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INITIAL-BOUNDARY-VALUE PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

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We consider an initial-boundary-value problem for a semilinear parabolic equation with nonlinear nonlocal boundary conditions. We prove the principle of comparison, establish the existence of local solutions, and study the problem of uniqueness and nonuniqueness.

1. Introduction

We consider nonnegative solutions of the initial-boundary-value problem for a semilinear parabolic equation

$$u_t = \Delta u + c(x,t)u^p, \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

with a nonlinear nonlocal boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \int\limits_{\Omega} k(x,y,t)u^l(y,t)\,dy, \qquad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and the following initial condition:

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $p > 0,\ l > 0,\ \Omega$ is a bounded domain in the space $\mathbb{R}^n,\ n \geq 1,$ with sufficiently smooth boundary $\partial\Omega$, and ν is the unit outer normal to $\partial\Omega$.

For the data of problem (1.1)–(1.3), we make the following assumptions:

$$c(x,t) \in C^{\alpha}_{\mathrm{loc}}(\overline{\Omega} \times [0,+\infty)), \quad 0 < \alpha < 1, \quad c(x,t) \geq 0,$$

$$k(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)), \quad k(x,y,t) \geq 0,$$

$$u_0(x) \in C^1(\overline{\Omega}), \qquad u_0(x) \geq 0, \quad x \in \Omega, \qquad \frac{\partial u_0(x)}{\partial \nu} = \int\limits_{\Omega} k(x,y,0)u_0^l(y)\,dy, \quad x \in \partial\Omega.$$

Numerous works are devoted to the investigation of initial-boundary-value problems for parabolic equations and systems with nonlinear nonlocal Dirichlet boundary conditions (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9] and the ref-

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erences therein). In particular, the initial-boundary-value problem for Eq. (1.1) with nonlocal boundary condition

$$u(x,t) = \int\limits_{\Omega} k(x,y,t)u^l(y,t)\,dy, \qquad x \in \partial\Omega, \quad t > 0,$$

was studied in the papers [2, 3] and [4, 5] for $c(x,t) \le 0$ and $c(x,t) \ge 0$, respectively.

Note that, for $p < 1$ and $l < 1$, the nonlinearities in Eq. (1.1) and in the boundary condition (1.2), respectively, do not satisfy the Lipschitz condition in the right semineighborhood of the point $u = 0$. The problems of uniqueness and nonuniqueness of the solutions of initial-boundary-value problems with non-Lipschitz nonlinearities were investigated by numerous authors for various parabolic equations and systems (see, e.g., [10, 11, 12, 13, 3, 5, 14] and the references therein).

In Sec. 2 of the present paper, we prove the principle of comparison for problem (1.1)–(1.3). In Sec. 3, we establish the existence of local solutions. The problems of uniqueness and nonuniqueness of the solution are investigated in Sec. 4.

2. Principle of Comparison

Let $Q_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T),\ \Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, and $T > 0$.

Definition 2.1. *A nonnegative function $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is called an upper solution of problem (1.1)–(1.3) in Q_T if*

$$u_t \ge \Delta u + c(x,t)u^p, \quad (x,t) \in Q_T, \tag{2.1}$$

$$\frac{\partial u(x,t)}{\partial \nu} \ge \int\limits_{\Omega} k(x,y,t)u^l(y,t)\,dy, \quad (x,t) \in S_T, \tag{2.2}$$

$$u(x,0) \ge u_0(x), \quad x \in \Omega. \tag{2.3}$$

A nonnegative function $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is called a lower solution of problem (1.1)–(1.3) in Q_T if inequalities (2.1)–(2.3) are true with the opposite signs. A function $u(x,t)$ is called a solution of problem (1.1)–(1.3) in Q_T if $u(x,t)$ is simultaneously an upper solution and a lower solution of problem (1.1)–(1.3) in Q_T.

Definition 2.2. *A solution $u(x,t)$ of problem (1.1)–(1.3) in Q_T is called maximum if, for any other solution $v(x,t)$ of problem (1.1)–(1.3) in Q_T, the inequality $v(x,t) \le u(x,t)$ holds in Q_T.*

Theorem 2.1. *Let $u_0(x) \not\equiv 0$ in Ω and let $u(x,t)$ be a solution of problem (1.1)–(1.3) in Q_T. Then $u(x,t) > 0$ for $(x,t) \in Q_T \cup S_T$.*

Proof. Since $u_0(x) \not\equiv 0$ in Ω and $u_t - \Delta u = c(x,t)u^p \ge 0$ in Q_T, according to the strong principle of maximum, $u(x,t) > 0$ in Q_T. We now show that

$$u(x,t) > 0 \quad \text{for} \quad (x,t) \in S_T.$$

Assume that there exists a point $(x_0, t_0) \in S_T$ such that $u(x_0, t_0) = 0$. Then, by virtue of Theorem 3.6 (see [15]), $\partial u(x_0, t_0)/\partial\nu < 0$, which contradicts condition (1.2).

Theorem 2.1 is proved.

Theorem 2.2. *Let $u(x,t)$ and $v(x,t)$ be upper and lower solutions of problem (1.1)–(1.3) in Q_T, respectively. In addition, suppose that $u(x,t) > 0$ or $v(x,t) > 0$ for $\min(p,l) < 1$ and $(x,t) \in Q_T \cup \Gamma_T$. Then $u(x,t) \geq v(x,t)$ for $(x,t) \in Q_T \cup \Gamma_T$.*

Proof. Assume that, for $t \in (0,T)$, a nonnegative function $\varphi(x,\tau)$ belongs to $C^{2,1}(\overline{Q}_t)$ and satisfies the homogeneous Neumann boundary condition. We multiply (2.1) by φ and integrate the obtained inequality over the domain Q_t. By using the Green formula and the formula of integration by parts, we get

$$\begin{aligned}\int_\Omega u(x,t)\varphi(x,t)\,dx \geq \int_\Omega u(x,0)\varphi(x,0)\,dx \\ + \int_0^t \int_\Omega (u(x,\tau)\varphi_\tau(x,\tau) + u(x,\tau)\Delta\varphi(x,\tau) + c(x,\tau)u^p(x,\tau)\varphi(x,\tau))\,dx\,d\tau \\ + \int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau)u^l(y,\tau)\,dy\,dS_x\,d\tau.\end{aligned} \tag{2.4}$$

On the other hand, the lower solution $v(x,t)$ satisfies inequality (2.4) with the opposite sign

$$\begin{aligned}\int_\Omega v(x,t)\varphi(x,t)\,dx \leq \int_\Omega v(x,0)\varphi(x,0)\,dx \\ + \int_0^t \int_\Omega (v(x,\tau)\varphi_\tau(x,\tau) + v(x,\tau)\Delta\varphi(x,\tau) + c(x,\tau)v^p(x,\tau)\varphi(x,\tau))\,dx\,d\tau \\ + \int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau)v^l(y,\tau)\,dy\,dS_x\,d\tau.\end{aligned} \tag{2.5}$$

Let $w(x,t) = v(x,t) - u(x,t)$. It follows from from inequalities (2.4) and (2.5) that

$$\begin{aligned}\int_\Omega w(x,t)\varphi(x,t)\,dx \leq \int_\Omega w(x,0)\varphi(x,0)\,dx \\ + \int_0^t \int_\Omega w(x,\tau)\Big(\varphi_\tau(x,\tau) + \Delta\varphi(x,\tau) + p\theta_1^{p-1}(x,\tau)c(x,\tau)\varphi(x,\tau)\Big)\,dx\,d\tau \\ + l\int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega \theta_2^{l-1}(y,\tau)k(x,y,\tau)w(y,\tau)\,dy\,dS_x\,d\tau,\end{aligned} \tag{2.6}$$

where θ_i, $i = 1,2$, are continuous positive functions in $\overline{Q}_t$ for $\min(p,l) < 1$ and continuous nonnegative functions in $\overline{Q}_t$, otherwise.

We define the function $\varphi(x,\tau)$ as a solution of the problem

$$\varphi_\tau + \Delta\varphi + p\theta_1^{p-1}(x,\tau)c(x,\tau)\varphi = 0, \quad (x,\tau) \in Q_t,$$

$$\frac{\partial\varphi(x,\tau)}{\partial\nu} = 0, \quad (x,\tau) \in S_t,$$

$$\varphi(x,t) = \psi(x), \quad x \in \Omega,$$

where $\psi(x) \in C_0^\infty(\Omega),\ 0 \le \psi \le 1$. According to the principle of comparison for linear parabolic equations, the solution $\varphi(x,\tau)$ of the analyzed problem is nonnegative and bounded. By virtue of (2.6) and the inequality $w(x,0) \le 0$, we obtain

$$\int_\Omega w(x,t)\psi(x)\,dx \le m(t)\int_0^t\int_\Omega w_+(x,\tau)\,dx\,d\tau, \tag{2.7}$$

where

$$w_+ = \max(0,w), \qquad m(t) = l|\partial\Omega| \sup_{\partial\Omega\times Q_t} k(x,y,\tau)\sup_{Q_t}\theta_2^{l-1}(x,\tau)\sup_{S_t}\varphi(x,\tau),$$

and $|\partial\Omega|$ is the Lebesgue measure of the set $\partial\Omega$. Note that $m(t) \le m(T_0)$ for $t \in (0,T_0]$ and any $T_0 \in (0,T)$.

We choose a sequence

$$\psi_n(x) \in C_0^\infty(\Omega), \quad 0 \le \psi_n \le 1,$$

that converges in $L^1(\Omega)$ to the function

$$\gamma_t(x) = \begin{cases} 1 & \text{for} \quad w(x,t) > 0, \\ 0 & \text{for} \quad w(x,t) \le 0. \end{cases}$$

Replacing $\psi(x)$ in (2.7) with $\psi_n(x)$ and passing to the limit as $n \to \infty$, we obtain

$$\int_\Omega w_+(x,t)\,dx \le m(T_0)\int_0^t\int_\Omega w_+(x,\tau)\,dx\,d\tau, \quad t \in (0,T_0].$$

In view of the arbitrariness of T_0 and the Gronwall lemma, $w_+(x,t) \le 0$ in Q_T.

Theorem 2.2 is proved.

Theorem 2.2 yields the following assertion:

Theorem 2.3. *Suppose that problem (1.1)–(1.3) has a solution in Q_T with a nonnegative initial condition for $\min(p,l) \ge 1$ and a positive initial condition, otherwise. Then the solution of problem (1.1)–(1.3) is unique.*

3. Existence of Local Solutions

In this section, we prove the existence of a local solution of problem (1.1)–(1.3) by using the formula of representation of the solution and the principle of contracting mappings.

Assume that the sequence $\{\varepsilon_m\}$ is such that $0 < \varepsilon_m < 1$ and $\varepsilon_m \to 0$ as $m \to \infty$. For $\varepsilon = \varepsilon_m$, $m = 1, 2, \dots,$ we introduce new functions $u_{0\varepsilon}(x)$ satisfying the following conditions:

$$u_{0\varepsilon}(x) \in C^1(\overline{\Omega}), \qquad u_{0\varepsilon}(x) \geq \varepsilon, \qquad u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x),$$

$$\varepsilon_i \geq \varepsilon_j, \qquad u_{0\varepsilon}(x) \to u_0(x) \quad \text{as} \quad \varepsilon \to 0,$$

$$\frac{\partial u_{0\varepsilon}(x)}{\partial \nu} = \int\limits_{\Omega} k(x, y, 0) u_{0\varepsilon}^l(y)\, dy, \quad x \in \partial\Omega.$$

Note that, for $\min(p, l) < 1$, the Lipschitz condition in the right semineighborhood of the point $u = 0$ is not satisfied for at least one nonlinearity in (1.1) and (1.2). For this reason, we consider an auxiliary problem for Eq. (1.1) with boundary condition (1.2) and the initial condition

$$u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega. \tag{3.1}$$

Theorem 3.1. *For some $T > 0$, problem (1.1), (1.2), (3.1) possesses a unique solution in Q_T.*

Proof. Let $G_N(x, y; t - \tau)$ be the Green function of the heat-conduction equation with the homogeneous Neumann boundary condition. Note that the function $G_N(x, y; t-\tau)$ has the following properties (see, e.g., [16]):

$$G_N(x, y; t - \tau) \geq 0, \qquad x, y \in \Omega, \quad 0 \leq \tau < t, \tag{3.2}$$

$$\int\limits_{\Omega} G_N(x, y; t - \tau)\, dy = 1, \qquad x \in \Omega, \quad 0 \leq \tau < t. \tag{3.3}$$

It is known that the function $u_\varepsilon(x, t)$ is a solution of problem (1.1), (1.2), (3.1) in Q_σ if and only if

$$\begin{aligned} u_\varepsilon(x, t) &= \int\limits_{\Omega} G_N(x, y; t) u_{0\varepsilon}(y)\, dy + \int\limits_0^t \int\limits_{\Omega} G_N(x, y; t - \tau) c(y, \tau) u_\varepsilon^p(y, \tau)\, dy\, d\tau \\ &+ \int\limits_0^t \int\limits_{\partial\Omega} G_N(x, \xi; t - \tau) \int\limits_{\Omega} k(\xi, y, \tau) u_\varepsilon^l(y, \tau)\, dy\, dS_\xi\, d\tau \equiv L u_\varepsilon(x, t), \quad (x, t) \in Q_\sigma. \end{aligned} \tag{3.4}$$

To prove the solvability of Eq. (3.4), we use the principle of contracting mappings. A sequence $\{u_{\varepsilon,n}(x, t)\}$, $n = 1, 2, \dots,$ is defined as follows:

$$u_{\varepsilon,1}(x, t) \equiv \varepsilon, \quad (x, t) \in \overline{Q}_\sigma, \tag{3.5}$$

and

$$u_{\varepsilon,n+1}(x, t) = L u_{\varepsilon,n}(x, t), \qquad (x, t) \in \overline{Q}_\sigma, \quad n = 1, 2, \dots . \tag{3.6}$$

Let

$$M_{0\varepsilon} = \sup_{\Omega} u_{0\varepsilon}(x).$$

By induction, we show that, for $M > \max(\varepsilon, M_{0\varepsilon})$ and some $\gamma \in (0, \sigma]$, the following relations are true:

$$\sup_{Q_\gamma} u_{\varepsilon,n}(x,t) \le M, \quad n = 1, 2, \dots. \tag{3.7}$$

For $n = 1$, inequality (3.7) is obvious. Thus, we assume that inequality (3.7) is true for $n = m$ and prove it for $n = m + 1$. Indeed, by virtue of (3.2)–(3.4) and (3.6), we find

$$\begin{aligned} u_{\varepsilon,m+1}(x,t) &= \int_\Omega G_N(x,y;t) u_{0\varepsilon}(y)\, dy + \int_0^t \int_\Omega G_N(x,y;t-\tau) c(y,\tau) u^p_{\varepsilon,m}(y,\tau)\, dy\, d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau) u^l_{\varepsilon,m}(y,\tau)\, dy\, dS_\xi\, d\tau \\ &\le M_{0\varepsilon} + M^p \nu(t) + M^l \mu(t), \end{aligned} \tag{3.8}$$

where $(x,t) \in Q_\gamma$ and

$$\nu(t) = \sup_\Omega \int_0^t \int_\Omega G_N(x,y;t-\tau) c(y,\tau)\, dy\, d\tau,$$

$$\mu(t) = \sup_\Omega \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)\, dy\ dS_\xi\, d\tau.$$

Note that (see [17]) there exist positive constants δ_1 and a_1 such that

$$\mu(t) \le a_1\sqrt{t} \quad \text{for} \quad 0 \le t \le \delta_1. \tag{3.9}$$

By virtue of (3.2) and (3.3), we get

$$\nu(t) \le a_2 t \quad \text{for} \quad 0 \le t \le \delta_2, \tag{3.10}$$

where δ_2 and a_2 are positive constants. We choose γ such that $0 < \gamma \le \min(\delta_1, \delta_2)$ and the inequality

$$\sup_{(0,\gamma)} (M^p \nu(t) + M^l \mu(t)) \le M - M_{0\varepsilon} \tag{3.11}$$

is true. Inequality (3.7) with $n = m + 1$ now follows from (3.8) and (3.11). By virtue of (3.2)–(3.6) and the properties of $u_{0\varepsilon}(x)$, we get

$$u_{\varepsilon,n}(x,t) \ge \varepsilon, \qquad (x,t) \in \overline{Q}_\gamma, \quad n = 1, 2, \dots\ . \tag{3.12}$$

Applying the Lagrange formula for $n = 2, 3, \dots,$ we obtain

$$\sup_{Q_\gamma} |u_{\varepsilon,n+1}(x,t) - u_{\varepsilon,n}(x,t)|$$

$$= \sup_{Q_\gamma} \Bigg| \int_0^t \int_\Omega G_N(x,\xi;t-\tau)c(y,\tau)(u^p_{\varepsilon,n}(\xi,\tau) - u^p_{\varepsilon,n-1}(\xi,\tau))\,d\xi\,d\tau$$

$$+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)(u^l_{\varepsilon,n}(y,\tau) - u^l_{\varepsilon,n-1}(y,\tau))\,dy\,dS_\xi\,d\tau \Bigg|$$

$$\le \sup_{Q_\gamma} \left(p\theta^{p-1}_{1,n}(x,t)\nu(t) + l\theta^{l-1}_{2,n}(x,t)\mu(t)\right) \sup_{Q_\gamma} |u_{\varepsilon,n}(x,t) - u_{\varepsilon,n-1}(x,t)|$$

$$\le \sup_{(0,\gamma)} \rho(t) \sup_{Q_\gamma} |u_{\varepsilon,n}(x,t) - u_{\varepsilon,n-1}(x,t)| \le (M+\varepsilon)\left(\sup_{(0,\gamma)} \rho(t)\right)^{n-1},$$

where $\theta_{i,n}(x,t),\ i=1,2,$ are functions continuous in $\overline{Q}_\gamma$ and such that $\alpha_1 \le \theta_{i,n}(x,t) \le M_1,\ (x,t) \in \overline{Q}_\gamma$, and

$$\rho(t) = p(\alpha_1^{p-1} + M_1^{p-1})\nu(t) + l(\alpha_1^{l-1} + M_1^{l-1})\mu(t), \quad t \in [0,\gamma].$$

Note that the positive constants α_1 and M_1 are independent of n. By virtue of (3.9) and (3.10), there exists a constant $T \in (0,\gamma)$ such that

$$\sup_{(0,T)} \rho(t) < 1.$$

Hence, the sequence $\{u_{\varepsilon,n}(x,t)\}$ is uniformly convergent in $\overline{Q}_T$ as $n \to \infty$. We define

$$u_\varepsilon(x,t) = \lim_{n\to\infty} u_{\varepsilon,n}(x,t).$$

By virtue of (3.7) and (3.12), we get

$$\varepsilon \le u_\varepsilon(x,t) \le M, \quad (x,t) \in \overline{Q}_T.$$

Passing in (3.6) to the limit as $n \to \infty$ and using the Lebesgue theorem on the limit transition under the integral sign, we conclude that the limit function $u_\varepsilon(x,t)$ satisfies Eq. (3.4). Therefore, $u_\varepsilon(x,t)$ is a solution of problem (1.1), (1.2), (3.1) in Q_T.

By contradiction, we prove the uniqueness of the solution of problem (1.1), (1.2), (3.1) in Q_T for small values of T. Assume that problem (1.1), (1.2), (3.1) has at least two solutions $u_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$ in Q_T. Reasoning as above, for small values of T, we get

$$\sup_{Q_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)| = \sup_{Q_T} \Bigg| \int_0^t \int_\Omega G_N(x,\xi;t-\tau)c(y,\tau)(u^p_\varepsilon(\xi,\tau) - v^p_\varepsilon(\xi,\tau))\,d\xi\,d\tau$$

$$+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)(u^l_\varepsilon(y,\tau) - v^l_\varepsilon(y,\tau))\,dy\,dS_\xi\,d\tau \Bigg|$$

$$\leq \sup_{Q_T} \Big(p\theta_1^{p-1}(x,t)\nu(t) + l\theta_2^{l-1}(x,t)\mu(t) \Big) \sup_{Q_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)|$$

$$\leq \alpha \sup_{Q_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)|,$$

where $\theta_i(x,t),\ i = 1, 2,$ are positive functions continuous in $\overline{Q}_T$ and $0 < \alpha < 1$. It is clear that $u_\varepsilon(x,t) = v_\varepsilon(x,t)$ in Q_T.

Theorem 3.1 is proved.

Theorem 3.2. *For some $T > 0$, problem (1.1)–(1.3) possesses a maximum solution in Q_T.*

Proof. Let u_ε be a solution of problem (1.1), (1.2), (3.1). It is easy to see that u_ε is an upper solution of problem (1.1)–(1.3). By Theorem 2.2, for $\varepsilon_1 \leq \varepsilon_2$, the inequality $u_{\varepsilon_1} \leq u_{\varepsilon_2}$ is true. By the Dini theorem (see [18]), for some $T > 0$, the sequence $\{u_\varepsilon(x,t)\}$ uniformly converges in $\overline{Q}_T$ as $\varepsilon \to 0$ to a function $u(x,t)$. Passing in (3.4) to the limit as $\varepsilon \to 0$ and using the Lebesgue theorem on the limit transition under the integral sign, we conclude that the function $u(x,t)$ satisfies the equation

$$u(x,t) = \int_\Omega G_N(x,y;t) u_0(y)\, dy + \int_0^t \int_\Omega G_N(x,y;t-\tau) c(y,\tau) u^p(y,\tau)\, dy\, d\tau$$

$$+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau) u^l(y,\tau)\, dy\, dS_\xi\, d\tau$$

in Q_T. Therefore, $u(x,t)$ is a solution of problem (1.1)–(1.3) in Q_T. It is easy to see that $u(x,t)$ is a maximum solution of problem (1.1)–(1.3) in Q_T.

Theorem 3.2 is proved.

4. Uniqueness and Nonuniqueness

In this section, we use some results from [5, 13].

Theorem 4.1. *Let $u_0(x) \equiv 0$ in Ω and let $u(x,t)$ be a maximum solution of problem (1.1)–(1.3) in Q_T. Assume that, for some $t_0 \in [0,T)$, at least one of the following two conditions is satisfied:*

$$0 < p < 1 \quad \textit{and} \quad c(x_0,t_0) > 0 \quad \textit{for some} \quad x_0 \in \Omega, \tag{4.1}$$

$$0 < l < 1 \quad \textit{and} \quad k(x,y_0,t_0) > 0 \quad \textit{for any} \quad x \in \partial\Omega \quad \textit{and some} \quad y_0 \in \partial\Omega. \tag{4.2}$$

Then the maximum solution $u(x,t)$ of problem (1.1)–(1.3) is nontrivial in Q_T.

Proof. Let condition (4.1) be satisfied. In view of the continuity of the function $c(x,t)$, there exist a neighborhood $U(x_0)$ of the point x_0 in Ω and a constant $T_1 \in (t_0,T)$ such that $c(x,t) \geq c_0 > 0,\ x \in U(x_0),\ t \in [t_0,T_1]$. Consider an auxiliary problem

$$\begin{gathered} u_t = \Delta u + c(x,t)u^p, \quad x \in U(x_0), \quad t_0 < t < T_1, \\ u(x,t) = 0, \quad x \in \partial U(x_0), \quad t_0 < t < T_1, \\ u(x,t_0) = 0, \quad x \in U(x_0). \end{gathered} \tag{4.3}$$

We now construct a lower solution of problem (4.3). Let

$$\underline{u}(x,t) = C(t-t_0)^{\frac{1}{1-p}} w(x,t),$$

where C is a positive constant and $w(x,t)$ is a solution of the problem

$$\begin{gathered} w_t = \Delta w, \quad x \in U(x_0), \quad t_0 < t < T_1, \\ w(x,t) = 0, \quad x \in \partial U(x_0), \quad t_0 < t < T_1, \\ w(x,t_0) = w_0(x), \quad x \in U(x_0). \end{gathered} \tag{4.4}$$

Here, $w_0(x)$ is a nontrivial nonnegative function continuous in $\overline{U(x_0)}$ and equal to zero on $\partial U(x_0)$. Note that $\underline{u}(x,t) = 0$ for $t = t_0$ or $x \in \partial U(x_0)$. By virtue of the strong maximum principle,

$$0 < w(x,t) < M_0 = \sup_{U(x_0)} w_0(x), \quad x \in U(x_0), \quad t_0 < t < T_1.$$

For all $(x,t) \in U(x_0) \times (t_0, T_1)$, the following relation is true:

$$\underline{u}_t - \Delta \underline{u} - c(x,t)\underline{u}^p = \frac{C}{1-p}(t-t_0)^{\frac{p}{1-p}} w - c(x,t)C^p(t-t_0)^{\frac{p}{1-p}} w^p \le 0,$$

where

$$C \le M_0^{-1}[c_0(1-p)]^{1/(1-p)}.$$

Let $u(x,t)$ be the maximum solution of problem (1.1)–(1.3) in Q_T with the trivial initial condition. By Theorem 3.2,

$$u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t),$$

where $u_\varepsilon(x,t)$ is a positive upper solution of problem (1.1)–(1.3) in Q_T. It is easy to see that $u_\varepsilon(x,t)$ is an upper solution of problem (4.3). According to the principle of comparison, for problem (4.3), we get

$$u_\varepsilon(x,t) \ge \underline{u}(x,t), \quad (x,t) \in \overline{U(x_0)} \times [t_0, T_1).$$

Passing to the limit in this inequality as $\varepsilon \to 0$, we obtain

$$u(x,t) \ge \underline{u}(x,t), \quad (x,t) \in \overline{U(x_0)} \times [t_0, T_1).$$

By using (1.2) and the strong maximum principle, we conclude that the maximum solution $u(x,t) > 0$ for $x \in \overline{\Omega}$, $t_0 < t < T_1$.

Assume that condition (4.2) is satisfied. Then there exist a neighborhood $V(y_0) \subset \overline{\Omega}$ of the point y_0 and a constant $T_2 \in (t_0, T)$ such that $k(x,y,t) > 0$ for $x \in \partial\Omega,\ y \in V(y_0),\ t_0 \le t \le T_2$.

We now perform the change of variables proposed in [19]. Let $\overline{x} \in \partial\Omega$ and let $\widehat{n}(\overline{x})$ be the unit inner normal to $\partial\Omega$ at the point $\overline{x}$. Since $\partial\Omega$ is a smooth surface, there exists a constant $\delta > 0$ such that the mapping

$\psi\colon \partial\Omega \times [0,\delta] \to \mathbb{R}^n$ given by the formula $\psi(\overline{x},s) = \overline{x} + s\widehat{n}(\overline{x})$ specifies new coordinates $(\overline{x},s)$ in the neighborhood $\partial\Omega$ in $\overline{\Omega}$. The results of direct calculations show that, in these coordinates, the operator Δ applied to the function $g(\overline{x},s) = g(s)$ independent of the variable $\overline{x}$ at the point $(\overline{x},s)$ has the form

$$\Delta g(\overline{x},s) = \frac{\partial^2 g}{\partial s^2}(\overline{x},s) - \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \frac{\partial g}{\partial s}(\overline{x},s), \tag{4.5}$$

where $H_j(\overline{x}),\ j = 1,\dots,n-1$, are the principal curvatures of $\partial\Omega$ at the point $\overline{x}$.

Let $\alpha > 1/(1-l)$, $0 < \xi_0 \le 1$, and let $t_0 < T_3 \le \min(T_2, t_0 + \delta^2)$. At points of the set $Q_{\delta,T_3} = \partial\Omega \times [0,\delta] \times (t_0,T_3)$ with coordinates $(\overline{x},s,t)$, we define a function

$$\underline{u}(\overline{x},s,t) = (t-t_0)^{\alpha} \left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+^3 .$$

Moreover, at points of the set $\overline{\Omega} \times [t_0,T_3) \setminus Q_{\delta,T_3}$, we take $\underline{u}(\overline{x},s,t) \equiv 0$. It is necessary to show that $\underline{u}(\overline{x},s,t)$ is a lower solution of problem (1.1)–(1.3) in $\Omega \times (t_0,T_4)$ for some $T_4 \in (t_0,T_3)$. Indeed, applying (4.5), we get

$$\begin{aligned}
&\underline{u}_t(\overline{x},s,t) - \Delta\underline{u}(\overline{x},s,t) - c(x,t)\underline{u}^p(\overline{x},s,t) \\
&\qquad = \alpha(t-t_0)^{\alpha-1}\left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+^3 \\
&\qquad + \frac{3}{2}s(t-t_0)^{\alpha-3/2}\left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+^2 - 6(t-t_0)^{\alpha-1}\left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+ \\
&\qquad - 3(t-t_0)^{\alpha-1/2}\left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+^2 \sum_{j=1}^{n-1}\frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} - c(x,t)\underline{u}^p(\overline{x},s,t) \ \le\ 0
\end{aligned}$$

in $\Omega \times (t_0,T_3)$ for sufficiently small values of ξ_0.

It is clear that the equalities

$$\frac{\partial\underline{u}}{\partial\nu}(\overline{x},0,t) = -\frac{\partial\underline{u}}{\partial s}(\overline{x},0,t) = 3(t-t_0)^{\alpha-\frac{1}{2}}\xi_0^2$$

are true. For $x \in \partial\Omega$ and sufficiently small values of $t - t_0$, we get

$$\begin{aligned}
&\frac{\partial\underline{u}}{\partial\nu}(x,t) - \int_{\Omega} k(x,y,t)\underline{u}^l(y,t)\,dy \\
&\qquad = 3(t-t_0)^{\alpha-\frac{1}{2}}\xi_0^2 - (t-t_0)^{\alpha l}\int_{\partial\Omega\times[0,\delta]} k(x,(\overline{y},s),t)|J(\overline{y},s)|\left(\xi_0 - \frac{s}{\sqrt{t-t_0}}\right)_+^{3l} d\overline{y}\,ds \\
&\qquad \le 3(t-t_0)^{\alpha-\frac{1}{2}}\xi_0^2 - (t-t_0)^{\alpha l+\frac{1}{2}}\int_{\partial\Omega} d\overline{y}\int_0^{\xi_0} k(x,(\overline{y},z\sqrt{t-t_0}),t)|J(\overline{y},z\sqrt{t-t_0})|\,(\xi_0 - z)_+^{3l}\,dz
\end{aligned}$$

$$\leq 3(t-t_0)^{\alpha-\frac{1}{2}}\xi_0^2 - C(t-t_0)^{\alpha l+\frac{1}{2}} \leq 0,$$

where $J(\overline{y}, s)$ is the Jacobian of transition to new coordinates and the constant C is independent of t. The remaining part of the proof is performed in exactly the same way as in the first part of the theorem.

Theorem 4.1 is proved.

Remark 4.1. Assume that the conditions of Theorem 4.1 are satisfied but (4.1) and (4.2) are replaced with the following conditions:

$$0 < p < 1 \quad \text{and} \quad c(x,t) \not\equiv 0 \quad \text{in} \quad Q_\tau \quad \text{for any} \quad \tau > 0 \tag{4.6}$$

and

$$\begin{gathered} 0 < l < 1 \quad \text{and there exist sequences } \{t_k\} \text{ and } \{y_k\}, \ k \in N, \\ \text{such that} \quad t_k > 0, \quad \lim_{k\to\infty} t_k = 0, \quad y_k \in \partial\Omega, \\ k(x, y_k, t_k) > 0 \quad \text{for any} \quad x \in \partial\Omega. \end{gathered} \tag{4.7}$$

Then the maximum solution of problem (1.1)–(1.3) is positive in $Q_T \cup S_T$.

Corollary 4.1. *Assume that the conditions of Theorem 4.1 are satisfied with (4.1) and (4.2) replaced by (4.6) and (4.7) and that*

$$c(x,t) \quad \textit{and} \quad k(x,y,t) \quad \textit{do not decrease in} \quad t \in [0,\overline{t}] \quad \textit{for some} \quad \overline{t} \in (0,T). \tag{4.8}$$

Then there exists only one positive solution of problem (1.1)–(1.3) in $Q_T \cup S_T$.

Proof. Let $u(x,t)$ be the maximum solution of problem (1.1)–(1.3) with $u_0(x) \equiv 0$ in Ω. It follows from Remark 4.1 that the inequality $u(x,t) > 0$ holds for $(x,t) \in Q_T \cup S_T$. Assume that there exists another positive solution $v(x,t)$ of problem (1.1)–(1.3) in $Q_T \cup S_T$ with the trivial initial condition. By virtue of (4.8), $v(x, t+\tau)$ is a positive upper solution of problem (1.1)–(1.3) in $Q_{\overline{t}-\tau}$ for $\tau \in (0, \overline{t})$. It follows from Theorem 2.2 that

$$u(x,t) \leq v(x, t+\tau)$$

for $(x,t) \in Q_{\overline{t}-\tau} \cup \Gamma_{\overline{t}-\tau}$. Passing to the limit as $\tau \to 0$, we obtain $u(x,t) \leq v(x,t)$ for $(x,t) \in Q_{\overline{t}} \cup \Gamma_{\overline{t}}$. By Definition 2.2 and Theorem 2.3, we conclude that $v(x,t) = u(x,t)$ for all $(x,t) \in Q_T \cup S_T$.

Corollary 4.1 is proved.

Theorem 4.2. *Assume that* $\min(p,l) < 1$, $u_0 \not\equiv 0$, *condition (4.8) is satisfied, and let at least one of conditions (4.6) and (4.7) be satisfied. Then the solution of problem (1.1)–(1.3) is unique.*

Proof. To prove uniqueness, it suffices to show that if v is a solution of problem (1.1)–(1.3), then

$$u(x,t) \leq v(x,t), \quad (x,t) \in Q_{T_1}, \tag{4.9}$$

where u is the maximum solution of problem (1.1)–(1.3).

We now consider three cases: $0 < l < 1$ and $0 < p \leq 1,$ $0 < l < 1$ and $p > 1,$ and $0 < p < 1$ and $l \geq 1.$ Let $0 < l < 1$ and $0 < p \leq 1$. Denote $z = u - v$. Then z satisfies the problem

$$\begin{gathered} z_t \leq \Delta z + c(x,t)z^p, \quad (x,t) \in Q_{T_1}, \\ \frac{\partial z(x,t)}{\partial \nu} \leq \int\limits_{\Omega} k(x,y,t) z^l(y,t)\, dy, \quad (x,t) \in S_{T_1}, \\ z(x,0) \equiv 0, \quad x \in \Omega. \end{gathered} \tag{4.10}$$

By virtue of Corollary 4.1, there exists a unique solution h of the problem

$$\begin{gathered} h_t = \Delta h + c(x,t)h^p, \quad (x,t) \in Q_{T_2}, \\ \frac{\partial h(x,t)}{\partial \nu} = \int\limits_{\Omega} k(x,y,t) h^l(y,t)\, dy, \quad (x,t) \in S_{T_2}, \\ h(x,0) \equiv 0, \quad x \in \Omega, \end{gathered}$$

such that $h(x,t) > 0,\ x \in \overline{\Omega},\ 0 < t < T_2$. Let $T_3 = \min(T_1, T_2)$. By using the arguments from the proofs of Corollary 4.1 and Theorem 2.2, we can show that $h \geq z$ and $u \geq h$. Further, we denote $a = h - z$ and apply the inequality (see, e.g., [20])

$$h^q - u^q + v^q \geq (h - u + v)^q,$$

where $0 < q \leq 1$ and $\max\{h, v\} \leq u \leq h + v$. This yields

$$\begin{gathered} a_t \geq \Delta a + c(x,t)a^p, \quad (x,t) \in Q_{T_3}, \\ \frac{\partial a(x,t)}{\partial \nu} \geq \int\limits_{\Omega} k(x,y,t) a^l(y,t)\, dy, \quad (x,t) \in S_{T_3}, \\ a(x,0) \equiv 0, \quad x \in \Omega. \end{gathered}$$

We now show that $a(x,t) > 0$ in Q_{T_3}. Indeed, assume the contrary. Then, by virtue of Theorem 2.1, there exists $\bar{t} \in (0, T_3)$ such that $a(x,t) \equiv 0$ in $Q_{\bar{t}}$. Hence,

$$\begin{aligned} \int\limits_{\Omega} k(x,y,t)(h^l(y,t) + v^l(y,t))\, dy &= \frac{\partial h(x,t)}{\partial \nu} + \frac{\partial v(x,t)}{\partial \nu} = \frac{\partial z(x,t)}{\partial \nu} + \frac{\partial v(x,t)}{\partial \nu} = \frac{\partial u(x,t)}{\partial \nu} \\ &= \int\limits_{\Omega} k(x,y,t) u^l(y,t)\, dy = \int\limits_{\Omega} k(x,y,t)(z(y,t) + v(y,t))^l\, dy \\ &= \int\limits_{\Omega} k(x,y,t)(h(y,t) + v(y,t))^l\, dy, \qquad (x,t) \in S_{\bar{t}}. \end{aligned}$$

Under condition (4.7), we arrive at a contradiction with the facts that $0 < l < 1,\ h(x,t) > 0$ and $v(x,t) > 0$ in $Q_{\bar{t}}$, and $k(x, y_k, t_k) > 0$ for any $x \in \partial\Omega$ and some $y_k \in \partial\Omega,\ 0 < t_k < \bar{t}$. Under condition (4.6), we arrive at a contradiction by using a different method. Indeed,

$$\begin{aligned} c(x,t)(h+v)^p &= c(x,t)(z+v)^p = c(x,t)u^p \\ &= u_t - \Delta u = (z+v)_t - \Delta(z+v) \\ &= (h+v)_t - \Delta(h+v) = c(x,t)(h^p + v^p), \quad (x,t) \in Q_{\bar{t}}, \end{aligned}$$

but this contradicts to the facts that $0 < p < 1,\ h(x,t) > 0$ and $v(x,t) > 0$ in $Q_{\bar{t}}$, and $c(x_1, t_1) > 0$ for some $x_1 \in \Omega$ and $t_1 \in (0, \bar{t})$.

Since $a(x,t) > 0$ in $Q_{\bar{t}}$, applying Corollary 4.1 and Theorem 2.2, we conclude that $a(x,t) \geq h(x,t)$ in $Q_{\bar{t}} \cup \Gamma_{\bar{t}}$. Hence, inequality (4.9) is true for $0 < l < 1$ and $0 < p \leq 1$.

Consider the second case where $0 < l < 1$ and $p > 1$. It is easy to see that there exists a constant $\beta > 0$ such that

$$u^p(x,t) - v^p(x,t) \leq \beta(u(x,t) - v(x,t)), \quad (x,t) \in Q_{T_4},$$

where $T_4 < T_2$. Let $z = u - v$. Then the function z satisfies problem (4.10) with $p = 1$ and $\beta c(x,t)$ instead of $c(x,t)$. The remaining part of the proof is performed in exactly the same way as in the first case with $p = 1$.

The third case is considered similarly.

Theorem 4.2 is proved.

Remark 4.2. Let $u_0 \not\equiv 0$ and let, for some $\tau > 0$, at least one of the following conditions be satisfied:

$$l \geq 1 \quad \text{and} \quad c(x,t) \equiv 0 \quad \text{in} \quad Q_\tau,$$

$$p \geq 1 \quad \text{and} \quad k(x,y,t) \equiv 0 \quad \text{in} \quad \partial\Omega \times Q_\tau,$$

$$c(x,t) \equiv 0 \quad \text{in} \quad Q_\tau \quad \text{and} \quad k(x,y,t) \equiv 0 \quad \text{in} \quad \partial\Omega \times Q_\tau.$$

Then, by virtue of Theorems 2.1 and 2.3, the solution of problem (1.1)–(1.3) is unique.

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