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GENERAL PROXIMAL-POINT ALGORITHM FOR MONOTONE OPERATORS

M. Eslamian[1] **and J. Vahidi**[2]

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We introduce a new general proximal-point algorithm for an infinite family of monotone operators in a real Hilbert space and establish strong convergence of the iterative process to a common null point of the infinite family of monotone operators. Our result generalizes and improves numerous results in the available literature.

1. Introduction

Let H be a real Hilbert space with scalar product $\langle .,. \rangle$ and let $A\colon D(A) \subset H \to H$ be a set-valued operator. Recall that A is called monotone if $\langle u-v, x-y\rangle \geq 0$ for any $[x,u],[y,v] \in G(A)$, where

$$G(A) = \big\{(x,u)\colon x \in D(A), u \in A(x)\big\}.$$

A monotone operator A is called maximal monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. Monotone operators prove to be a key class of objects in the modern Optimization and Analysis (see, e.g., the monographs [1–4] and the references therein). On the other hand, a variety of problems, including convex programming and variational inequalities, can be formulated as the problems of finding zeros of monotone operators. Hence, the problem of finding a solution $z \in H$ of $0 \in Az$ has been investigated by numerous researchers. A popular method used to solve $0 \in Az$ by iterations is the proximal-point algorithm proposed by Rockafellar [5], which is recognized as a powerful and successful algorithm in finding zeros of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal-point algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J^A_{c_n}(x_n + e_n), \tag{1.1}$$

where $J^A_r = (I + rA)^{-1}$ for all $r > 0$ is the resolvent of A and $\{e_n\}$ is a sequence of errors. Rockafellar proved the weak convergence of algorithm (1.1). However, as shown by Güler [6], the proximal-point algorithm is not necessarily strongly convergent. Since that time, numerous authors have conducted worthwhile research aimed at modifying the proximal-point algorithm in order to guarantee its strong convergence (see, e.g., [7–10]). In particular, Xu [11] introduced the following iterative scheme:

$$x_{n+1} = t_n x_0 + (1-t_n) J^A_{r_n} x_n + e_n, \tag{1.2}$$

where x_0 is the starting point and $\{e_n\}$ is the error sequence. For summable $\{e_n\}$, it was proved that $\{x_n\}$ is

[1] Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr and School of Mathematics, Institute for Research in Fundamental Science, Tehran, Iran.
[2] Department of Applied Mathematics, Iran University of Science and Technology, Tehran, Iran.

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strongly convergent if $r_n \to \infty$ and $\{t_n\} \subset (0,1)$ with

$$\lim_{n\to\infty} t_n = 0, \quad \sum_{n=0}^{\infty} t_n = \infty.$$

Boikanyo and Morosanu [12] generalized this algorithm (1.2) with error sequences from l^p for $1 \leq p < 2$. Recently, Xu [13] proposed the following regularization for the proximal-point algorithm:

$$x_{n+1} = J_{r_n}^A\big(t_n x_0 + (1-t_n)x_n + e_n\big) \tag{1.3}$$

which essentially includes the so-called prox-Tikhonov algorithm introduced by Lehdili and Moudafi [14] as a special case. Boikanyo and Morosanu [15] mentioned that the proximal-point algorithm (1.3) is equivalent to algorithm (1.2). These algorithms were further studied and analyzed by many authors (see [16–23]).

In the present paper, we introduce a general proximal-point algorithm aimed at finding a common null point for an infinite family of monotone operators. We establish the strong convergence of the iterative process to a common zero of the family of monotone operators. Our result generalizes some results of Xu [11], Tian and Song [17], Boikanyo and Morosanu [16], Yao and Noor [23], and many other researchers.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ weakly converges to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ strongly converges to x. Let K be a nonempty, closed, and convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in K denoted by $P_K x$ and such that

$$\|x - P_K x\| \leq \|x - y\| \quad \forall y \in K.$$

The operator P_K is called the metric projection of H onto K. We also know that, for $x \in H$ and $z \in K$, $z = P_K x$ if and only if

$$\langle x - z, y - z \rangle \leq 0 \quad \forall y \in K.$$

It is known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. In what follows, we use the following notions on $S\colon K \to H$.

(i) S is nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in K.$$

(ii) S is firmly nonexpansive if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 - \big\|(x - Sx) - (y - Sy)\big\|^2 \quad \forall x, y \in K.$$

It is well known that P_K is a nonexpansive mapping.

The resolvent operator has the following properties:

Lemma 2.1 [1]. *For a $\lambda > 0$,*

(i) A is monotone if and only if the resolvent J_λ^A of A is single-valued and firmly nonexpansive;

(ii) A is maximal monotone if and only if J_λ^A of A is single-valued and firmly nonexpansive and its domain is the entire H;

(iii) $0 \in A(x^\star) \iff x^\star \in \mathrm{Fix}(J_\lambda^A)$, where $\mathrm{Fix}(J_\lambda^A)$ denotes the fixed-point set of J_λ^A.

Since the fixed-point set of a nonexpansive operator is closed and convex, the projection onto the solution set $Z = A^{-1}(0) = \{x \in D(A) \colon 0 \in Ax\}$ is well defined whenever $Z \neq \varnothing$. For more details, see [24].

Lemma 2.2 [1] (The resolvent identity). *For $\lambda, \mu > 0$, the following identity holds:*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right), \quad x \in H.$$

Let B be a strongly positive bounded linear operator on H, i.e., there is a constant $\overline{\gamma} > 0$ such that

$$\langle Bx, x \rangle \geq \overline{\gamma}\, \|x\|^2 \quad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping S :

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - \langle x, b \rangle.$$

Marino and Xu [25] introduced the following iterative process for finding a fixed point of a nonexpansive mapping based on the viscosity approximation method introduced by Moudafi [26]:

$$x_{n+1} = a_n \gamma f(x_n) + (I - a_n B) S x_n \quad \forall n \geq 0. \tag{2.1}$$

They proved that, under some appropriate condition imposed on the parameters, the sequence $\{x_n\}$ generated by (2.1) strongly converges to the unique solution of the variational inequality

$$\big\langle (B - \gamma f) x^\star, x - x^\star \big\rangle \geq 0 \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where h is a potential function for γf $\big($i.e., $h'(x) = \gamma f(x)\ \forall x \in H\big)$.

Lemma 2.3 [25]. *Assume that B is a strongly positive bounded linear operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then*

$$\|I - \rho B\| \leq 1 - \rho \overline{\gamma}.$$

Lemma 2.4. *The following inequality holds in a Hilbert space H:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5 [27]. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H. Then, for any given*

$$\{\lambda_n\}_{n=1}^{\infty} \subset (0,1) \quad \textit{with} \quad \sum_{n=1}^{\infty} \lambda_n = 1$$

and for any positive integers $i,\ j$ with $i<j$,

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2.$$

Lemma 2.6 [11]. *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n \delta_n + \beta_n, \quad n \geq 0,$$

where $\{\gamma_n\},\ \{\beta_n\}$ and $\{\delta_n\}$ satisfy the conditions:

(i) $\gamma_n \subset [0,1],\ \sum_{n=1}^{\infty} \gamma_n = \infty,$

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty,$

(iii) $\beta_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} \beta_n < \infty.$

Then $\lim_{n\to\infty} \alpha_n = 0.$

Lemma 2.7 [28]. *Let $\{t_n\}$ be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \leq t_{n_i+1}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n\colon\ t_k < t_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, t_n\} \leq t_{\tau(n)+1}.$$

3. Main Result

We now formulate our main result.

Theorem 3.1. *Let $A_i,\ i \in \mathbb{N}$, be an infinite family of monotone operators in a Hilbert space H with $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(\{0\}) \neq \varnothing$. Assume that K is a nonempty closed convex subset of H such that*

$$\bigcap_{i=1}^{\infty} \overline{D(A_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rA_i)$$

for all $r > 0$. Moreover, assume that f is a b-contraction of K into itself and B is a strongly positive bounded linear operator on H with a coefficient $\overline{\gamma}$ and

$$0 < \gamma < \frac{\overline{\gamma}}{b}.$$

Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and

$$y_n = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n, \quad n \geq 0,$$

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n B) y_n \quad \forall n \geq 0,$$

where $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ satisfy the following conditions:

- *(i)* $\{\beta_n\} \subset (0,1),\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty,$
- *(ii)* $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0,$
- *(iii)* $\{\alpha_{n,i}\} \subset (0,1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \in \mathbb{N}$.

Then the sequence $\{x_n\}$ strongly converges to $z \in Z$, which solves the variational inequality;

$$\langle (B - \gamma f)z, x - z \rangle \geq 0 \quad \forall x \in Z.$$

Proof. Since $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(\{0\})$ is closed and convex, we conclude that the projection P_Z is well defined. Since $\lim_{n\to\infty} \beta_n = 0$, we can assume that $\beta_n \in (0, \|B\|^{-1})$ for all $n \geq 0$. Applying Lemma 2.3, we find

$$\|I - \beta_n B\| \leq 1 - \beta_n \overline{\gamma}. \tag{3.1}$$

Further, we show that $\{x_n\}$ is bounded. By Lemma 2.1, the operators $J_{r_n}^{A_i}$ are nonexpansive and, hence, we get

$$\begin{aligned}
\|y_n - z\| &\leq \|\alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n - z\| \\
&\leq \alpha_{n,0}\|x_n - z\| + \sum_{i=1}^{\infty} \alpha_{n,i} \|J_{r_n}^{A_i} x_n - z\| \\
&\leq \alpha_{n,0}\|x_n - z\| + \sum_{i=1}^{\infty} \alpha_{n,i} \|x_n - z\| \leq \|x_n - z\|.
\end{aligned}$$

By using inequality (3.1), we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n(\gamma f(x_n) - Bz) + ((I - \beta_n B)(y_n - z)\| \\
&\leq \beta_n \|\gamma f(x_n) - Bz\| + \|I - \beta_n B\| \|y_n - z\|
\end{aligned}$$

$$\leq \beta_n \gamma \|f(x_n) - f(z)\| + \beta_n \|\gamma f(z) - Bz\| + (1 - \beta_n \overline{\gamma}) \|x_n - z\|$$

$$\leq \beta_n \gamma b \|x_n - z\| + \beta_n \|\gamma f(z) - Bz\| + (1 - \beta_n \overline{\gamma}) \|x_n - z\|$$

$$\leq (1 - \beta_n(\overline{\gamma} - \gamma b)) \|x_n - z\| + \beta_n \|\gamma f(z) - Bz\|$$

$$\leq \max\left\{ \|x_n - z\|, \frac{1}{\overline{\gamma} - \gamma b} \|\gamma f(z) - Bz\| \right\}.$$

By induction, we conclude that

$$\|x_n - z\| \leq \max\left\{ \|x_0 - z\|,\ \frac{1}{\overline{\gamma} - \gamma b} \|\gamma f(z) - Bz\| \right\} \quad \forall n \geq 0.$$

This shows that $\{x_n\}$ is bounded and, hence, the same is true for $\{f(x_n)\}$. Further, we show that, for each $i \in \mathbb{N}$,

$$\lim_{n\to\infty} \left\| x_n - J_{r_n}^{A_i} x_n \right\| = 0.$$

By using Lemma 2.5, for any $i \in \mathbb{N}$ we get

$$\begin{aligned}
\|y_n - z\|^2 &\leq \|\alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n - z\|^2 \\
&\leq \alpha_{n,0} \|x_n - z\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|J_{r_n}^{A_i} x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i} \left\| J_{r_n}^{A_i} x_n - x_n \right\|^2 \\
&\leq \alpha_{n,0} \|x_n - z\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} \|x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i} \left\| J_{r_n}^{A_i} x_n - x_n \right\|^2 \\
&\leq \|x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i} \left\| J_{r_n}^{A_i} x_n - x_n \right\|^2. \qquad (3.2)
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| \beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z) \right\|^2 \\
&\leq \left\| \beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z) \right\|^2 \\
&\leq \beta_n^2 \|\gamma f(x_n) - Bz\|^2 + (1 - \beta_n \overline{\gamma})^2 \|y_n - z\|^2 \\
&\quad + 2\beta_n (1 - \beta_n \overline{\gamma}) \|\gamma f(x_n) - Bz\| \|y_n - z\| \\
&\leq \beta_n^2 \|\gamma f(x_n) - Bz\|^2 + 2\beta_n (1 - \beta_n \overline{\gamma}) \|\gamma f(x_n) - Bz\| \|x_n - z\| \\
&\quad + (1 - \beta_n \overline{\gamma})^2 \|x_n - z\|^2 - (1 - \beta_n \overline{\gamma})^2 \alpha_{n,0}\alpha_{n,i} \|J_{r_n}^{A_i} x_n - x_n\|^2. \qquad (3.3)
\end{aligned}$$

Thus, for every $i \in \mathbb{N}$, we get

$$(1-\beta_n\overline{\gamma})^2\alpha_{n,0}\alpha_{n,i}\left\|J_{r_n}^{A_i}x_n - x_n\right\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\beta_n(1-\beta_n\overline{\gamma})\|\gamma f(x_n) - Bz\|\|x_n - z\| + \beta_n^2\|\gamma f(x_n) - Bz\|^2. \tag{3.4}$$

Note that the Banach contraction-mapping principle guarantees that $P_{\mathcal{Z}}(I - B + \gamma f)$ has a unique fixed point z, which is the unique solution of the variational inequality

$$\big\langle (B-\gamma f)z, x - z\big\rangle \geq 0 \quad \forall x \in Z.$$

We finally analyze inequality (3.4) by considering the following two cases:

Case 1. Assume that $\big\{\|x_n - z\|\big\}$ is a monotone sequence. In other words, for sufficiently large n_0, $\{\|x_n - z\|\}_{n\geq n_0}$ is either nondecreasing or nonincreasing. Since $\|x_n - z\|$ is bounded, we conclude that $\|x_n - z\|$ is convergent. Moreover, since $\lim_{n\to\infty}\beta_n = 0$, and $\{f(x_n)\}$ and $\{x_n\}$ are bounded, it follows from (3.4) that

$$\lim_{n\to\infty}(1-\beta_n\overline{\gamma})^2\alpha_{n,0}\alpha_{n,i}\left\|J_{r_n}^{A_i}x_n - x_n\right\|^2 = 0.$$

This implies that

$$\lim_{n\to\infty}\left\|J_{r_n}^{A_i}x_n - x_n\right\| = 0.$$

By using the resolvent identity (Lemma 2.2), for any $r > 0$, we conclude that

$$\begin{aligned}\left\|x_n - J_r^{A_i}x_n\right\| &\leq \left\|x_n - J_{r_n}^{A_i}x_n\right\| + \left\|J_{r_n}^{A_i}x_n - J_r^{A_i}x_n\right\| \\ &\leq \left\|x_n - J_{r_n}^{A_i}x_n\right\| + \left\|J_r^{A_i}\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^{A_i}x_n\right) - J_r^{A_i}x_n\right\| \\ &\leq \left\|x_n - J_{r_n}^{A_i}x_n\right\| + \left\|\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^{A_i}x_n - x_n\right\| \\ &\leq \left\|x_n - J_{r_n}^{A_i}x_n\right\| + \left|1-\frac{r}{r_n}\right|\left\|J_{r_n}^{A_i}x_n - x_n\right\| \to 0, \quad n\to\infty.\end{aligned}$$

Further, we show that

$$\limsup_{n\to\infty}\langle (B-\gamma f)z, z - x_n\rangle \leq 0.$$

We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}(\langle (B-\gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty}(\langle (B-\gamma f)z, z - x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ weakly convergent to x^*. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. We now show that $x^* \in Z$. Indeed,

$$\left\|x_{n_i} - J_r^{A_i} x^*\right\| \leq \left\|x_{n_i} - J_r^{A_i} x_{n_i}\right\| + \left\|J_r^{A_i} x_{n_i} - J_r^{A_i} x^*\right\|$$

$$\leq \left\|x_{n_i} - J_r^{A_i} x_{n_i}\right\| + \|x_{n_i} - x^*\|.$$

This yields

$$\limsup_{i\to\infty} \left\|x_{n_i} - J_r^{A_i} x^*\right\| \leq \limsup_{i\to\infty} \|x_{n_i} - x^*\|.$$

By the Opial property of Hilbert space H we obtain $x^* = J_r^{A_i} x^*$, $i \in \mathbb{N}$. Hence $x^* \in Z$. Therefore, it follows from $z = P_Z(I - B + \gamma f)z$ and $x^* \in \Psi$ that

$$\limsup_{n\to\infty} \langle (B - \gamma f)z, z - x_n\rangle = \lim_{i\to\infty} (\langle (B - \gamma f)z, z - x_{n_i}\rangle = (\langle (B - \gamma f)z, z - x^*\rangle \leq 0.$$

Since

$$x_{n+1} - z = \beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z),$$

by using Lemma 2.4 and the inequality (3.2), we find

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|(I - \beta_n B)(y_n - z)\|^2 + 2\beta_n\langle \gamma f(x_n) - Bz, x_{n+1} - z\rangle \\
&\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 \\
&\quad + 2\beta_n\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 + 2\beta_n b\gamma\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 + \beta_n b\gamma(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\beta_n\langle \gamma f z - Bz, x_{n+1} - z\rangle \\
&\leq \big((1 - \beta_n\overline{\gamma})^2 + \beta_n b\gamma\big)\|x_n - z\|^2 + \beta_n\gamma b\|x_{n+1} - z\|^2 \\
&\quad + 2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle.
\end{aligned}$$

This means that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \frac{1 - 2\beta_n\overline{\gamma} + (\beta_n\overline{\gamma})^2 + \beta_n\gamma b}{1 - \beta_n\gamma b}\|x_n - z\|^2 \\
&\quad + \frac{2\beta_n}{1 - \eta_n\gamma b}\langle \gamma f z - Bz, x_{n+1} - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n \gamma b}\right)\|x_n - z\|^2 \\
&\quad + \frac{(\beta_n \overline{\gamma})^2}{1 - \eta_n \gamma b}\|x_n - z\|^2 + \frac{2\beta_n}{1 - \beta_n \gamma b}\langle \gamma f z - Bz, x_{n+1} - z\rangle \\
&\leq \left(1 - \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n \gamma b}\right)\|x_n - z\|^2 \\
&\quad + \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n \gamma b}\left(\frac{(\beta_n \overline{\gamma}^2)P}{2(\overline{\gamma} - \gamma b)} + \frac{1}{\overline{\gamma} - \gamma b}\right)\langle \gamma f z - Bz, x_{n+1} - z\rangle \\
&= (1 - \gamma_n)\|x_n - z\|^2 + \gamma_n \delta_n,
\end{aligned}
$$

where

$$P = \sup\{\|x_n - z\|^2 :\, n \geq 0\}, \qquad \gamma_n = \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n \gamma b},$$

and

$$\delta_n = \frac{(\beta_n \overline{\gamma}^2)P}{2(\overline{\gamma} - \gamma b)} + \frac{1}{\overline{\gamma} - \gamma b}\langle \gamma f z - Bz, x_{n+1} - z\rangle.$$

It is easy to see that

$$\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{as} \quad \gamma_n \to 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \delta_n \leq 0.$$

Thus, by applying Lemma 2.6, we conclude that the sequence $\{x_n\}$ strongly converges to z.

Case 2. Assume that $\{\|x_n - z\|\}$ is not a monotone sequence. Then we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some sufficiently large n_0) as follows:

$$\tau(n) = \max\{k \in \mathbb{N}; k \leq n :\, \|x_k - z\| < \|x_{k+1} - z\|\}.$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and, for all $n \geq n_0$,

$$\|x_{\tau(n)} - z\| < \|x_{\tau(n)+1} - z\|.$$

Thus, it follows from (3.3) that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 - \|x_n - z\|^2 &\leq \beta_n^2 \|\gamma \mathfrak{f}(x_n) - Bz\|^2 + ((\beta_n \overline{\gamma})^2 - 2\beta_n \overline{\gamma})\|x_n - z\|^2 \\
&\quad + 2\beta_n(1 - \beta_n \overline{\gamma})\|\gamma \mathfrak{f}(x_n) - Bz\|\|x_n - z\|.
\end{aligned}
$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\{f(x_n)\}$ and $\{x_n\}$ are bounded, we obtain

$$\lim_{n\to\infty} \left(\|x_{\tau(n)+1} - z\|^2 - \|x_{\tau(n)} - z\|^2\right) = 0. \tag{3.5}$$

By using the same argument as in Case 1, we find

$$\lim_{n\to\infty}\left\|J_{r_n}^{A_i}x_{\tau(n)}-x_{\tau(n)}\right\|=0$$

and

$$\|x_{\tau(n)+1}-z\|^2\le(1-\gamma_{\tau(n)})\|x_{\tau(n)}-z\|^2+\gamma_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty}\delta_{\tau(n)}\le 0$. Since $\|x_{\tau(n)}-z\|\le\|x_{\tau(n)+1}-z\|$, we have

$$\gamma_{\tau(n)}\|x_{\tau(n)}-z\|^2\le\gamma_{\tau(n)}\delta_{\tau(n)}.$$

Further, since $\gamma_{\tau(n)}>0$ we deduce

$$\|x_{\tau(n)}-z\|^2\le\delta_{\tau(n)}.$$

It follows from $\limsup_{n\to\infty}\delta_{\tau(n)}\le 0$ that $\lim_{n\to\infty}\|x_{\tau(n)}-z\|=0$. Together with (3.5), this implies that $\lim_{n\to\infty}\|x_{\tau(n)+1}-z\|=0$. Thus, by Lemma 2.7, we conclude that

$$0\le\|x_n-z\|\le\max\left\{\|x_{\tau(n)}-z\|,\|x_n-z\|\right\}\le\|x_{\tau(n)+1}-z\|.$$

Therefore, $\{x_n\}$ strongly converges to $z=P_{\mathcal{Z}}(I-B+\gamma f)z$.

Theorem 3.1 is proved.

Theorem 3.2. *Let $A_i,\ i\in\mathbb{N}$, be an infinite family of maximal monotone operators in a real Hilbert space H with $Z=\bigcap_{i=1}^{\infty}A_i^{-1}(\{0\})\neq\varnothing$. Assume that f is a b-contraction of H into itself and A is a strongly positive bounded linear operator on H with a coefficient $\overline{\gamma}$ and*

$$0<\gamma<\frac{\overline{\gamma}}{b}.$$

Let $\{x_n\}$ be a sequence generated by $x_0\in H$ and

$$y_n=\alpha_{n,0}x_n+\sum_{i=1}^{\infty}\alpha_{n,i}J_{r_n}^{A_i}x_n,\quad n\ge 0,$$

$$x_{n+1}=\beta_n\gamma f(x_n)+(I-\beta_nB)y_n\quad\forall n\ge 0,$$

where

$$\sum_{i=0}^{\infty}\alpha_{n,i}=1$$

and $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\beta_n\}\subset(0,1)$, $\lim_{n\to\infty}\beta_n=0$, *and* $\sum_{n=1}^{\infty}\beta_n=\infty$;

(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$;

(iii) $\{\alpha_{n,i}\} \subset (0, 1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \in \mathbb{N}$.

Then the sequence $\{x_n\}$ *strongly converges to* $z \in Z$, *which solves the variational inequality*

$$\langle (B - \gamma f)z, x - z\rangle \geq 0 \quad \forall x \in Z.$$

Proof. Since A_i are maximal monotone operators, we conclude that A_i are monotone and satisfy the condition $R(I + rA_i) = H$ for all $r > 0$. Setting $K = H$ in Theorem 3.1, we obtain the desired result.

Further, setting $B = I$ and $\gamma = 1$ in Theorem 3.1, for a finite family of monotone operators, we immediately arrive at the following result:

Corollary 3.1. *Let* $A_i,\ i = 1, 2, \ldots, m$, *be a finite family of monotone operators in a Hilbert space* H *with*

$$Z = \bigcap_{i=1}^{m} A_i^{-1}(\{0\}) \neq \varnothing.$$

Suppose that K *is a nonempty closed convex subset of* H *such that*

$$\bigcap_{i=1}^{m} \overline{D(A_i)} \subset K \subset \bigcap_{i=1}^{m} R(I + rA_i)$$

for all $r > 0$. *Assume that* f *is a* b*-contraction of* K *into itself. Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in H$ *and*

$$y_n = \alpha_{n,0}x_n + \sum_{i=1}^{m} \alpha_{n,i} J_{r_n}^{A_i} x_n, \quad n \geq 0,$$

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n \quad \forall n \geq 0,$$

where

$$\sum_{i=0}^{m} \alpha_{n,i} = 1$$

and $\{\alpha_{n,i}\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$,

(iii) $\{\alpha_{n,i}\} \subset (0, 1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for* $i = 1, 2, \ldots, m$.

Then the sequence $\{x_n\}$ *strongly converges to* $z \in Z$, *which solves the variational inequality*

$$\langle z - fz, x - z\rangle \geq 0 \quad \forall x \in Z.$$

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