\{-h_l^*(z)\}\frac{Q_l(z)}{P_l(z)}. \end{aligned}\tag{2.9}$$

Let $z = re^{i\theta}$, where $r \notin E_1 \cup E_2 \cup [0,1]$. Substituting (2.2)–(2.4), (2.6)–(2.8) in (2.9), we find

$$\begin{aligned}\exp\{ar^k\} \le &\sum_{j\in I\setminus\{l\}} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\}\Big(\exp\{a\cos(k\theta+\theta_j)r^k + r^{k-1+\varepsilon}\} + 1\Big)\\ &+ \sum_{j\notin I} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\}\\ &\times\Big(\exp\{(b+\varepsilon)r^k + r^{k-1+\varepsilon}\} + 1\Big) + \exp\{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}\}\\ \le\ & n\exp\{(c+\varepsilon)r^k + 2r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\}\\ \le\ & n\exp\{(c+2\varepsilon)r^k\}. \end{aligned}\tag{2.10}$$

Dividing both sides of (2.10) by $\exp\{ar^k\}$ and letting $r \to \infty$, we get $1 \le 0$. A contradiction. Hence, $\sigma(f) \ge k+1$.

Proof of Corollary 1.1. Assume that $\sigma(f) < k+1$. By using the same method as in the proof of Theorem 1.1, we also obtain (2.1)–(2.7).

By Lemma 2.1, there is a set $E_3 \subset (1,\infty)$ with finite logarithmic measure such that, for all z satisfying

$$|z| = r \notin E_3 \cup [0,1],$$

we get

$$|B_j(z)| \le \exp\{r^{\beta_1+\varepsilon}\}, \quad 1 \le j \le m, \tag{2.11}$$

where $\beta_1 = \max\{\sigma(B_j),\ 1 \le j \le m\} < k$.

Applying Lemma 2.2 to $f(z)$, we conclude that there is a set $E_4 \subset (1,\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin E_4 \cup [0,1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}, \qquad 1 \le j \le n+m, \quad j \ne l. \tag{2.12}$$

By virtue of (1.3) and (2.1), we find

$$\begin{aligned} -\exp\{a_{lk}z^k\} = &\sum_{j\in I\setminus\{l\}} \exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\} + \frac{Q_j(z)}{P_l(z)}\right) \\ &+ \sum_{j\notin I} \exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\} + \frac{Q_j(z)}{P_l(z)}\right) \\ &+ \sum_{j=n+1}^{n+m} B_j(z)\frac{f(z+c_j)}{f(z+c_l)} + \exp\{-h_l^*(z)\}\frac{Q_l(z)}{P_l(z)}. \end{aligned} \tag{2.13}$$

Let $z = re^{i\theta}$, where $r \notin E_1 \cup E_2 \cup E_3 \cup E_4 \cup [0,1]$. Substituting (2.2)–(2.7), (2.11) and (2.12) in (2.13), we obtain

$$\begin{aligned} \exp\{ar^k\} \le &\sum_{j\in I\setminus\{l\}} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\}\Big(\exp\{a\cos(k\theta+\theta_j)r^k + r^{k-1+\varepsilon}\} + 1\Big) \\ &+ \sum_{j\notin I} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\}\Big(\exp\{(b+\varepsilon)r^k + r^{k-1+\varepsilon}\} + 1\Big) \\ &+ m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\} + \exp\{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}\} \\ \le &\; n\exp\{(c+\varepsilon)r^k + 2r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\} + m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\} \\ \le &\; n\exp\{(c+2\varepsilon)r^k\} + m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\}. \end{aligned} \tag{2.14}$$

Dividing both sides of (2.14) by $\exp\{ar^k\}$ and letting $r \to \infty$, we conclude that $1 \le 0$. This is a contradiction. Hence, $\sigma(f) \ge k+1$ is true.

Proof of Theorem 1.2. Consider the following two cases:

Case 1: $d = 0$.

Contrary to our assertion, suppose that $\lambda(f) < \sigma(f)$. Then $f(z)$ can be represented as

$$f(z) = H(z)e^{h(z)}, \tag{2.15}$$

where $H(z)(\not\equiv 0)$ is the canonical product (or polynomial) formed by the roots of $f(z)$ such that

$$\lambda(H) = \sigma(H) = \lambda(f) < \sigma(f)$$

and

$$h(z) = a_k z^k + a_{k-1} z^{k-1} + \ldots + a_0; \tag{2.16}$$

here, $k \in \mathbb{N}^+$ satisfies $k = \sigma(f) > \lambda(f)$ and $a_k (\neq 0), a_{k-1}, \ldots, a_0$ are constants.

Substituting (2.15) in (1.2), we obtain

$$A_n(z) H(z + c_n) \exp\{h(z + c_n)\} + \ldots + A_1(z) H(z + c_1) \exp\{h(z + c_1)\} = 0,$$

or

$$\begin{aligned} &A_n(z) \exp\{h(z + c_n) - h(z + c_1)\} H(z + c_n) \\ &\quad \ldots + A_2(z) \exp\{h(z + c_2) - h(z + c_1)\} H(z + c_2) + A_1(z) H(z + c_1) = 0. \end{aligned} \tag{2.17}$$

Since $\sigma(f) > \max\{\sigma(A_j)\colon 1 \le j \le n\} + 1$, we conclude that $\deg h(z) = k \ge 2$. By virtue of (2.16), we get

$$h(z + c_j) - h(z + c_1) = k a_k (c_j - c_1) z^{k-1} + h_j^*(z), \tag{2.18}$$

where $h_j^*(z)$ are polynomials with $\deg h_j^* \le k - 2,\ j = 2, \ldots, n$.

We set

$$I = \left\{ i\colon |c_i - c_1| = \max_{2 \le j \le n} |c_j - c_1| \right\}.$$

In what follows, we consider two cases:

Case 1.1. I contains exactly one term.

Without loss of generality, assume that $I = \{n\}$. By $\sigma(A_j) < \sigma(f) - 1 = k - 1,\ j = 1, \ldots, n,$ and (2.18), we find

$$\sigma\left(A_j \exp\{h(z + c_j) - h(z + c_1)\}\right) = \deg(h(z + c_j) - h(z + c_1)) = k - 1, \quad j = 2, \ldots, n.$$

By the definition of I and $I = \{n\}$, we conclude that, in Eq. (2.17), the type $k\,|a_k(c_n - c_1)|$ of the coefficient $A_n \exp\{h(z + c_n) - h(z + c_1)\}$ is strictly greater than the types $k|a_k(c_j - c_1)|$ of the coefficients $A_j \exp\{h(z + c_j) - h(z + c_1)\},\ j = 2, \ldots, n - 1$. Therefore, by applying Theorem B to equation (2.17), we get

$$\sigma(H) \ge (k - 1) + 1 = k = \sigma(f).$$

Thus, we arrive at a contradiction. Hence, $\lambda(f) = \sigma(f)$.

Case 1.2. I contains more than one term.

Without loss of generality, we can assume that $I = \{s, s + 1, \ldots, n\},\ 2 \le s < n$. We set

$$a_k = |a_k| e^{i\theta_0}, \qquad \theta_j = \arg(c_j - c_1), \quad j = s, \ldots, n.$$

It follows from the definition of I that

$$|c_j - c_1| < |c_n - c_1|, \quad j = 1, \ldots, s-1,$$

$$|c_j - c_1| = |c_n - c_1|, \quad j = s, \ldots, n.$$

Since c_j are different constants, θ_j are also different constants. Thus, we can choose $\theta \in [0, 2\pi)$ such that

$$\cos((k-1)\theta + \theta_0 + \theta_n) = 1. \tag{2.19}$$

By $\theta_j \neq \theta_n,\ j = s, \ldots, n-1,$ and (2.19), we conclude that

$$\cos((k-1)\theta + \theta_0 + \theta_j) < 1, \quad j = s, \ldots, n-1. \tag{2.20}$$

Denote

$$a = |a_k(c_n - c_1)|, \qquad \beta = \max_{1 \le j < s}\{|a_k(c_j - c_1)|\},$$
$$b = \max_{s \le j \le n-1}\{a\cos((k-1)\theta + \theta_0 + \theta_j), \beta\}, \qquad \alpha = \max_{1 \le j \le n}\{\sigma(A_j), \lambda(f) - 1, k-2\}. \tag{2.21}$$

Obviously,

$$\beta < a, \qquad b < a, \qquad \alpha < k-1. \tag{2.22}$$

By Lemma 2.1, for any given $\varepsilon,\ 0 < \varepsilon < \min\{a - b, 1\}$, there exists a set $E_1 \subset (1, \infty)$ of finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we can write

$$\left|\frac{A_j(z)}{A_n(z)}\right| \le \exp\{r^{\alpha+\varepsilon}\}, \quad j = 1, \ldots, n-1. \tag{2.23}$$

It is known that both $\exp\{-h_n^*\}$ and $\exp\{h_j^* - h_n^*\}$ are of regular order $\le k-2 \le \alpha$. Then for large z, $|z| = r$, we obtain

$$|\exp\{-h_n^*\}| \le \exp\{r^{\alpha+\varepsilon}\}, \qquad \left|\exp\{h_j^* - h_n^*\}\right| \le \exp\{r^{\alpha+\varepsilon}\}, \quad j = 2, \ldots, n-1. \tag{2.24}$$

Applying Lemma 2.2 to $H(z)$, we conclude that there exists a set $E_2 \subset (1, \infty)$ of finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we get

$$\left|\frac{H(z + c_j)}{H(z + c_n)}\right| \le \exp\{r^{\alpha+\varepsilon}\}, \quad j = 1, \ldots, n-1. \tag{2.25}$$

By (2.17), we obtain

$$-\exp\{ka_k(c_n - c_1)z^{k-1}\} = \sum_{j=s}^{n-1} \frac{A_j}{A_n}\frac{H(z + c_j)}{H(z + c_n)}\exp\{h_j^* - h_n^*\}\exp\{ka_k(c_j - c_1)z^{k-1}\}$$

$$+\sum_{j=2}^{s-1}\frac{A_j}{A_n}\frac{H(z+c_j)}{H(z+c_n)}\exp\{h_j^*-h_n^*\}\exp\{ka_k(c_j-c_1)z^{k-1}\}$$

$$+\frac{A_1}{A_n}\frac{H(z+c_1)}{H(z+c_n)}\exp\{-h_n^*\}. \tag{2.26}$$

We take $z=re^{i\theta}$, where $r\notin[0,1]\cup E_1\cup E_2$. Substituting (2.19)–(2.25) into (2.26), we find

$$\exp\{kar^{k-1}\}\le(n-2)\exp\{3r^{\alpha+\varepsilon}\}\exp\{kbr^{k-1}\}+\exp\{3r^{\alpha+\varepsilon}\}$$

$$\le(n-1)\exp\{kbr^{k-1}+3r^{\alpha+\varepsilon}\}.$$

Thus,

$$1\le(n-1)\exp\{3r^{\alpha+\varepsilon}+kbr^{k-1}-kar^{k-1}\}.$$

Letting $r\to\infty$, by (2.22), we get $1\le 0$. However, this is impossible. Hence, $\lambda(f)=\sigma(f)$.

Case 2: $d\neq 0$.

We set $g(z)=f(z)-d$. Then

$$f(z)=g(z)+d \tag{2.27}$$

and

$$\sigma(g)=\sigma(f)>\max\{\sigma(A_j)\colon 1\le j\le n\}+1. \tag{2.28}$$

Substituting (2.27) in (1.2), we get

$$A_n(z)g(z+c_n)+\ldots+A_1(z)g(z+c_1)=-d(A_n(z)+\ldots+A_1(z)). \tag{2.29}$$

If $A_n(z)+\ldots+A_1(z)\not\equiv 0$, by virtue of (2.28), (2.29), and Theorem D, we obtain $\lambda(g)=\sigma(g)$, i.e., $\lambda(f-d)=\sigma(f)$.

If $A_n(z)+\ldots+A_1(z)\equiv 0$, then $g(z)$ is an entire solution of the difference equation

$$A_n(z)g(z+c_n)+\ldots+A_1(z)g(z+c_1)=0.$$

In view of (2.28) and Case 1 considered above, we conclude that $\lambda(g)=\sigma(g)$, i.e., $\lambda(f-d)=\sigma(f)$.

The analysis of Cases 1 and 2 demonstrates that $f(z)$ takes every finite value d infinitely many times and $\lambda(f-d)=\sigma(f)$.

Proof of Corollary 1.4. Without loss of generality, we can assume that $n\ge m$. By Corollary 1.2, we know that $\sigma(f)\ge n+1$. If $\sigma(f)>n+1$, then, by Theorem 1.2, $\lambda(f)=\sigma(f)$. Hence, we can assume that $\sigma(f)=n+1$.

Suppose that $\lambda(f)<\sigma(f)$, then $f(z)$ can be represented as

$$f(z)=g(z)e^{h(z)}, \tag{2.30}$$

where $g(z)(\not\equiv 0)$ is the canonical product (or polynomial) formed by the roots of $f(z)$ such that

$$\sigma(g) = \lambda(g) = \lambda(f) < \sigma(f) = n + 1,$$

and

$$h(z) = d_{n+1}z^{n+1} + d_n z^n + \ldots + d_0 \tag{2.31}$$

is a polynomial, where $d_{n+1} \neq 0, d_n \ldots, d_0$ are constants.

Substituting (2.30), (2.31) in (1.4) and dividing by $e^{h(z)}$, we obtain

$$A_2(z)e^{h(z+c_2)-h(z)+h_2(z)}g(z+c_2) + A_1(z)e^{h(z+c_1)-h(z)+h_1(z)}g(z+c_1) + A_0(z)g(z) = 0. \tag{2.32}$$

By virtue of (2.31), we can write

$$\begin{aligned} h(z+c_1) - h(z) + h_1(z) &= ((n+1)c_1 d_{n+1} + a_n)z^n + h_1^*(z), \\ h(z+c_2) - h(z) + h_2(z) &= (n+1)c_2 d_{n+1} z^n + b_m z^m + h_2^*(z), \end{aligned} \tag{2.33}$$

where $h_1^*(z)$ and $h_2^*(z)$ are polynomials of degree not greater than $n-1$.

Consider the following two cases:

Case 1: $n > m$.

By $(n+1)c_2 d_{n+1} \neq 0$, we can write

$$\deg(h(z+c_2) - h(z) + h_2(z)) = n \geq \deg(h(z+c_1) - h(z) + h_1(z)).$$

Combining this with (2.32) and Corollary 1.2, we get $\sigma(g) \geq n+1$. A contradiction. Hence, $\lambda(f) = \sigma(f) = n+1$.

Case 2: $n = m$.

If $(n+1)c_1 d_{n+1} + a_n \neq 0$, then it follows from (2.33) that

$$\deg(h(z+c_1) - h(z) + h_1(z)) = n \geq \deg(h(z+c_2) - h(z) + h_2(z)).$$

Combining this with (2.32) and Corollary 1.2, we conclude that $\sigma(g) \geq n+1 = \sigma(f)$. A contradiction. Therefore, $\lambda(f) = \sigma(f) = n+1$.

If $(n+1)c_1 d_{n+1} + a_n = 0$, then, for $c_1 \neq 0$, we find

$$(n+1)c_2 d_{n+1} + b_m = -\frac{a_n}{c_1}c_2 + b_m = \frac{c_1 b_m - c_2 a_n}{c_1} \neq 0.$$

Hence,

$$\deg(h(z+c_2) - h(z) + h_2(z)) = n > \deg(h(z+c_1) - h(z) + h_1(z)).$$

Together with (2.32) and Corollary 1.2, we have $\sigma(g) \geq n+1$. A contradiction. Thus, $\lambda(f) = \sigma(f) = n+1$.

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