DOI 10.1007/s11253-016-1278-2
*Ukrainian Mathematical Journal, Vol. 68, No. 7, December, 2016 (Ukrainian Original Vol. 68, No. 7, July, 2016)*

# ASYMPTOTIC BEHAVIOR OF THE EXTREME VALUES OF RANDOM VARIABLES. DISCRETE CASE

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UDC 519.21

We study the exact asymptotics of almost surely extreme values of discrete random variables.

## 1. Introduction

Consider a sequence $\xi, \xi_1, \xi_2, \ldots$ of independent identically distributed random variables with distribution function $F(t) = \mathbf{P}(\xi < t)$. Let

$$z_n = \max_{1 \le i \le n} \xi_i.$$

Starting from the classical work [1], the asymptotic behavior of $z_n$ is fairly completely investigated (see, e.g., [2, 3, 4, 5, 6, 7, 8]). Thus, necessary and sufficient conditions for the asymptotic stability in probability

$$\exists (\alpha_n) \quad z_n - \alpha_n \xrightarrow{P} 0, \quad n \to \infty, \tag{1}$$

were established as early as in [1].

Let $\xi$ be a discrete random variable with distribution $(k, p_k),\ k \ge 1$. In what follows, we assume that

$$\mathbf{P}(\xi = k) = p_k > 0, \qquad \sum_{k=1}^{\infty} p_k = 1,$$

$$R(k) = -\ln\big(1 - F(k)\big) = -\ln\left(\sum_{i \ge k} p_i\right), \qquad r(k) = R(k) - R(k-1).$$

For discrete random variables, Anderson [9] obtained an interesting correction of (1): For the existence of a sequence of integers $(k_n)$ such that

$$\lim_{n \to \infty} \mathbf{P}\Big(\{z_n = k_n\} \bigcup \{z_n = k_n + 1\}\Big) = 1, \tag{2}$$

it is necessary and sufficient that

$$r(k) \to \infty, \quad k \to \infty. \tag{3}$$

As far as the asymptotics of $z_n$ almost surely is concerned, we do not know any other works except [10] dealing with the investigation of the discrete case. In [10], the following theorem was proved:

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 68, No. 7, pp. 945–956, July, 2016. Original article submitted August 5, 2015; revision submitted April 27, 2016.

 

**Theorem A.** *Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of discrete independent identically distributed random variables with distribution $(k, p_k),\ k \geq 1,$ let the function $r(k)$ specified above monotonically increase, and let*

$$\sum_{k\geq 1} r(k+1)\exp(-r(k)) < \infty. \tag{4}$$

*Then*

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = 1, \tag{5}$$

*where*

$$A_n = \{z_n = a_n - 1\}\bigcup\{z_n = a_n\}\bigcup\{z_n = a_n + 1\},$$

$$a_n = \max\left(k\colon\ \sum_{i\geq k} p_i \geq \frac{1}{n}\right).$$

In this theorem, it is not clear to what extent condition (4) is close to the condition necessary for the validity of equality (5).

Note that the geometric and Poisson distributions do not satisfy condition (4). However, the asymptotic behavior of $z_n$ for the geometric distribution is known [10], whereas the same problem for the Poisson distribution remains open.

Another natural question arises from the result established in [9]: Is it possible to replace the events $A_n$ in equality (5) by $\tilde{A}_n = \{z_n = k_n\}\bigcup\{z_n = k_n + 1\}$?

In the present paper, we try to give answers to these questions.

## 2. Main Results

**Theorem 1.** *Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of discrete independent identically distributed random variables with distribution $(k, p_k),\ k \geq 1,$ let $r(k)$ be a monotone function, and let condition (3) be satisfied. The equality*

$$\mathbf{P}(z_n > a_n + 1 \ \textit{infinitely often}) = 0 \tag{6}$$

*is true if and only if*

$$\sum_{k\geq 1}\exp(-r(k)) < \infty. \tag{7}$$

*Moreover, if (7) is true, then*

$$\mathbf{P}(z_n = a_n + 1 \ \textit{infinitely often}) = 1. \tag{8}$$

**Theorem 2.** *Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of discrete independent identically distributed random variables with distribution $(k, p_k),\ k \geq 1,$ let $r(k)$ be a monotone function, and let condition (3) be satisfied. The equality*

$$\mathbf{P}(z_n < a_n - 1 \ \textit{infinitely often}) = 0 \tag{9}$$

*is true if and only if*

$$\sum_{k\geq 1}\exp(-e^{r(k)}) < \infty. \tag{10}$$

*Moreover, if (10) is true, then*

$$\mathbf{P}(z_n = a_n - 1 \ \textit{infinitely often}) = 1. \tag{11}$$

These theorems enable us to somewhat weaken condition (4) in Theorem A.

***Corollary 1.*** *If, under the conditions of Theorem 1, equality (7) holds, then relations (5), (8), and (11) are true.*

***Remark 1.*** It seems likely that, under the conditions of Corollary 1,

$$\mathbf{P}(z_n = a_n \ \text{infinitely often}) = 1.$$

In what follows, we establish even stronger assertions but under severer conditions.

It follows from Corollary 1 that the events $A_n$ in equality (5) cannot be replaced by

$$A_n' = \{z_n = a_n\} \bigcup \{z_n = a_n - 1\} \quad \text{or} \quad A_n'' = \{z_n = a_n\} \bigcup \{z_n = a_n + 1\}.$$

Nevertheless, it is possible to prove a certain analog of the result from [9], which is true almost surely.

We fix certain $\alpha$,

$$0 < \alpha < \frac{1}{2},$$

and consider the following subsets of natural numbers:

$$J_\alpha = \bigl\{n \ge 1 \ \exists k \ge 1\colon \ R(k) + \alpha r(k+1) \le \ln n < R(k) + (1-\alpha) r(k+1)\bigr\},$$

$$J_\alpha^- = \bigl\{n \ge 1 \ \exists k \ge 1\colon \ R(k) \le \ln n < R(k) + \alpha r(k+1)\bigr\}.$$

We set

$$\kappa_n = \begin{cases} a_n - 1 & \text{for} \quad n \in J_\alpha^-, \\ a_n, & \text{otherwise.} \end{cases}$$

**Theorem 3.** *Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of discrete independent identically distributed random variables with distribution $(k, p_k)$, $k \ge 1$, let $r(k)$ be a monotone function, and let condition (4) be satisfied. Then the equalities*

$$\mathbf{P}\Bigl(\exists n_\alpha \ \forall n > n_\alpha\colon \ \{z_n = \kappa_n\} \bigcup \{z_n = \kappa_n + 1\}\Bigr) = 1, \tag{12}$$

$$\mathbf{P}\bigl(\exists n_\alpha \ \forall n > n_\alpha, n \in J_\alpha\colon \ z_n = a_n\bigr) = 1 \tag{13}$$

*are true.*

It follows from the above-mentioned theorems that the asymptotic behavior of $z_n$ in the discrete case under conditions of the form (7) strongly differs from its behavior in the continuous case (see, e.g., [4, 8]).

If $R(k)$ increases slower than a linear function, then the situation changes. Thus, in [10], it is shown that the asymptotics of $z_n$ for the geometric distribution and the corresponding exponential distribution are equivalent. The following theorem generalizes this result to a certain class of discrete random variables for which the variations of $r(k)$ are regular:

**Theorem 4.** *Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of discrete independent identically distributed random variables with distribution $(k, p_k),\ k \geq 1,$ let $R(t)$ be a differentiable function, and let*

$$R'(t) = \tilde{r}(t) = t^b L(t), \quad -1 < b \leq 0,$$

*where $L(t)$ is a slowly varying function as $t \to \infty$. Moreover, let $|L(t)| \leq C$ for $b = 0$.*

*Then*

$$\limsup_{n\to\infty} \frac{\tilde{r}(a_n)(z_n - a_n)}{\ln\ln n} = 1, \tag{14}$$

$$\liminf_{n\to\infty} \frac{\tilde{r}(a_n)(z_n - a_n)}{\ln\ln\ln n} = -1 \tag{15}$$

*almost surely, where $a_n$ is given in Theorem A.*

## 3. Proof of Main Results

We first present several important statements.

**Lemma 1.** *Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables with distribution function $F(t)$ and let $(u_n)$ be a nondecreasing sequence of real numbers. Then the probability*

$$\mathbf{P}\left(z_n \geq u_n \ \textit{infinitely often}\right)$$

*is equal to 0 or 1 depending on whether the series*

$$\sum_{n=1}^{\infty} \big(1 - F(u_n)\big) \tag{16}$$

*converges or diverges.*

In fact, Lemma 1 is reduced to the Borel–Cantelli lemma (see [5, p. 190]).

**Lemma 2.** *Let $\xi,\ \xi_1,\ \xi_2,\ \ldots$ be a sequence of independent identically distributed random variables and let $(u_n)$ be a nondecreasing sequence of real numbers such that*

$$\mathbf{P}(\xi > u_n) \to 0, \qquad n\mathbf{P}(\xi > u_n) \to \infty$$

*as $n \to \infty$. Then the probability*

$$\mathbf{P}\big(z_n \leq u_n \ \textit{infinitely often}\big)$$

*is equal to 0 or 1 depending on whether the series*

$$\sum_{n=1}^{\infty} \mathbf{P}(\xi > u_n) \exp\big(-n\mathbf{P}(\xi > u_n)\big) \tag{17}$$

*converges or diverges.*

*In addition, if* $\mathbf{P}(\xi > u_n) \to c > 0$, *then* $\mathbf{P}\big(z_n \le u_n \text{ infinitely often}\big) = 0$ *and if*

$$\liminf_{n\to\infty} n\mathbf{P}(\xi > u_n) < \infty,$$

*then*

$$\mathbf{P}\big(z_n \le u_n \text{ infinitely often}\big) = 1.$$

This statement was established in [6, 7] (see also [5, pp. 190, 191]).

**Lemma 3** [11]. *Assume that* $H(x)$ *regularly varies as* $x \to \infty$, $c_n \to \infty$, *and* $d_n \to \infty$ *and that* $c_n/d_n \to 1$ *as* $n \to \infty$.

*Then*

$$\frac{H(c_n)}{H(d_n)} \to 1.$$

Note that a more general result was proved in [11].

***Proof of Theorem 1.*** Let $m$ be a fixed integer, $m \ge 1$. The definition of $a_n$ yields the following implication:

$$\{a_n = k\} \Leftrightarrow \{\exp(R(k)) \le n < \exp(R(k+1))\}.$$

Denote

$$I_k = \big\{n \ge 1\colon\ \exp(R(k)) \le n < \exp(R(k+1))\big\}, \quad k = 1, 2, \dots.$$

By using condition (3), we obtain

$$\begin{aligned}\sum_{n\ge1} \mathbf{P}(\xi \ge a_n + m) &= \sum_{k\ge1} \mathbf{P}(\xi \ge k + m) \sum_{n\in I_k} 1 \\ &= \sum_{k\ge1} \mathbf{P}(\xi \ge k+m) \exp(R(k+1))(1 + o(1)) \\ &= \sum_{k\ge1} \exp(-R(k+m) + R(k+1))(1 + o(1)).\end{aligned} \tag{18}$$

Here and in what follows, we assume that

$$\sum_{n\in I_k} f(n) = 0$$

for $I_k = \varnothing$.

It is clear that, for $m = 1,$ series (18) is divergent and, hence, by Lemma 1,

$$\mathbf{P}(z_n \geq a_n + 1 \text{ infinitely often}) = 1. \tag{19}$$

Assume that condition (7) is satisfied. We choose $m = 2.$ Then series (18) can be rewritten in the form

$$\sum_{k \geq 1} \exp(-r(k+2))(1 + o(1)). \tag{20}$$

It is clear that the convergence of the last series is equivalent to condition (7). By using Lemma 1 once again, we get

$$\mathbf{P}(z_n \geq a_n + 2 \text{ infinitely often}) = 0. \tag{21}$$

Since the quantities $z_n$ and $a_n$ take integer values, equalities (19) and (21) (taken together) are equivalent to equalities (6) and (8).

If condition (7) is not satisfied, then series (20) and, hence, (18) are divergent. By Lemma 1, this yields the relation

$$\mathbf{P}(z_n \geq a_n + 2 \text{ infinitely often}) = 1,$$

which contradicts equality (6).

Theorem 1 is proved.

***Proof of Theorem 2.*** We act in the same way as above but apply Lemma 2 instead of Lemma 1. We set

$$d_k = \mathbf{P}(\xi > k - 2) = \exp(-R(k-1)), \qquad s_k = \sum_{n \in I_k} \exp(-n d_k),$$

where the set $I_k$ is specified in the proof of Theorem 1. We now estimate series (17) for $u_n = a_n - 2:$

$$\sum_{n \geq 1} \mathbf{P}(\xi > a_n - 2) \exp(-n\mathbf{P}(\xi > a_n - 2))$$
$$= \sum_{k \geq 1} \mathbf{P}(\xi > k - 2) \sum_{n \in I_k} \exp(-n\mathbf{P}(\xi > k - 2)) = \sum_{k \geq 1} d_k s_k. \tag{22}$$

It is known that if $f(t)$ is a nonincreasing function, then

$$\sum_{k=m+1}^{n+1} f(k) \leq \int\limits_{m}^{n+1} f(t)dt \leq \sum_{k=m}^{n} f(k). \tag{23}$$

Assume that condition (10) is satisfied. After elementary calculations, we obtain the following estimate from (23):

$$s_k \leq \int\limits_{\exp(R(k))-1}^{\exp(R(k+1))} \exp(-d_k t)dt \leq \frac{1}{d_k} \exp(-e^{r(k)})(1 + o(1)). \tag{24}$$

Estimates (22) and (24) and condition (10) show that series (17) converges for $u_n = a_n - 2.$

The other conditions of Lemma 2 are also satisfied. Indeed, for $n \in I_k,\ a_n = k,\ k \to \infty$,

$$n\mathbf{P}(\xi > a_n - 2) = n\exp(-R(k-1)) \geq \exp(R(k))\exp(-R(k-1)) = \exp(r(k)) \to \infty.$$

Hence, by Lemma 2,

$$\mathbf{P}(z_n \leq a_n - 2 \ \text{ infinitely often}) = 0, \tag{25}$$

which is equivalent to (9).

We now show that

$$\mathbf{P}(z_n \leq a_n - 1 \ \text{ infinitely often}) = 1. \tag{26}$$

To this end, we define $n \in I_k$ by the equality $n = \big[\exp(R(k))\big] + 1$. Then, as $k \to \infty$, we get

$$n\mathbf{P}(\xi > a_n - 1) = \big([\exp(R(k))] + 1\big)\exp(-R(k)) \to 1.$$

By using Lemma 2 once again, we obtain (26).

Equalities (25) and (26) taken together are equivalent to equalities (9) and (11).

Assume that condition (10) is not satisfied. By using the right inequality in (23), we establish the lower bound

$$s_k \geq \int\limits_{\exp(R(k))+1}^{\exp(R(k+1)-1)} \exp(-d_k t)dt = \frac{1}{d_k}\exp(-e^{r(k)})(1+o(1)).$$

Hence, series (22) is divergent. By Lemma 2, this means that

$$\mathbf{P}(z_n \leq a_n - 2 \ \text{ infinitely often}) = 1,$$

which contradicts (9).

Theorem 2 is proved.

***Proof of Theorem 3.*** For a discrete random variable $\xi$ with distribution $(k, p_k),\ k \geq 1$, we construct a continuous random variable $\xi^c$ “close,” in a certain sense, to $\xi$. Similar structures are known (see, e.g., [10]).

We set

$$R^c(t) = R(k) + (t-k)r(k+1) \quad \text{for} \quad t \in [k, k+1), \quad k \geq 1,$$

$$r^c(t) = r(k+1) \quad \text{for} \quad t \in [k, k+1), \quad k \geq 1.$$

We define a distribution function as follows:

$$\begin{gathered} F^c(t) = 1 - \exp(-R^c(t)), \quad t \geq 1, \\ F^c(1) = 0. \end{gathered} \tag{27}$$

Let $\xi^c$ be a random variable with distribution function $F^c(t),\ \xi^d = [\xi^c]$. Thus, for any $k \geq 1$, we find

$$\mathbf{P}(\xi^d = k) = F^c(k+1) - F^c(k) = \exp(-R(k)) - \exp(-R(k+1)) = p_k,$$

i.e., the random variable $\xi^d$ is identically distributed with $\xi$. Without loss of generality, we can assume that $\xi_i \equiv \xi_i^d$, where $\xi_i^d$ are independent copies of $\xi^d$.

It is known [10] that, under the conditions of Theorem 3,

$$z_n^c - a_n^c \to 0 \quad \text{almost surely}, \tag{28}$$

where $z_n^c$ is constructed on the basis of independent identically distributed random variables with the distribution function $F^c(t)$,

$$a_n^c = \inf(y\colon\ F^c(y) \geq 1 - 1/n).$$

Moreover,,

$$a_n^c = k + \frac{\ln n - R(k)}{r(k+1)} \quad \text{for} \quad \ln n \in [R(k), R(k+1)) \tag{29}$$

and

$$[a_n^c] = \max(k\colon\ \ln n \geq R(k)) = \max\left(k\colon\ \sum_{i \geq k} p_i \geq \frac{1}{n}\right) = a_n. \tag{30}$$

If $n \in J_\alpha,\ \ln n \in [R(k), R(k+1))$, then

$$\alpha \leq \frac{\ln n - R(k)}{r(k+1)} \leq 1 - \alpha$$

and, therefore,

$$k + \alpha \leq a_n^c \leq k + 1 - \alpha. \tag{31}$$

By using relations (28), (30), and (31), we get

$$\exists n_\alpha\, \forall n > n_\alpha, n \in J_\alpha\colon\ z_n = [z_n^c] = [a_n^c] = a_n \quad \text{almost surely},$$

i.e., (13) is proved.

In deducing equality (12), we use similar reasoning. Indeed, it is easy to see that, for $n \in J_\alpha^-,\ \ln n \in [R(k), R(k+1))$,

$$k \leq a_n^c < k + \alpha, \quad [a_n^c] = a_n = k,$$

and, hence,

$$\exists n_\alpha\, \forall n > n_\alpha\colon\ z_n = [z_n^c] = a_n \qquad \text{or} \qquad = a_n - 1 \quad \text{almost surely}.$$

On the other hand, for $n \notin J_\alpha^-,\ \ln n \in [R(k), R(k+1))$,

$$k + \alpha \leq a_n^c < k + 1, \quad [a_n^c] = a_n = k.$$

This yields

$$\exists n_\alpha \ \ \forall n > n_\alpha : \ z_n = [z_n^c] = a_n \qquad \text{or} \qquad = a_n + 1 \quad \text{almost surely.}$$

Hence, equality (12) is true.

Theorem 3 is proved.

***Proof of Theorem 4.*** Under the conditions of the theorem, we consider a continuous random variable $\tilde{\xi}$ with the distribution function

$$\tilde{F}(t) = 1 - \exp(-R(t)), \quad t \geq 1, \qquad \tilde{F}(1) = 0.$$

Since, for any integer $k,$ we have

$$\{\tilde{\xi} < k\} = \big\{[\tilde{\xi}] < k\big\},$$

it is possible to conclude that

$$\mathbf{P}\big([\tilde{\xi}] < k\big) = 1 - \exp(-R(k)) = \mathbf{P}(\xi < k). \tag{32}$$

We set

$$\tilde{a}_n = \inf\big(y : \ \tilde{F}(y) \geq 1 - 1/n\big)$$

and

$$\tilde{z}_n = \max_{1 \leq i \leq n} \tilde{\xi}_i,$$

where $\tilde{\xi}_i$ are independent copies of $\tilde{\xi}$.

According to the results obtained in [8], we conclude that if, under the conditions of Theorem 4, the integral

$$\int_1^\infty \frac{d\tilde{F}(y)}{1 - \tilde{F}(zy)} < \infty \quad \forall z \in (0,1) \tag{33}$$

is bounded, then the following law of iterated logarithm is true for the maximum scheme:

$$\limsup_{n\to\infty} \frac{\tilde{r}(\tilde{a}_n)(\tilde{z}_n - \tilde{a}_n)}{\ln\ln n} = 1, \tag{34}$$

$$\liminf_{n\to\infty} \frac{\tilde{r}(\tilde{a}_n)(\tilde{z}_n - \tilde{a}_n)}{\ln\ln\ln n} = -1 \tag{35}$$

almost surely.

Assume that estimate (34) is true. The definitions of $a_n$ and $\tilde{a}_n$ directly lead to the equality

$$\tilde{a}_n = a_n + \theta, \quad 0 \leq \theta \leq 1. \tag{36}$$

In addition, under the conditions of Theorem 4, we get

$$|\tilde{r}(t)| \leq C \quad \forall t \geq 1. \tag{37}$$

Combining relations (32) and (33)–(37) and using Lemma 3, we arrive at equalities (14) and (15).

It remains to check inequality (33). We have

$$\int_1^\infty \frac{d\tilde{F}(y)}{1-\tilde{F}(zy)} = \int_1^\infty \tilde{r}(t)\exp(-(R(t)-R(zt))dt$$

$$= \int_1^\infty \tilde{r}(t)\exp(-t(1-z)r(\theta t))dt, \tag{38}$$

where $z \le \theta \le 1$.

Since

$$\tilde{r}(\theta t) = (\theta t)^b L(\theta t), \quad -1 < b \le 0,$$

and estimate (37) is true, we conclude that integral (38) is bounded.

Theorem 4 is proved.

## 4. Examples

We now consider several examples of application of the obtained results to some distributions. In what follows, we assume that $(\xi_n)$ is a sequence of independent copies of the random variable $\xi$.

***Example 1.*** Consider a normal random variable $\gamma$ with the distribution function $\Phi(t)$,

$$\Phi(t) = \int_{-\infty}^t \varphi(s)\,ds, \quad \varphi(s) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{s^2}{2}\right).$$

Let

$$\xi = \begin{cases} [\gamma] & \text{for} \quad \gamma \ge 1, \\ 1 & \text{for} \quad \gamma < 1. \end{cases}$$

For this random variable, for large $k$, we get

$$R(k) = \frac{k^2}{2} + \ln k + \ln\sqrt{2\pi} + o(1),$$

$$r(k) = k - \frac{1}{2} + o(1)$$

(see [10]). It is easy to see that the function $r(k)$ satisfies the conditions of Theorems 1–3. Hence, the random variable $\xi$ satisfies equalities (6), (8), (9), and (11)–(13) with $a_n = \left[\sqrt{2\ln n}\right]$.

***Example 2*** (Poisson distribution)**.** Let

$$\mathbf{P}(\xi = k) = p_k = \frac{\lambda^k}{k!}e^{-\lambda}, \quad \lambda \ge 0.$$

It is known [12, p. 38] that, for the Poisson distribution, we have

$$\frac{1}{p_k}\sum_{i>k} p_i \to 0 \quad \text{as} \quad k \to \infty.$$

By using the Stirling formula

$$n! \sim \sqrt{2\pi} n^{n+1/2} \exp(-n),$$

we easily obtain the equalities

$$R(k) = -\lambda + \frac{1}{2}\ln 2\pi - k(\ln\lambda + 1) + \left(k + \frac{1}{2}\right)\ln k + o(1),$$

$$r(k) = \ln k + O(1).$$

It is clear that condition (10) is satisfied. Hence, for the Poisson distribution, equalities (9) and (11) are true. However, neither condition (7), nor (all the more) condition (4) are satisfied.

The following auxiliary statement is useful for the analysis of this important case:

**Lemma 4.** *Let $\xi$ be a discrete random variable with distribution $(k, p_k)$, $k \geq 1$, let $r(k)$ be a monotone function, and let condition (3) be satisfied. If the series*

$$\sum_{k=1}^{\infty} \exp(-jr(k)) \tag{39}$$

*converges for $j = m$ and diverges for $j = m - 1$, then*

$$\mathbf{P}(z_n \geq a_n + m + 1 \ \textit{infinitely often}) = 0,$$

$$\mathbf{P}(z_n \geq a_n + m \ \textit{infinitely often}) = 1.$$

In fact, to prove Lemma 4, it is necessary to repeat the reasoning used in the proof of Theorem 1.

For the Poisson distribution, according to the presented estimates, series (39) is convergent for $j = m = 2$ and divergent for $j = m = 1$. According to Lemma 4, we get

$$\mathbf{P}(z_n > a_n + 2 \ \text{infinitely often}) = 0,$$

$$\mathbf{P}(z_n = a_n + 2 \ \text{infinitely often}) = 1.$$

Hence, for the Poisson distribution, we obtain

$$\mathbf{P}\Big(\exists n_0 \quad \forall n \geq n_0 \colon \{z_n = a_n - 1\} \bigcup \{z_n = a_n\} \bigcup \{z_n = a_n + 1\} \bigcup \{z_n = a_n + 2\}\Big) = 1,$$

whereas equality (5) is not true.

It is known that $a_n$ can be specified by the equality

$$a_n = k \quad \text{for} \quad \ln n \in [R(k), R(k+1)).$$

By using the above-mentioned estimate for $R(k)$, we arrive at the relation

$$\lim_{n\to\infty} \frac{a_n \ln\ln n}{\ln n} = 1.$$

Unfortunately, the author does not know any elementary exact estimation for the quantity $a_n$ in the case of Poisson distribution.

***Example 3.*** Let

$$R(k) = C(k-1)^{\beta}, \qquad 0 < \beta \le 1, \quad C > 0, \quad k \ge 1.$$

Then

$$\tilde{r}(k) = C\beta(k-1)^{\beta-1},$$

$$a_n = \left[1 + \left(\frac{\ln n}{C}\right)^{1/\beta}\right],$$

$$\tilde{r}(a_n) \sim C^{1/\beta}\beta(\ln n)^{1-1/\beta},$$

and, according to Theorem 4,

$$\limsup_{n\to\infty} \frac{z_n C^{1/\beta}\beta(\ln n)^{1-1/\beta} - \beta \ln n}{\ln\ln n} = 1,$$

$$\liminf_{n\to\infty} \frac{z_n C^{1/\beta}\beta(\ln n)^{1-1/\beta} - \beta \ln n}{\ln\ln\ln n} = -1 \quad \text{almost surely.}$$

***Example 4*** (geometric distribution)**.** Setting $\beta = 1$ and

$$C = \ln\frac{1}{1-q}, \quad 0 < q < 1,$$

in Example 3, we arrive at the geometric distribution

$$\mathbf{P}(\xi = k) = p_k = q(1-q)^{k-1}, \quad k \ge 1,$$

$$R(k) = -\ln\sum_{i\ge k} q(1-q)^{k-1} = (k-1)\ln\frac{1}{1-q},$$

$$r(k) = \tilde{r}(k) = \ln\frac{1}{1-q},$$

$$a_n = \left[1 + \left(\ln\frac{1}{1-q}\right)^{-1}\ln n\right].$$

Hence,

$$\limsup_{n\to\infty} \frac{z_n \ln \dfrac{1}{1-q} - \ln n}{\ln\ln n} = 1,$$

$$\liminf_{n\to\infty} \frac{z_n \ln \dfrac{1}{1-q} - \ln n}{\ln\ln\ln n} = -1 \quad \text{almost surely.}$$

***Example 5.*** Consider a one-channel queuing system $M/M/1$ (infinite queue, a Poisson flow of applications with a parameter $\lambda$, the exponential service time with a parameter $\mu$, and

$$\rho = \frac{\lambda}{\mu}$$

is the load of a queuing system, $0 < \rho < 1$). By $\xi_i$ we denote the maximum length of the queue in the $i$th busy period. It is known [9] that $\xi_i$ has the distribution

$$F(n) = \mathbf{P}(\xi_i < n) = \frac{1-\rho^{n-1}}{1-\rho^n}.$$

By using this equality, we easily obtain

$$R(k) = (k-1)\ln\frac{1}{\rho} + \ln\frac{1}{1-\rho} + \ln(1-\rho^k),$$

$$r(k) = \tilde{r}(k) = \ln\frac{1}{\rho} + o(1),$$

$$a_n = 1 + \left[\frac{\ln n - \ln \dfrac{1}{1-\rho} + o(1)}{\ln \dfrac{1}{\rho}}\right] = \frac{\ln n}{\ln \dfrac{1}{\rho}} + O(1).$$

Let $z_n = \max_{1\le i\le n} \xi_i$ be the maximum length of the queue for $n$ busy periods.

Since the conditions of Theorem 4 are satisfied, we get

$$\limsup_{n\to\infty} \frac{z_n \ln \dfrac{1}{\rho} - \ln n}{\ln\ln n} = 1,$$

$$\liminf_{n\to\infty} \frac{z_n \ln \dfrac{1}{\rho} - \ln n}{\ln\ln\ln n} = -1$$

almost surely. The last example was investigated in [9] in a somewhat different context.

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