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ON WEAKLY PERIODIC GIBBS MEASURES FOR THE POTTS MODEL WITH EXTERNAL FIELD ON THE CAYLEY TREE

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We study the Potts model with external field on a Cayley tree of order $k \geq 2$. For the antiferromagnetic Potts model with external field and $k \geq 6$ and $q \geq 3$, it is shown that the weakly periodic Gibbs measure, which is not periodic, is not unique. For the Potts model with external field equal to zero, we also study weakly periodic Gibbs measures. It is shown that, under certain conditions, the number of these measures cannot be smaller than $2^q - 2$.

1. Introduction

The notion of Gibbs measure for the Potts model on a Cayley tree is introduced in the ordinary way (see [1, 2, 3, 4]). The ferromagnetic Potts model with three components on the Cayley tree of the second order was studied in [5]. It was shown that there exists a critical temperature T_c such that, for $T < T_c$, one can find three translation-invariant Gibbs measures and an uncountably many Gibbs measures that are not translation invariant. The results obtained in [5] were generalized in [6] for the Potts model with finitely many states on the Cayley tree of any (finite) order.

In [7], the uniqueness of translation-invariant Gibbs measure on the Cayley tree was proved for the antiferromagnetic Potts model with external field. The work [8] is devoted to the investigation of the Potts model with countably many states and a nonzero external field on the Cayley tree. It was proved that this model possesses a unique translation-invariant Gibbs measure.

All translation-invariant Gibbs measures were determined in [9]. In particular, it was shown that, for sufficiently low temperatures, their number is equal to $2^q - 1$. It was proved that there exist $[q/2]$ critical temperatures and the exact number of translation-invariant Gibbs measures was found for each intermediate temperature. Moreover, there are works generalizing the Potts model to the case of competing interactions (see [14, 20, 21]).

The notion of weakly periodic Gibbs measure was introduced and some measures of this kind were obtained for the Ising model in [10, 11]. In [19], we also studied weakly periodic Gibbs measures for the Ising model and determined weakly periodic Gibbs measures different from the measures obtained in [10, 11]. In [12], for the Potts model, we studied weakly periodic ground states and weakly periodic Gibbs measures. The weakly periodic Gibbs measures obtained in [12] were also translation-invariant.

In [13], we proved the existence of weakly periodic Gibbs measures for the Potts model that are not translation invariant.

The present paper is devoted to the investigation of weakly periodic (nonperiodic) Gibbs measures for the Potts model with external field on the Cayley tree. In Sec. 2, we present the main definitions and known facts. The results obtained for weakly periodic Gibbs measures are presented in Sec. 3. The proofs of all results can be found in Sec. 4.

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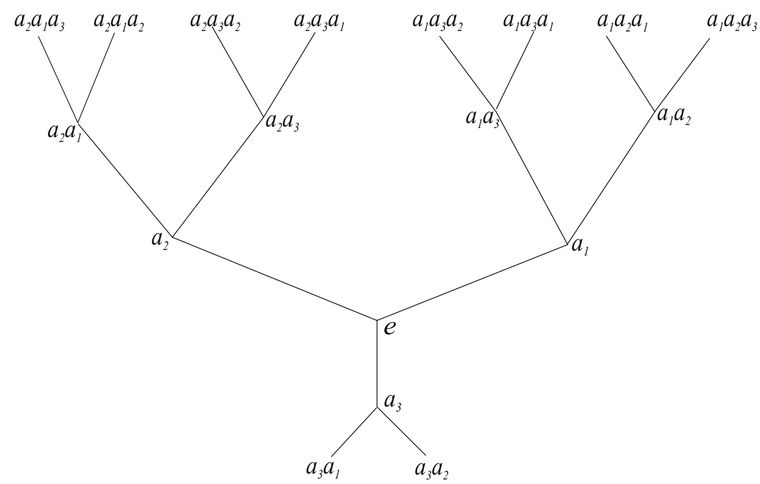

Fig. 1. Cayley tree τ^2 and elements of the group representation of the vertices.

2. Definitions and Known Facts

Let $\tau^k = (V, L),\ k \geq 1,$ be a Cayley tree of order $k,$ i.e., an infinite tree in which exactly $k+1$ edges leave every vertex; here, V is the set of vertices and L is the set of edges $\tau^k.$

Let G_k be the free product of $k+1$ cyclic groups $\{e, a_i\}$ of order two with generators $a_1, a_2, \ldots, a_{k+1},$ respectively, i.e., $a_i^2 = e.$

There exists a one-to-one correspondence between the set of vertices V of the Cayley tree of order k and the group G_k (see [7, 15, 16]).

This correspondence is constructed as follows: Every fixed vertex $x_0 \in V$ is associated with the identity element e of the group $G_k.$ Since, without loss of generality, we can assume that the analyzed graph can be regarded as plane, every neighboring vertex of the point x_0 (i.e., e) is associated with the generator $a_i,\ i = 1, 2, \ldots, k+1,$ in the positive direction (see Fig. 1).

At each vertex $a_i,$ we now define a word of length two $a_i a_j$ for the neighboring vertices of $a_i.$ Since one of the neighboring vertices of a_i is $e,$ we set $a_i a_i = e.$ Then the remaining vertices neighboring with a_i can be enumerated in a unique way following the rule of enumeration presented above. Further, for the neighboring vertices of the vertex $a_i a_j,$ we define a word of length three as follows: Since one of the vertices neighboring with $a_i a_j$ is $a_i,$ we set $a_i a_j a_j = a_i.$ Then the enumeration of the other neighboring vertices is unique and has the form $a_i a_j a_l,\ i, j, l = 1, 2, \ldots, k+1.$ This agrees with the previous step because

$$a_i a_j a_j = a_i a_j^2 = a_i.$$

Hence, we can establish the one-to-one correspondence between the set of vertices of the Cayley tree τ^k and the group $G_k.$

The representation constructed above is called a right representation because, in this case, if x and y are neighboring vertices and g and $h \in G_k$ are the corresponding elements of the group, then either $g = h a_i$ or $h = g a_j$ for some i or $j.$ A left representation is defined in a similar way.

In the group G_k (respectively, in the Cayley tree), we consider a transformation of left (right) shift defined as follows: For $g \in G_k,$ we set

$$T_g(h) = gh, \qquad (T_g(h) = hg) \quad \forall h \in G_k.$$

The collection of all left (right) shifts on G_k is isomorphic to the group $G_k.$

Any transformation S of the group G_k induces a transformation $\widehat{S}$ on the set of vertices V of the Cayley tree τ^k. Therefore, we identify V with G_k.

Theorem 1. *A group of left (right) shifts on the right (left) representation of the Cayley tree is a translation group (see [7, 16]).*

For any point $x^0 \in V$, we set

$$W_n = \{x \in V \mid d(x^0, x) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad \text{and} \quad L_n = \big\{\langle x, y\rangle \in L \mid x, y \in V_n\big\},$$

where $d(x, y)$ is the distance between x and y on the Cayley tree, i.e., the number of edges in the path connecting x and y.

By $S(x)$ we denote the set of "direct descendants" of the point $x \in G_k$, i.e., if $x \in W_n$, then

$$S(x) = \{y \in W_{n+1} \colon d(x, y) = 1\}.$$

We now consider a model in which the spin variables take values from the set $\Phi = \{1, 2, \dots, q\}$, $q \geq 2$, and are located on the vertices of the tree. Then the *configuration* σ on V is defined as a function

$$x \in V \to \sigma(x) \in \Phi.$$

The configurations σ_n and ω_n on V_n and W_n, respectively, are defined in a similar way. The set of all configurations on V (respectively, on V_n and W_n) coincides with $\Omega = \Phi^V$ $\big($respectively, with $\Omega_{V_n} = \Phi^{V_n}$ and $\Omega_{W_n} = \Phi^{W_n}\big)$. It is easy to see that

$$\Phi^{V_n} = \Phi^{V_{n-1}} \times \Phi^{W_n}.$$

The union of configurations $\sigma_{n-1} \in \Phi^{V_{n-1}}$ and $\omega_n \in \Phi^{W_n}$ is defined by the following relation (see [14]):

$$\sigma_{n-1} \vee \omega_n = \big\{\{\sigma_{n-1}(x), x \in V_{n-1}\}, \{\omega_n(y), y \in W_n\}\big\}.$$

The Hamiltonian of the Potts model with external field α is defined as follows:

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V} \delta_{1\sigma(x)}, \tag{1}$$

where J, $\alpha \in \mathbb{R}$.

We define the finite-dimensional distribution of the probability measure μ in the volume V_n as follows:

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right\}, \tag{2}$$

where $\beta = 1/T$, $T > 0$ is temperature, Z_n^{-1} is the normalization factor, $\big\{h_x = (h_{1,x}, \dots, h_{q,x}) \in R^q,\ x \in V\big\}$ is a collection of vectors, and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V_n} \delta_{1\sigma(x)}.$$

We say that the probability distribution (2) is consistent if, for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$,

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{3}$$

Here, $\sigma_{n-1} \vee \omega_n$ is the union of configurations, i.e., $\sigma_{n-1} \vee \omega_n \in \Phi^{V_n}$ such that

$$(\sigma_{n-1} \vee \omega_n) \,|_{V_{n-1}} = \sigma_{n-1} \quad \text{and} \quad (\sigma_{n-1} \vee \omega_n) \,|_{W_n} = \omega_n.$$

In this case, there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$, we get

$$\mu\big(\{\sigma \,|_{V_n} = \sigma_n\}\big) = \mu_n(\sigma_n).$$

This measure is called a split Gibbs measure corresponding to Hamiltonian (1) and the vector-valued function h_x, $x \in V$.

The following assertion describes the condition for h_x guaranteeing the consistency of $\mu_n(\sigma_n)$:

Theorem 2 [7]**.** *The probability distribution*

$$\mu_n(\sigma_n), \qquad n = 1, 2, \ldots,$$

in (2) is consistent if and only if, for any $x \in V$,

$$h_x = \sum_{y \in S(x)} F(h_y, \theta, \alpha), \tag{4}$$

where

$$F\colon\, h = (h_1, \ldots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta, \alpha) = (F_1, \ldots, F_{q-1}) \in \mathbb{R}^{q-1}$$

is defined as

$$F_i = \alpha\beta\delta_{1i} + \ln\left(\frac{(\theta-1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}}\right), \qquad \theta = \exp(J\beta),$$

and $S(x)$ is the set of direct descendants of the point x.

Let $G_k/G_k^* = \{H_1, \ldots, H_r\}$ be a quotient group, where G_k^* is a normal divisor of finite index $r \geq 1$.

Definition 1. *A collection of vectors $h = \{h_x,\, x \in G_k\}$ is called G_k^*-periodic if $h_{yx} = h_x$ $\forall x \in G_k$, $y \in G_k^*$; the G_k-periodic collections are called translation invariant.*

For $x \in G_k$, we denote $x_\downarrow = \{y \in G_k\colon\, \langle x, y\rangle\} \setminus S(x)$.

Definition 2. *A collection of vectors $h = \{h_x,\, x \in G_k\}$ is called G_k^*-weakly periodic if $h_x = h_{ij}$ for $x \in H_i$, $x_\downarrow \in H_j$ $\forall x \in G_k$.*

Definition 3. *A measure μ is called G_k^*-periodic (weakly periodic) if it corresponds to a G_k^*-periodic (weakly periodic) collection of vectors h.*

3. Weakly Periodic Measures

The level of difficulty of the problem of description of weakly periodic Gibbs measures depends on the structure and index of the normal divisor used to impose the condition of periodicity. In [17], it is shown that the group G_k does not have normal divisors of odd index other than 1. Therefore, we consider normal divisors of even index. In the present paper, we restrict ourselves to the case of index 2.

Let q be arbitrary, i.e., $\sigma\colon V \to \Phi = \{1, 2, 3, \ldots, q\}$. In the present paper, we consider the case $q \geq 2$. Let $A \subset \{1, 2, \ldots, k+1\}$. It is known that any normal divisor of index 2 of the group G_k has the form

$$H_A = \left\{ x \in G_k\colon \sum_{i \in A} w_x(a_i) \text{ is even} \right\},$$

where $w_x(a_i)$ is the number of letters a_i in the word $x \in G_k$ [7]. Note that, in the case $|A| = k+1$, where $|A|$ denotes the number of elements in the set A, i.e., $A = N_k$, the notion of weak periodicity coincides with the notion of ordinary periodicity. Indeed, for $|A| = k+1$, we get

$$H_A = \left\{ x \in G_k\colon \sum_{i \in A} w_x(a_i) \text{ is even} \right\} = \big\{ x \in G_k\colon |x| \text{ is even} \big\} = G_k^{(2)}.$$

Thus, $x_\downarrow \in G_k \setminus G_k^{(2)}$ for $x \in G_k^{(2)}$ and $x_\downarrow \in G_k^{(2)}$ for $x \in G_k \setminus G_k^{(2)}$. In view of Definitions 1–3, this implies that, in this case, the notion of weak periodicity coincides with the notion of ordinary periodicity. Hence, we consider $A \subset N_k$ such that $A \neq N_k$.

Let

$$G_k / H_A = \{H_A, G_k \setminus H_A\}$$

be a quotient group. For simplicity, we denote $H_0 = H_A$ and $H_1 = G_k \setminus H_A$. The H_A-weakly periodic collections of the vectors $h = \{h_x \in R^{q-1}\colon x \in G_k\}$ have the form

$$h_x = \begin{cases} h_1 & \text{for} \quad x_\downarrow \in H_0, \quad x \in H_0, \\ h_2 & \text{for} \quad x_\downarrow \in H_0, \quad x \in H_1, \\ h_3 & \text{for} \quad x_\downarrow \in H_1, \quad x \in H_0, \\ h_4 & \text{for} \quad x_\downarrow \in H_1, \quad x \in H_1. \end{cases}$$

Here, $h_i = (h_{i1}, h_{i2}, \ldots, h_{iq-1})$, $i = 1, 2, 3, 4$. Thus, by virtue of (4), we get

$$\begin{aligned} h_1 &= (k - |A|)F(h_1, \theta) + |A| F(h_2, \theta), \\ h_2 &= (|A| - 1)F(h_3, \theta) + (k + 1 - |A|)F(h_4, \theta), \\ h_3 &= (|A| - 1)F(h_2, \theta) + (k + 1 - |A|)F(h_1, \theta), \\ h_4 &= (k - |A|)F(h_4, \theta) + |A| F(h_3, \theta). \end{aligned} \tag{5}$$

We introduce the following notation:

$$e^{h_{ij}} = z_{ij}, \quad i = 1,2,3,4, \quad j = 1,2,\ldots,q-1.$$

Then the last system of equations can be rewritten in the form

$$\begin{aligned}
z_{1j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{1j}+\sum_{i=1}^{q-1} z_{1i}+1}{\sum_{i=1}^{q-1} z_{1i}+\theta}\right)^{k-|A|}\left(\frac{(\theta-1)z_{2j}+\sum_{i=1}^{q-1} z_{2i}+1}{\sum_{i=1}^{q-1} z_{2i}+\theta}\right)^{|A|},\\
z_{2j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{3j}+\sum_{i=1}^{q-1} z_{3i}+1}{\sum_{i=1}^{q-1} z_{3i}+\theta}\right)^{|A|-1}\left(\frac{(\theta-1)z_{4j}+\sum_{i=1}^{q-1} z_{4i}+1}{\sum_{i=1}^{q-1} z_{4i}+\theta}\right)^{k+1-|A|},\\
z_{3j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{2j}+\sum_{i=1}^{q-1} z_{2i}+1}{\sum_{i=1}^{q-1} z_{2i}+\theta}\right)^{|A|-1}\left(\frac{(\theta-1)z_{1j}+\sum_{i=1}^{q-1} z_{1i}+1}{\sum_{i=1}^{q-1} z_{1i}+\theta}\right)^{k+1-|A|},\\
z_{4j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{4j}+\sum_{i=1}^{q-1} z_{4i}+1}{\sum_{i=1}^{q-1} z_{4i}+\theta}\right)^{k-|A|}\left(\frac{(\theta-1)z_{3j}+\sum_{i=1}^{q-1} z_{3i}+1}{\sum_{i=1}^{q-1} z_{3i}+\theta}\right)^{|A|},
\end{aligned} \tag{6}$$

where $j = 1,2,3,\ldots,q-1$.

Consider a mapping $A\colon \mathbb{R}^{4(q-1)} \to \mathbb{R}^{4(q-1)}$ defined as follows:

$$\begin{aligned}
z'_{1j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{1j}+\sum_{i=1}^{q-1} z_{1i}+1}{\sum_{i=1}^{q-1} z_{1i}+\theta}\right)^{k-|A|}\left(\frac{(\theta-1)z_{2j}+\sum_{i=1}^{q-1} z_{2i}+1}{\sum_{i=1}^{q-1} z_{2i}+\theta}\right)^{|A|},\\
z'_{2j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{3j}+\sum_{i=1}^{q-1} z_{3i}+1}{\sum_{i=1}^{q-1} z_{3i}+\theta}\right)^{|A|-1}\left(\frac{(\theta-1)z_{4j}+\sum_{i=1}^{q-1} z_{4i}+1}{\sum_{i=1}^{q-1} z_{4i}+\theta}\right)^{k+1-|A|},\\
z'_{3j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{2j}+\sum_{i=1}^{q-1} z_{2i}+1}{\sum_{i=1}^{q-1} z_{2i}+\theta}\right)^{|A|-1}\left(\frac{(\theta-1)z_{1j}+\sum_{i=1}^{q-1} z_{1i}+1}{\sum_{i=1}^{q-1} z_{1i}+\theta}\right)^{k+1-|A|},\\
z'_{4j} &= \exp(\alpha\beta\delta_{1j})\left(\frac{(\theta-1)z_{4j}+\sum_{i=1}^{q-1} z_{4i}+1}{\sum_{i=1}^{q-1} z_{4i}+\theta}\right)^{k-|A|}\left(\frac{(\theta-1)z_{3j}+\sum_{i=1}^{q-1} z_{3i}+1}{\sum_{i=1}^{q-1} z_{3i}+\theta}\right)^{|A|},
\end{aligned} \tag{7}$$

where $j = 1,2,3,\ldots,q-1$.

We introduce the notation

$$I_m = \{(z_1, z_2, \ldots, z_{q-1}) \in \mathbb{R}^{q-1}:\ z_1 = z_2 = \ldots = z_m,\ z_{m+1} = \ldots = z_{q-1} = 1\}, \tag{8}$$

$$M_m = \{(z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}) \in \mathbb{R}^{4(q-1)}:\ z^{(i)} \in I_m,\ i = 1, 2, 3, 4\}. \tag{9}$$

Here, $m = 1, 2, \ldots, q-1.$

Lemma 1.

1. *For $\alpha \neq 0,$ the set M_1 is invariant under the mapping $A.$*
2. *For $\alpha = 0,$ the sets $M_m,\ m = 1, 2, \ldots, q-1,$ are invariant under the mapping $A.$*

We first consider the case $\alpha \neq 0.$ Denote

$$z_i = z_{i1}, \quad i = 1, 2, 3, 4, \quad \text{and} \quad \lambda = \exp(\alpha\beta).$$

Then, on the invariant set $M_1,$ the system of equations (6) is reduced to the following system of equations:

$$\begin{aligned}
z_1 &= \lambda \left(\frac{\theta z_1 + q - 1}{\theta + q - 2 + z_1}\right)^{k-|A|} \left(\frac{\theta z_2 + q - 1}{\theta + q - 2 + z_2}\right)^{|A|}, \\
z_2 &= \lambda \left(\frac{\theta z_3 + q - 1}{\theta + q - 2 + z_3}\right)^{|A|-1} \left(\frac{\theta z_4 + q - 1}{\theta + q - 2 + z_4}\right)^{k+1-|A|}, \\
z_3 &= \lambda \left(\frac{\theta z_2 + q - 1}{\theta + q - 2 + z_2}\right)^{|A|-1} \left(\frac{\theta z_1 + q - 1}{\theta + q - 2 + z_1}\right)^{k+1-|A|}, \\
z_4 &= \lambda \left(\frac{\theta z_4 + q - 1}{\theta + q - 2 + z_4}\right)^{k-|A|} \left(\frac{\theta z_3 + q - 1}{\theta + q - 2 + z_3}\right)^{|A|}.
\end{aligned} \tag{10}$$

Further, denote

$$f(z) = \frac{\theta z + q - 1}{\theta + q - 2 + z}.$$

The following lemma is evident:

Lemma 2. *The function $f(z)$ is strictly decreasing for $0 < \theta < 1$ and strictly increasing for $1 < \theta.$*

Proposition 1. *Let $\mathbf{z} = (z_1, z_2, z_3, z_4)$ be a solution of the system of equations (10). If $z_i = z_j$ for some $i \neq j,$ then*

$$z_1 = z_2 = z_3 = z_4.$$

Consider the antiferromagnetic Ising model with external field, i.e., $0 < \theta < 1.$ Assume that $|A| = k.$ Then the system of equations (10) has the form

$$\begin{aligned} z_1 &= \lambda\big(f(z_2)\big)^k, \\ z_2 &= \lambda\big(f(z_3)\big)^{k-1}\big(f(z_4)\big), \\ z_3 &= \lambda\big(f(z_2)\big)^{k-1}\big(f(z_1)\big), \\ z_4 &= \lambda\big(f(z_3)\big)^k. \end{aligned} \tag{11}$$

The investigation of the system of equations (11) is reduced to the investigation of the system of equations

$$\begin{aligned} z_2 &= \lambda\big(f(z_3)\big)^{k-1} f\left(\lambda(f(z_3))^k\right), \\ z_3 &= \lambda\big(f(z_2)\big)^{k-1} f\left(\lambda(f(z_2))^k\right). \end{aligned} \tag{12}$$

Denote

$$\psi(z) = \lambda\left(f(z)\right)^{k-1} f\left(\lambda(f(z))^k\right). \tag{13}$$

Then the system of equations (12) can be rewritten in the form

$$\begin{aligned} z_2 &= \psi(z_3), \\ z_3 &= \psi(z_2). \end{aligned} \tag{14}$$

The number of solutions of the system of equations (14) is equal to the number of solutions of the equation $\psi(\psi(z)) = z$. The following lemma is true:

Lemma 3. *Let $\gamma\colon [0,1] \to [0,1]$ be a continuous function with a fixed point $\xi \in (0,1)$. Suppose that the function γ is differentiable at the point $\xi \in (0,1)$ and $\gamma'(\xi) < -1$. Then there exist x_0 and x_1 such that $0 \le x_0 < \xi < x_1 \le 1$ and $\gamma(x_0) = x_1,\ \gamma(x_1) = x_0$ (see [18, p. 70]).*

Note that the equation

$$z = \lambda f^k(z)$$

has a unique solution z_* (see [4, p. 109]).

Proposition 2. *For $k \ge 6$ and $\lambda \in (\lambda_1, \lambda_2)$, the system of equations (14) has three solutions of the form $(z_*, z_*),\ (z_2^*, z_3^*)$, and (z_3^*, z_2^*), where $\lambda_i = b_i^k,\ i = 1, 2$, and*

$$\begin{aligned} b_1 &= \frac{(k-1-\sqrt{k^2-6k+1})(1-\theta)(\theta+q-1)z_*^{(k-1)/k}}{2(\theta+q-2+z_*)^2}, \\ b_2 &= \frac{(k-1+\sqrt{k^2-6k+1})(1-\theta)(\theta+q-1)z_*^{(k-1)/k}}{2(\theta+q-2+z_*)^2}. \end{aligned} \tag{15}$$

By virtue of Theorem 2, the following theorem is true:

Theorem 3. *For $|A| = k,\ k \geq 6$, and $\alpha \in (\alpha_1, \alpha_2)$, the antiferromagnetic Potts model with external field has at most two H_A-weakly periodic (nonperiodic) Gibbs measures, where $\alpha_i = kT \ln b_i$ and T is temperature.*

Remark 1.

1. The condition $k \geq 6$ is necessary in order that the inequality $k^2 - 6k + 1 \geq 0$ be true. Hence, for $2 \leq k \leq 5$, the method proposed in the present paper cannot be applied and the problem remains open.
2. The problem of investigation of weakly periodic (nonperiodic) Gibbs measures for normal divisors of the other even indices in the Potts model with nonzero external field remains open.

Consider the case $\alpha = 0$. Then the system of equations (6) on the invariant set M_m is reduced to the system of equations

$$\begin{aligned} z_1 &= \left(\frac{(\theta+m+1)z_1+q-m}{mz_1+\theta+q-m-1}\right)^{k-|A|}\left(\frac{(\theta+m+1)z_2+q-m}{mz_2+\theta+q-m-1}\right)^{|A|}, \\ z_2 &= \left(\frac{(\theta+m+1)z_3+q-m}{mz_3+\theta+q-m-1}\right)^{|A|-1}\left(\frac{(\theta+m+1)z_4+q-m}{mz_4+\theta+q-m-1}\right)^{k+1-|A|}, \\ z_3 &= \left(\frac{(\theta+m+1)z_2+q-m}{mz_2+\theta+q-m-1}\right)^{|A|-1}\left(\frac{(\theta+m+1)z_1+q-m}{mz_1+\theta+q-m-1}\right)^{k+1-|A|}, \\ z_4 &= \left(\frac{(\theta+m+1)z_4+q-m}{mz_4+\theta+q-m-1}\right)^{k-|A|}\left(\frac{(\theta+m+1)z_3+q-m}{mz_3+\theta+q-m-1}\right)^{|A|}. \end{aligned} \tag{16}$$

Denote

$$f_m(z) = \frac{(\theta+m+1)z+q-m}{mz+\theta+q-m-1}.$$

It is easy to see that the function $f_m(z)$ is strictly decreasing for $0 < \theta < 1$ and strictly increasing for $1 < \theta$. By analogy with Proposition 1, we can prove the following assertion:

Proposition 3. *Let $\mathbf{z} = (z_1, z_2, z_3, z_4)$ be a solution of the system of equations (16). If $z_i = z_j$ for some $i \neq j$, then $z_1 = z_2 = z_3 = z_4$.*

Consider the case where $0 < \theta < 1$ and $|A| = k$. In this case, the system of equations (16) takes the form

$$\begin{aligned} z_1 &= (f_m(z_2))^k, \\ z_2 &= (f_m(z_3))^{k-1}(f_m(z_4)), \\ z_3 &= (f_m(z_2))^{k-1}(f_m(z_1)), \\ z_4 &= (f_m(z_3))^k. \end{aligned} \tag{17}$$

The following theorem is true:

Theorem 4. *Let $|A| = k$ and $k \geq 6$. If one of the following conditions is satisfied:*

(i) $\dfrac{4k}{k+1+\sqrt{k^2-6k+1}} \leq q < \dfrac{4k}{k+1-\sqrt{k^2-6k+1}}$ *and* $0 < \theta < \theta_2$*;*

(ii) $q \leq \dfrac{4k}{k+1+\sqrt{k^2-6k+1}}$ *and* $\theta_1 < \theta < \theta_2$*,*

then there exist at least $2^q - 2$ *weakly periodic (nonperiodic) Gibbs measures, where*

$$\theta_1 = \frac{4\,k - kq - q - q\sqrt{k^2 - 6\,k + 1}}{4k}, \qquad \theta_2 = \frac{4\,k - kq - q + q\sqrt{k^2 - 6\,k + 1}}{4k}.$$

Remark 2.

1. The H_A-weak periodic measures appearing in Theorems 1 and 3 are new and enable us to describe a continuum set of nonperiodic Gibbs measures unknown earlier.
2. If, instead of (9), we consider M_{q-1}, then Theorem 4 coincides with Theorem 3 in [13].
3. In the case $q = 2$, the Potts model describes the Ising model. For $|A| = k$ and $q = 2$, Theorem 4 coincides with Theorem 4 in [19]. The case where $|A| = 1$ and $q = 2$ was studied in [10, 11].
4. The problem of investigation of weakly periodic (nonperiodic) Gibbs measures for normal divisors of the other even indices in the Potts model with zero external field remains open.

4. Proofs

Proof of Lemma 1. 1. Let $\mathbf{z} = (z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}) \in M_1$. Then

$$z^{(i)} \in I_1, \qquad i = 1, 2, 3, 4.$$

By definition (8), we obtain $z^{(i)} = (z_i, 1, 1, \ldots, 1)$, where $z_i \neq 1$, $i = 1, 2, 3, 4$. In view of this result and (7), we find

$$z'_{1j} = \left(\frac{\theta + q - 2 + z_1}{\theta + q - 2 + z_1}\right)^{k-|A|} \left(\frac{\theta + q - 2 + z_2}{\theta + q - 2 + z_2}\right)^{|A|} = 1, \quad j = 2, 3, \ldots, q-1,$$

$$z'_{2j} = \left(\frac{\theta + q - 2 + z_3}{\theta + q - 2 + z_3}\right)^{|A|-1} \left(\frac{\theta + q - 2 + z_4}{\theta + q - 2 + z_4}\right)^{k+1-|A|} = 1, \quad j = 2, 3, \ldots, q-1,$$

$$z'_{3j} = \left(\frac{\theta + q - 2 + z_2}{\theta + q - 2 + z_2}\right)^{|A|-1} \left(\frac{\theta + q - 2 + z_4}{\theta + q - 2 + z_4}\right)^{k+1-|A|} = 1, \quad j = 2, 3, \ldots, q-1,$$

$$z'_{4j} = \left(\frac{\theta + q - 2 + z_4}{\theta + q - 2 + z_4}\right)^{k-|A|} \left(\frac{\theta + q - 2 + z_3}{\theta + q - 2 + z_3}\right)^{|A|} = 1, \quad j = 2, 3, \ldots, q-1.$$

Hence, $A(\mathbf{z}) \in L_1$.

The second part of the lemma is proved similarly.

Proof of Proposition 1. We derive the following equalities from the system of equations (10):

$$\frac{z_1}{z_2} = \left(\frac{f(z_1)}{f(z_4)}\right)^{k-|A|}\left(\frac{f(z_2)}{f(z_3)}\right)^{|A|-1}\left(\frac{f(z_2)}{f(z_4)}\right), \tag{18}$$

$$\frac{z_1}{z_3} = \left(\frac{f(z_2)}{f(z_1)}\right), \tag{19}$$

$$\frac{z_1}{z_4} = \left(\frac{f(z_1)}{f(z_4)}\right)^{k-|A|}\left(\frac{f(z_2)}{f(z_3)}\right)^{|A|}, \tag{20}$$

$$\frac{z_2}{z_3} = \left(\frac{f(z_3)}{f(z_2)}\right)^{|A|-1}\left(\frac{f(z_4)}{f(z_1)}\right)^{k-|A|+1}, \tag{21}$$

$$\frac{z_2}{z_4} = \left(\frac{f(z_4)}{f(z_3)}\right), \tag{22}$$

$$\frac{z_3}{z_4} = \left(\frac{f(z_1)}{f(z_4)}\right)^{k-|A|}\left(\frac{f(z_2)}{f(z_3)}\right)^{|A|-1}\left(\frac{f(z_1)}{f(z_3)}\right). \tag{23}$$

Let $\mathbf{z} = \{z_1, z_2, z_3, z_4\}$ be the solution of the system of equations (10) and let $z_1 = z_2$. In view of the strict monotonicity of the function $f(z)$ and equality (19), we get $z_1 = z_2 = z_3$. In this case, we obtain $z_1 = z_4$ from (21) and, therefore, $z_1 = z_2 = z_3 = z_4$.

Let $z_1 = z_3$. In view of the strict monotonicity of the function $f(z)$ and equality (19), we get $z_1 = z_2 = z_3$. In this case, it follows from (21) that $z_1 = z_4$ and, thus, $z_1 = z_2 = z_3 = z_4$.

Further, let $z_1 = z_4$. Thus, in view of the strict monotonicity of the function $f(z)$ and equality (20), we find $z_2 = z_3$. In this case, it follows from (22) that

$$z_2 f(z_2) = z_4 f(z_4). \tag{24}$$

We now consider a function

$$\phi(z) = zf(z) = z\frac{\theta z + q - 1}{\theta + q - 2 + z}$$

and find its derivative

$$\phi'(z) = \frac{\theta z^2 + 2\theta(\theta + q - 2)z + (q-1)(\theta + q - 2)}{(\theta + q - 2 + z)^2}.$$

It follows from $\theta > 0,\ z > 0$, and $q \geq 2$ that the function $\phi(z)$ is strictly increasing. Hence, (24) is true only for $z_2 = z_4$.

The other cases are proved similarly.

The proposition is proved.

Proof of Proposition 2. It is easy to see that function (13) satisfies the following assertions:

(i) $\psi(z_*) = z_*$,

(ii) the function $\psi(z)$ is defined on $\mathbb{R}_+$,

(iii) $\psi(z)$ is bounded and differentiable at the point z_*.

Hence, by Lemma 1, for $\psi'(z_*) < -1$, the system of equations (14) has three solutions of the form (z_*, z_*), (z_2^*, z_3^*), and (z_3^*, z_2^*). The inequality $\psi'(z_*) < -1$ is equivalent to the following inequality:

$$k\frac{(1-\theta)^2(\theta+q-1)^2 z_*^{2\frac{k-1}{k}}}{(\theta+z_*+q-2)^4} + b(k-1)\frac{(1-\theta)(\theta+q-1)z_*^{\frac{k-1}{k}}}{(\theta+z_*+q-2)^2} + b^2 < 0,$$

where $b = \sqrt[k]{\lambda}$. Therefore,

$$(b-b_1)(b-b_2) < 0,$$

where b_1 and b_2 are given by (15).

Proposition 2 is proved.

Proof of Theorem 4. The investigation of the system of equations (17) is reduced to the investigation of the system of equations

$$\begin{aligned} z_2 &= \big(f_m(z_3)\big)^{k-1} f_m\big((f_m(z_3))^k\big), \\ z_3 &= \big(f_m(z_2)\big)^{k-1} f_m\big((f_m(z_2))^k\big). \end{aligned} \tag{25}$$

Introducing the notation

$$\varphi(z) = (f_m(z))^{k-1} f_m((f_m(z))^k), \tag{26}$$

we reduce the system of equations (25) to the system

$$\begin{aligned} z_2 &= \varphi(z_3), \\ z_3 &= \varphi(z_2). \end{aligned} \tag{27}$$

It is easy to see that function (26) satisfies the following assertions:

(i) $\varphi(1) = 1$,

(ii) the function $\varphi(z)$ is defined on $\mathbb{R}_+$,

(iii) $\varphi(z)$ is bounded and differentiable at the point $z = 1$.

Thus, by Lemma 3, for $\varphi'(1) < -1$, the system of equations (27) has three solutions of the form (1, 1), (z_2^*, z_3^*), and (z_3^*, z_2^*). The inequality $\varphi'(1) < -1$ is equivalent to the inequality

$$k\frac{(\theta-1)^2}{(\theta+q-1)^2} + (k-1)\frac{\theta-1}{\theta+q-1} + 1 < 0. \tag{28}$$

Hence,

$$2k(\theta-\theta_1)(\theta-\theta_2) < 0,$$

where θ_1 and θ_2 are given by (12).

It is clear that if $k < 5$, then θ_1 and θ_2 are complex. If $k = 5$, then $\theta_1 = \theta_2$ and inequality (28) has no solutions.

We now consider the case $k \geq 6$. Assume that θ_1 and θ_2 are both negative. Then inequality (28) does not have solutions. If $\theta_1 \leq 0$ and $0 < \theta_2 < 1$, i.e.,

$$\frac{4k}{k+1+\sqrt{k^2-6k+1}} \leq q < \frac{4k}{k+1-\sqrt{k^2-6k+1}},$$

then inequality (28) possesses a solution $\theta \in (0, \theta_2)$. This proves the first assertion of Theorem 3.

We now prove the second assertion. Let $0 \leq \theta_1 < 1$ and $0 < \theta_2 < 1$. Then the following inequality is true:

$$q \leq \frac{4k}{k+1+\sqrt{k^2-6k+1}}.$$

In this case, inequality (28) possesses a solution $\theta_1 < \theta < \theta_2$. It is easy to see that θ_1 and θ_2 cannot be greater than 1. By virtue of Theorem 2, for any m, under the conditions of Theorem 4, we get two weakly periodic (nonperiodic) Gibbs measures. It follows from (8) that m is the number of coordinates of a vector from $\mathbb{R}^{q-1}$ unequal to 1. It is clear that the number of these vectors is equal to

$$\sum_{m=1}^{q-1} C_{q-1}^{m} = 2^{q-1} - 1.$$

Hence, under the conditions of Theorem 4, we obtain $2(2^{q-1} - 1) = 2^q - 2$ weakly periodic (nonperiodic) Gibbs measures.

Theorem 4 is proved.

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