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DETERMINATION OF THE LOWER COEFFICIENT IN A PARABOLIC EQUATION WITH STRONG POWER DEGENERATION

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We establish conditions for the existence and uniqueness of the classical solution to the inverse problem of identification of the time-dependent lowest coefficient of the first derivative in a one-dimensional degenerate parabolic equation. The Dirichlet boundary conditions and the integral condition of overdetermination are imposed. We study the case of strong power degeneration.

Introduction

In the present paper, we study the coefficient inverse problem for a one-dimensional parabolic equation with strong power degeneration. In addition to the solution of the direct problem, the time-dependent coefficient of the first derivative of an unknown function with respect to the space variable is also regarded as unknown.

The inverse problems of determination of the time-dependent lowest coefficient in a one-dimensional parabolic equation without degeneration were studied in [1–4] in domains with fixed boundaries and in [5–7] in domains with free boundaries. The conditions of unique solvability of these problems for different collections of boundary conditions (Dirichlet and Neumann) and conditions of overdetermination were established.

The direct problems for parabolic equations with degeneration are studied fairly completely. At the same time, the inverse problems for these equations are, in fact, not investigated at all. Thus, we can mention only the works [8, 9] devoted to the inverse problems for parabolic equations with degeneration caused by a function depending on the space variable.

The inverse problems of determination of the coefficient

$$a = a(t), \quad t \in [0, T],$$

in the parabolic equation with power degeneration

$$u_t = a(t)t^{\beta} u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t)$$

were considered by Ivanchov and Saldina [10, 11]. They showed that, unlike the case of weak power degeneration $(0 < \beta < 1)$, for the solvability of the problem in the case of strong power degeneration $(\beta \geq 1)$, certain conditions should be imposed on the lowest coefficients of the equation as $t \to 0$.

The conditions of solvability of the inverse problems of determination of the time-dependent lowest coefficient in a parabolic equation with weak power degeneration were obtained in [12, 13] in the case of domains with fixed boundaries and in [14, 15] in the case of domains with free boundaries. The well-posedness of the inverse problem of simultaneous determination of two time-dependent coefficients in a weakly degenerate parabolic equation was established in [16].

Our aim is to establish conditions for the unique solvability of the inverse problem of determination of the lowest coefficient in a parabolic equation in the case of strong power degeneration.

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1. Statement of the Problem and Main Results

In a domain $Q_T = \{(x,t)\colon 0 < x < h,\ 0 < t < T\}$, we consider the inverse problem of determination of the coefficient $b = b(t)$ in the equation

$$u_t = a(t)t^\beta u_{xx} + b(t)u_x + c(x,t)u + f(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \tag{2}$$

the Dirichlet boundary conditions

$$u(0,t) = \mu_1(t), \qquad u(h,t) = \mu_2(t), \quad t \in [0,T], \tag{3}$$

and the integral condition of overdetermination

$$\int\limits_0^h u(x,t)dx = \mu_3(t), \quad t \in [0,T]. \tag{4}$$

We study the case of strong degeneration where $\beta \geq 1$.

The following theorem establishes the conditions required for the existence and uniqueness of the solution of problem (1)–(4):

Theorem 1. *Assume that the following conditions are satisfied:*

(i) $\varphi \in C^1[0,h],\ a \in C[0,T],\ \mu_i \in C^1[0,T],\ i = 1,2,3,$ *and* $c, f \in C(\overline{Q}_T)$ *and satisfy the Hölder condition with respect to the variable* x *with exponent* $\alpha,\ 0 < \alpha < 1$*;*

(ii) $a(t) > 0,\ \mu_2(t) - \mu_1(t) \neq 0,\ t \in [0,T],\ |f(x,t)| \leq A_1 t^\gamma,\ |c(x,t)| \leq A_2 t^\gamma,\ (x,t) \in \overline{Q}_T,$ *and* $|\mu_3'(t)| \leq A_3 t^\gamma,\ t \in [0,T],$ *where* $A_i,\ i = 1,2,3,$ *are positive constants and* $\gamma > \dfrac{\beta-1}{2}$ *is an arbitrary fixed number;*

(iii) $\mu_1(0) = \varphi(0),\ \mu_2(0) = \varphi(h),$ *and* $\displaystyle\int_0^h \varphi(x)dx = \mu_3(0).$

Then there exists a unique solution

$$(b,u) \in C[0,T_0] \times C^{2,1}(Q_{T_0}) \cap C(\overline{Q}_{T_0}), \qquad |b(t)| \leq M_0 t^\eta,$$

of problem (1)–(4) with $\eta = \min\left\{\gamma, \dfrac{\beta+1}{2}\right\}$, *where the numbers* $M_0 > 0$ *and* $T_0,\ 0 < T_0 \leq T,$ *are determined by the initial data of this problem.*

2. Reduction of Problem (1)–(4) to an Equivalent System of Equations

We temporarily assume that the function $b = b(t)$ is known.

To reduce the direct problem (1)–(3) to a system of integral equations for the functions $u = u(x,t)$ and $v = v(x,t)$, where $v(x,t) \equiv u_x(x,t)$, we use the Green functions $G_k(x,t,\xi,\tau),\ k = 1,2,$ of the first $(k = 1)$

and second $(k=2)$ boundary-value problems for the heat-conduction equation

$$u_t = a(t)t^{\beta}u_{xx}. \tag{5}$$

It is known [17, p. 12] that these functions are given by the formulas

$$G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{+\infty} \left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right) \right.$$

$$\left. +(-1)^k \exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right)\right), \tag{6}$$

$$0 \le x, \quad \xi \le h, \quad 0 \le \tau < t \le T, \quad k=1,2,$$

where

$$\theta(t) = \int_0^t a(\sigma)\sigma^{\beta}d\sigma.$$

As a result, we obtain

$$u(x,t) = \int_0^h G_1(x,t,\xi,0)\varphi(\xi)d\xi + \int_0^t G_{1\xi}(x,t,0,\tau)a(\tau)\tau^{\beta}\mu_1(\tau)d\tau$$

$$- \int_0^t G_{1\xi}(x,t,h,\tau)a(\tau)\tau^{\beta}\mu_2(\tau)d\tau + \int_0^t\int_0^h G_1(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau$$

$$+ \int_0^t\int_0^h G_1(x,t,\xi,\tau)(b(\tau)v(\xi,\tau)+c(\xi,\tau)u(\xi,\tau))d\xi d\tau = \sum_{i=1}^5 I_i, \tag{7}$$

$$v(x,t) = \int_0^h G_2(x,t,\xi,0)\varphi'(\xi)d\xi - \int_0^t G_2(x,t,0,\tau)\mu_1'(\tau)d\tau$$

$$+ \int_0^t G_2(x,t,h,\tau)\mu_2'(\tau)d\tau + \int_0^t\int_0^h G_{1x}(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau$$

$$+ \int_0^t\int_0^h G_{1x}(x,t,\xi,\tau)(b(\tau)v(\xi,\tau)+c(\xi,\tau)u(\xi,\tau))d\xi d\tau = \sum_{i=1}^5 J_i. \tag{8}$$

Note that Eq. (8) is derived from Eq. (7) as a result of differentiation with respect to the space variable. Moreover, we use the relations

$$G_{1x}(x,t,\xi,\tau) = -G_{2\xi}(x,t,\xi,\tau) \qquad \text{and} \qquad G_{2\tau}(x,t,\xi,\tau) = -a(\tau)\tau^{\beta}G_{2\xi\xi}(x,t,\xi,\tau),$$

which can be easily verified with the help of relation (6).

To obtain an equation for the function $b = b(t),$ we integrate Eq. (1). By using (2)–(4), we get

$$b(t) = \left(\mu_3'(t) - a(t)t^{\beta}(v(h,t) - v(0,t)) - \int_0^h (f(x,t) + c(x,t)u(x,t))\,dx\right)(\mu_2(t) - \mu_1(t))^{-1}, \tag{9}$$

$$t \in [0,T].$$

We now analyze the behavior of the integrals on the right-hand sides of relations (7) and (8). Since

$$\int_0^h G_1(x,t,\xi,0)\,d\xi + \int_0^t G_{1\xi}(x,t,0,\tau)a(\tau)\tau^{\beta}\,d\tau - \int_0^t G_{1\xi}(x,t,h,\tau)a(\tau)\tau^{\beta}\,d\tau = 1,$$

we have

$$\sum_{i=1}^{3} |I_i| \le C_1, \tag{10}$$

where C_1 is a positive constant depending on the estimates for the functions $\varphi(x),\ \mu_1(t),$ and $\mu_2(t).$ In view of the fact that

$$G_1(x,t,\xi,\tau) \le G_2(x,t,\xi,\tau), \qquad \int_0^h G_2(x,t,\xi,0)\,d\xi = 1, \tag{11}$$

we obtain

$$|I_4| \le C_2, \qquad |J_1| \le C_3, \quad (x,t) \in \overline{Q}_T, \tag{12}$$

where C_2 and C_3 are positive constants determined by the initial data of the problem.

To estimate the other integrals on the right-hand sides of equalities (7) and (8), we use the following estimates for the Green functions [17, p. 12]:

$$G_2(x,t,\xi,\tau) \le C_4\left(1 + \frac{1}{\sqrt{\theta(t) - \theta(\tau)}}\right), \qquad \int_0^h G_{1x}(x,t,\xi,\tau)\,d\xi \le \frac{C_5}{\sqrt{\theta(t) - \theta(\tau)}}. \tag{13}$$

By using the definition of the function $\theta = \theta(t),$ we establish the behavior of the expression

$$I \equiv \int_0^t \frac{d\tau}{\sqrt{\theta(t) - \theta(\tau)}} = \int_0^t \frac{d\tau}{\sqrt{\int_\tau^t a(\sigma)\sigma^{\beta}d\sigma}} \le \sqrt{\frac{1+\beta}{A_0}}\int_0^t \frac{d\tau}{\sqrt{t^{1+\beta} - \tau^{1+\beta}}},$$

where $A_0 \equiv \min_{[0,T]} a(t)$ as $t \to 0$. In the last integral, we perform the change of variables $z = \dfrac{\tau}{t}$ and obtain

$$I \leq \sqrt{\frac{1+\beta}{A_0}} t^{\frac{1-\beta}{2}} \int\limits_0^1 \frac{dz}{\sqrt{1-z^{1+\beta}}} \leq \sqrt{\frac{1+\beta}{A_0}} t^{\frac{1-\beta}{2}} \int\limits_0^1 \frac{dz}{\sqrt{1-z}} \leq C_6 t^{\frac{1-\beta}{2}}. \tag{14}$$

This means that the integrals $J_2,\ J_3,$ and J_4 behave as $t^{\frac{1-\beta}{2}}$ when $t \to 0$.

Denote

$$U(t) = \max_{x\in[0,h]} |u(x,t)| \qquad \text{and} \qquad V(t) = \max_{(x,\tau)\in[0,h]\times[0,t]} |v(x,\tau)|, \quad t \in [0,T].$$

By using (11), (13), and the conditions of the theorem, we obtain the following inequalities from relations (7)–(9):

$$U(t) \leq C_7 + C_8 \int\limits_0^t \left(|b(\tau)| V(\tau) + \tau^\gamma U(\tau)\right) d\tau, \quad t \in [0,T], \tag{15}$$

$$V(t) \leq \frac{C_9}{t^{\frac{\beta-1}{2}}} + C_{10} \int\limits_0^t \frac{|b(\tau)| V(\tau) + \tau^\gamma U(\tau)}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}}\, d\tau, \quad t \in (0,T], \tag{16}$$

$$|b(t)| \leq C_{11} t^\gamma + C_{12} t^\beta V(t) + C_{13} t^\gamma U(t), \quad t \in [0,T]. \tag{17}$$

It follows from relations (15)–(17) that the function $u = u(x,t)$ is continuous in $\overline{Q}_T$, the function $v = v(x,t)$ behaves as $t^{\frac{1-\beta}{2}}$ when $t \to 0$, and $b(t)$ tends to zero as $t \to 0$ as the power function t^η.

We solve the system of inequalities (15)–(17). Denote

$$\widetilde{b}(t) = \max_{0\leq\tau\leq t} |b(\tau)|.$$

It is clear that this function satisfies inequality (17). Since the functions $\widetilde{b} = \widetilde{b}(t)$ and $V = V(t)$ are nondecreasing, we rewrite

$$U(t) \leq C_7 + C_8 T \widetilde{b}(t) V(t) + C_8 \int\limits_0^t \tau^\gamma U(\tau)\, d\tau, \quad t \in [0,T].$$

By using Lemma 2.2.2 in [17, p. 23], for the last inequality, we get

$$U(t) \leq C_{14}\Big(1 + \widetilde{b}(t) V(t)\Big), \quad t \in [0,T]. \tag{18}$$

Thus, inequality (17) takes the form

$$\widetilde{b}(t)(1 - C_{15} t^\gamma V(t)) \leq C_{16} t^\gamma + C_{12} t^\beta V(t), \quad t \in [0,T]. \tag{19}$$

In view of the behavior of the function $V = V(t)$ as $t \to 0$, we can state that there exists a number $t_1,\ 0 < t_1 \le T$, such that the inequality

$$1 - C_{15} t_1^{\gamma} V(t) \ge \frac{1}{2} \tag{20}$$

holds. This means that inequalities (18) and (19) can be rewritten in the form

$$|b(t)| \le \widetilde{b}(t) \le C_{17} t^{\gamma} + C_{18} t^{\beta} V(t), \quad t \in [0, t_1], \tag{21}$$

$$U(t) \le C_{19} \Big(1 + t^{\gamma} V(t) + t^{\beta} V^2(t)\Big), \quad t \in [0, t_1]. \tag{22}$$

Substituting (21) and (22) in (16), we arrive at the inequality

$$V(t) \le \frac{C_{20}}{t^{\frac{\beta-1}{2}}} + C_{21} \int\limits_0^t \frac{\tau^{\gamma} V(\tau) + \tau^{\beta} V^2(\tau)}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}}\, d\tau. \tag{23}$$

Denote

$$V_1(t) = V(t) t^{\frac{\beta-1}{2}}.$$

We multiply inequality (23) by $t^{\frac{\beta-1}{2}}$. Since $\beta \ge 1$, we get

$$\frac{1}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}} \le \frac{1}{t^{\frac{\beta}{2}} \sqrt{t - \left(\frac{\tau}{t}\right)^{\beta} \tau}} \le \frac{1}{t^{\frac{\beta}{2}} \sqrt{t - \tau}}.$$

As a result, we obtain the following inequality from (23):

$$V_1(t) \le C_{20} + \frac{C_{21}}{\sqrt{t}} \int\limits_0^t \frac{\tau^{\gamma - \frac{\beta-1}{2}} V_1(\tau) + \tau V_1^2(\tau)}{\sqrt{t - \tau}}\, d\tau, \quad t \in [0, t_1]. \tag{24}$$

Let $\gamma \le \dfrac{\beta+1}{2}$. Then

$$V_1(t) \le C_{20} + \frac{C_{22}}{\sqrt{t}} \int\limits_0^t \frac{\tau^{\gamma - \frac{\beta-1}{2}} (V_1(\tau) + 1)^2}{\sqrt{t - \tau}}\, d\tau.$$

Thus, we denote

$$V_2(t) = V_1(t) + 1$$

and obtain

$$V_2(t) \le C_{23} + \frac{C_{22}}{\sqrt{t}} \int\limits_0^t \frac{\tau^{\gamma - \frac{\beta-1}{2}} V_2^2(\tau)}{\sqrt{t - \tau}}\, d\tau, \quad t \in [0, t_1]. \tag{25}$$

Further, we square both sides of (25) and apply the Cauchy and Cauchy–Buniakowski inequalities [18, pp. 49, 382]:

$$V_2^2(t) \le 2C_{23}^2 + 4C_{22}^2 t^{\gamma-\frac{\beta-2}{2}} \int\limits_0^t \frac{\tau^{\gamma-\frac{\beta+1}{2}} V_2^4(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

In the last inequality, we replace t by σ, multiply the inequality by $\dfrac{\sigma^{\gamma-\frac{\beta-1}{2}}}{\sqrt{t-\sigma}}$, and integrate it with respect to σ from 0 to t. As a result, we get

$$\int\limits_0^t \frac{\sigma^{\gamma-\frac{\beta-1}{2}} V_2^2(\sigma)}{\sqrt{t-\sigma}}\, d\sigma \le C_{24} t^{\gamma-\frac{\beta-2}{2}} + C_{25} t^{2\gamma-\beta+\frac{3}{2}} \int\limits_0^t \tau^{\gamma-\frac{\beta+1}{2}} V_2^4(\tau)\, d\tau.$$

By using the last inequality in (25), we obtain

$$V_2(t) \le C_{26} + C_{27} \int\limits_0^t \frac{V_2^4(\tau)}{\tau^{\frac{\beta+1}{2}-\gamma}}\, d\tau. \tag{26}$$

Denote

$$\chi(t) = C_{26} + C_{27} \int\limits_0^t \frac{V_2^4(\tau)}{\tau^{\frac{\beta+1}{2}-\gamma}}\, d\tau. \tag{27}$$

Thus, it follows from (26) that $V_2(t) \le \chi(t)$. Differentiating (27) and using the last inequality, we get

$$\chi'(t) \le \frac{C_{27}}{t^{\frac{\beta+1}{2}-\gamma}}\, \chi^4(t),$$

whence

$$\chi(t) \le \frac{C_{26} \sqrt[3]{\gamma - \frac{\beta-1}{2}}}{\sqrt[3]{\gamma - \frac{\beta-1}{2} - 3C_{26}^3 C_{27} t^{\gamma-\frac{\beta-1}{2}}}}.$$

Taking the number $t_2,\ 0 < t_2 \le T$, such that

$$\gamma - \frac{\beta-1}{2} - 3C_{26}^3 C_{27} t_2^{\gamma-\frac{\beta-1}{2}} > 0, \tag{28}$$

we conclude that $\chi(t) \le C_{28}$ or

$$V_2(t) \le C_{28}, \quad t \in [0, t_2].$$

In the case where $\gamma > \dfrac{\beta+1}{2}$, inequality (24) is reduced to the form

$$V_2(t) \le C_{29} + C_{30} \int\limits_0^t \frac{V_2^2(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

Reasoning as in the solution of (25), we obtain

$$V_2(t) \le C_{31}, \quad t \in [0, t_3],$$

where the number $t_3,\ 0 < t_3 \le T,$ is determined by the constants C_{29} and C_{30}. Thus,

$$|v(x,t)| \le \frac{M_1}{t^{\frac{\beta-1}{2}}}, \quad (x,t) \in [0,h] \times (0,t_4], \tag{29}$$

where $M_1 = \max\{C_{28}, C_{31}\}$ and $t_4 = \min\{t_1, t_2, t_3\}$. According to (17) and (22), we find

$$|u(x,t)| \le M_2, \quad (x,t) \in [0,h] \times [0,t_4], \tag{30}$$

$$|b(t)| \le M_0 t^\eta, \quad t \in [0,t_4]. \tag{31}$$

This means that we have established the *a priori* estimates (29)–(31) for the solutions of the system of equations (7)–(9).

Thus, problem (1)–(4) is reduced to the equivalent system of equations (7)–(9). We understand the indicated equivalence as follows: if a pair of functions (b,u) is a solution of problem (1)–(4) for $(x,t) \in [0,h] \times [0,t_4]$, then the triple of functions

$$(u,v,b) \in C(\overline{Q}_{t_4}) \times C([0,h] \times (0,t_4]) \times C[0,t_4], \quad |b(t)| \le M_0 t^\eta, \quad t \in [0,t_4],$$

satisfies equality (7)–(9) and, conversely, if (u,v,b) is a solution of system (7)–(9), then (b,u) belongs to the class $C[0,t_4] \times C^{2,1}(Q_{t_4}) \cap C(\overline{Q}_{t_4})$ and satisfies problem (1)–(4) and the estimate $|b(t)| \le M_0 t^\eta,\ t \in [0,t_4]$.

Indeed, the first part of this assertion follows from the procedure used to deduce the system of equations (7)–(9). To prove the converse assertion, we differentiate both sides of equality (7) and obtain

$$u_x(x,t) = \int_0^h G_2(x,t,\xi,0)\varphi'(\xi)\,d\xi - \int_0^t G_2(x,t,0,\tau)\mu_1'(\tau)\,d\tau$$

$$+ \int_0^t G_2(x,t,h,\tau)\mu_2'(\tau)\,d\tau + \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau) f(\xi,\tau)\,d\xi\,d\tau$$

$$+ \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)(b(\tau)v(\xi,\tau) + c(\xi,\tau)u(\xi,\tau))\,d\xi\,d\tau.$$

Since the right-hand sides of the obtained equality and equality (8) coincide, we get

$$v(x,t) \equiv u_x(x,t), \quad (x,t) \in [0,h] \times (0,t_4].$$

Moreover, the established behavior of the functions $b = b(t)$ and $v = v(x,t)$ implies that the product $b(t)v(x,t)$ is a continuous function in $\overline{Q}_{t_4}$. By using this result in equality (7), we arrive at an integrodifferential equation for the function $u = u(x,t)$ and conclude that the function u belongs to the class $C^{2,1}(Q_{t_4}) \cap C(\overline{Q}_{t_4})$ and is a solution of problem (1)–(3) [17, p. 49]. Thus, we can rewrite Eq. (9) in the form

$$a(t)t^{\beta}(u_x(h,t) - u_x(0,t) + b(t)(u(h,t) - u(0,t) + \int\limits_0^h (f(x,t) + c(x,t)u(x,t))\,dx = \mu_3'(t)$$

or

$$\int\limits_0^h u_t(x,t)\,dx = \mu_3'(t), \quad t \in [0, t_4].$$

By using the consistency condition

$$\int\limits_0^h \varphi(x)\,dx = \mu_3(0),$$

we arrive at condition (4), which completes the proof of equivalence of problem (1)–(4) and the system of equations (7)–(9).

3. Proof of Existence of a Solution of Problem (1)–(4)

By using the Schauder theorem on fixed point of a completely continuous operator, we prove the existence of solution for the system of equations (7)–(9) equivalent to problem (1)–(4).

We introduce new functions

$$p(t) = b(t)t^{-\eta} \quad \text{and} \quad w(x,t) = t^{\frac{\beta-1}{2}} v(x,t)$$

and rewrite the system of equations (7)–(9) in the form

$$u(x,t) = \sum_{i=1}^{4} I_i + \int\limits_0^t \int\limits_0^h G_1(x,t,\xi,\tau) \times \left(\tau^{\eta - \frac{\beta-1}{2}} p(\tau) w(\xi,\tau) + c(\xi,\tau)u(\xi,\tau)\right) d\xi\,d\tau, \quad (x,t) \in \overline{Q}_{t_4}, \tag{32}$$

$$w(x,t) = t^{\frac{\beta-1}{2}} \sum_{i=1}^{4} J_i + t^{\frac{\beta-1}{2}} \int\limits_0^t \int\limits_0^h G_{1x}(x,t,\xi,\tau) \times \left(\tau^{\eta - \frac{\beta-1}{2}} p(\tau) w(\xi,\tau) + c(\xi,\tau)u(\xi,\tau)\right) d\xi\,d\tau, \quad (x,t) \in \overline{Q}_{t_4}, \tag{33}$$

$$p(t) = \left(\mu_3'(t) - a(t)t^{\frac{\beta+1}{2}}(w(h,t) - w(0,t)) - \int\limits_0^h (f(x,t) + c(x,t)u(x,t))\,dx\right) \times t^{-\eta}(\mu_2(t) - \mu_1(t))^{-1}, \quad t \in [0, t_4]. \tag{34}$$

We take arbitrary (u, w, p) for which inequalities (29)–(31) are true. By using (29)–(31), we estimate the right-hand sides of equations (32)–(34) as follows:

$$|P_1(u,w,p)| \equiv \left| \sum_{i=1}^{4} I_i + \int_0^t \int_0^h G_1(x,t,\xi,\tau)\Big(\tau^{\eta-\frac{\beta-1}{2}} p(\tau)w(\xi,\tau) + c(\xi,\tau)u(\xi,\tau)\Big)\, d\xi\, d\tau \right|$$

$$\leq C_7 + C_{32}t^{\eta-\frac{\beta-3}{2}} + C_{33}t^{\gamma+1},$$

$$|P_2(u,w,p)| \equiv \left| t^{\frac{\beta-1}{2}} \sum_{i=1}^{4} J_i + t^{\frac{\beta-1}{2}} \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)\Big(\tau^{\eta-\frac{\beta-1}{2}} p(\tau)w(\xi,\tau) + c(\xi,\tau)u(\xi,\tau)\Big)\, d\xi\, d\tau \right|$$

$$\leq C_9 + C_{34}t^{\eta-\frac{\beta-1}{2}} + C_{35}t^{\gamma}.$$

Note that the constants C_7 and C_9 in the obtained estimates are smaller than M_2 and M_1, respectively. We choose a number $t_5,\ 0 < t_5 \leq T,$ such that the inequalities

$$C_7 + C_{32}t_5^{\eta-\frac{\beta-3}{2}} + C_{33}t_5^{\gamma+1} \leq M_2, \tag{35}$$

$$C_9 + C_{34}t_4^{\eta-\frac{\beta-1}{2}} + C_{35}t_5^{\gamma} \leq M_1 \tag{36}$$

are true. This yields

$$|P_1(u,w,p)| \leq M_2, \quad (x,t) \in [0,h] \times [0,t_5], \tag{37}$$

$$|P_2(u,w,p)| \leq M_1, \quad (x,t) \in [0,h] \times [0,t_5]. \tag{38}$$

In addition, by virtue of (37) and (38), it follows from (34) that

$$|P_3(u,w,p)| \equiv \Bigg| \Bigg(\mu_3'(t) - a(t)t^{\frac{\beta+1}{2}}(w(h,t) - w(0,t))$$

$$- \int_0^h (f(x,t) + c(x,t)u(x,t))dx \Bigg) t^{-\eta}(\mu_2(t) - \mu_1(t))^{-1} \Bigg| \leq M_0, \quad t \in [0,t_5]. \tag{39}$$

In the Banach space

$$\mathbb{B} = (C(\overline{Q}_{T_0}))^2 \times (C[0,T_0]),$$

we define a set

$$N = \{(u,w,p) \in (C(\overline{Q}_{T_0}))^2 \times (C[0,T_0]) \colon |u(x,t)| \leq M_2,\ |w(x,t)| \leq M_1,\ |p(t)| \leq M_0\},$$

where $T_0 = \min\{t_4, t_5\}$. It is clear that the set N is closed and convex. We rewrite system (32)–(34) in the form of an operator equation

$$\omega = P\omega,$$

where $\omega = (u, w, p)$ and the operator $P = (P_1, P_2, P_3)$ is given by the right-hand sides of equations (32)–(34). As follows from the reasoning presented above, the operator P maps the set N into itself. The complete continuity of the operator P is proved in exactly the same way as in [11] and [17, p. 27]. Thus, by virtue of the Schauder theorem on fixed point of a completely continuous operator, there exists a continuous solution of system (32)–(34) and, hence, a solution of problem (1)–(4) for $x \in [0, h]$ and $t \in [0, T_0]$.

4. Proof of Uniqueness of a Solution of Problem (1)–(4)

We prove the uniqueness of solution of problem (1)–(4) by contradiction with the help of the system of equations (32)–(34). Assume that there are two solutions $(u_i, w_i, p_i),\ i = 1, 2,$ of the system of equations (32)–(34). Thus, we get the following system for the differences $u = u_1 - u_2,\ w = w_1 - w_2,$ and $p = p_1 - p_2$:

$$u(x,t) = \int_0^t \int_0^h G_1(x,t,\xi,\tau)$$

$$\times \left(\tau^{\eta - \frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + \tau^{\eta - \frac{\beta-1}{2}} w_2(\xi,\tau) p(\tau) + c(\xi,\tau) u(\xi,\tau)\right) d\xi\, d\tau, \quad (x,t) \in \overline{Q}_{T_0}, \qquad (40)$$

$$w(x,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)$$

$$\times \left(\tau^{\eta - \frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + \tau^{\eta - \frac{\beta-1}{2}} w_2(\xi,\tau) p(\tau) + c(\xi,\tau) u(\xi,\tau)\right) d\xi\, d\tau, \quad (x,t) \in \overline{Q}_{T_0}, \qquad (41)$$

$$p(t) = \left(a(t) t^{\frac{\beta+1}{2}} (w(h,t) - w(0,t)) - \int_0^h c(x,t) u(x,t)\, dx \right)$$

$$\times t^{-\eta} (\mu_2(t) - \mu_1(t))^{-1}, \quad t \in [0, T_0]. \qquad (42)$$

Substituting equality (42) in (40) and (41), we obtain

$$u(x,t) = \int_0^t \int_0^h G_1(x,t,\xi,\tau) \Bigg(\tau^{\eta - \frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + w_2(\xi,\tau) \Bigg(\tau a(\tau)(w(h,\tau) - w(0,\tau))$$

$$- \tau^{-\frac{\beta-1}{2}} \int_0^h c(\zeta,\tau) u(\zeta,\tau) d\zeta \Bigg) (\mu_2(\tau) - \mu_1(\tau))^{-1} \Bigg) d\xi\, d\tau, \quad (x,t) \in \overline{Q}_{T_0}, \qquad (43)$$

$$w(x,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau)\Bigg(\tau^{\eta-\frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + w_2(\xi,\tau)\Bigg(\tau a(\tau)(w(h,\tau) - w(0,\tau))$$

$$- \tau^{-\frac{\beta-1}{2}} \int_0^h c(\zeta,\tau) u(\zeta,\tau) d\zeta\Bigg)(\mu_2(\tau) - \mu_1(\tau))^{-1}\Bigg) d\xi\, d\tau, \quad (x,t) \in \overline{Q}_{T_0}. \tag{44}$$

We supplement the system of equations (43), (44) with the following two equations for the functions $w(h,t)$ and $w(0,t)$:

$$w(h,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^h G_{1x}(h,t,\xi,\tau)\Bigg(\tau^{\eta-\frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + w_2(\xi,\tau)\Bigg(\tau a(\tau)(w(h,\tau) - w(0,\tau))$$

$$- \tau^{-\frac{\beta-1}{2}} \int_0^h c(\zeta,\tau) u(\zeta,\tau) d\zeta\Bigg)(\mu_2(\tau) - \mu_1(\tau))^{-1}\Bigg) d\xi\, d\tau, \quad t \in [0,T_0], \tag{45}$$

$$w(0,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^h G_{1x}(0,t,\xi,\tau)\Bigg(\tau^{\eta-\frac{\beta-1}{2}} p_1(\tau) w(\xi,\tau) + w_2(\xi,\tau)\Bigg(\tau a(\tau)(w(h,\tau) - w(0,\tau))$$

$$- \tau^{-\frac{\beta-1}{2}} \int_0^h c(\zeta,\tau) u(\zeta,\tau) d\zeta\Bigg)(\mu_2(\tau) - \mu_1(\tau))^{-1}\Bigg) d\xi\, d\tau, \quad t \in [0,T_0]. \tag{46}$$

As a result, we arrive at the system of homogeneous integral Volterra equations of the second kind (43)–(46). In view of (11) and (13), we conclude that the kernels of this system have integrable singularities. Hence, the system possesses solely the trivial solution.

The theorem is proved.

Remark. It follows from the proof of the theorem that the behavior of the function $b = b(t)$ as $t \to 0$ guarantees the convergence of the integral J_5 and makes it possible to establish the behavior of the function $v = v(x,t)$ as a solution of the integral equation (8).

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