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ONE PROBLEM CONNECTED WITH THE HELGASON SUPPORT PROBLEM

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We solve the problem of description of the set of continuous functions in annular subdomains of the n-dimensional sphere with zero integrals over all $(n-1)$-dimensional spheres covering the inner spherical cap. As an application, we establish a spherical analog of the Helgason support theorem and new uniqueness theorems for functions with zero spherical means.

1. Introduction

Let $\mathbb{R}^n$ be a real Euclidean space of dimension $n \geq 2$ with Euclidean norm $|\cdot|$. According to the well-known Helgason support theorem [1], any function $f \in C(\mathbb{R}^n)$ that satisfies the estimates

$$\sup_{x \in \mathbb{R}^n} |x|^k |f(x)| < \infty, \quad k = 1, 2, \ldots, \tag{1}$$

and has zero integrals over all hyperplanes in $\mathbb{R}^n$ disjoint with a certain compact convex set K is equal to zero in $\mathbb{R}^n \setminus K$. Examples (see [1], [2], Chap. 1, [3], Chap. 1.8) show that the rapid decrease in f specified in this statement cannot be omitted or essentially weakened. Various modifications and generalizations of this result can be found in [2, 3].

The key point in the proof of the Helgason theorem is the following lemma on functions with zero spherical means:

Lemma A [1]. *Let a function $f \in C(\mathbb{R}^n)$ be such that condition (1) is satisfied and the integral of f over any sphere containing a ball $|x| \leq 1$ is equal to zero. Then $f(x) = 0$ for $|x| > 1$.*

In view of the importance of this fact in the study of the Radon transform, Helgason [4] proposed to generalize Lemma A to an arbitrary complete simply connected Riemann manifold M of negative curvature. For manifolds M satisfying an additional condition of analyticity, this was done by Grinberg and Quinto in [5] who used the technique of microlocal analysis and analytic wave front.

On the other hand, in view of the necessity of condition (1) in Lemma A, we encounter a problem of description of continuous functions in the domain $\alpha < |x| < \beta$ with zero integrals over all spheres covering the ball $|x| \leq \alpha$. The statement and solution of its two-dimensional version belongs to Globevnik [6]. The generalizations to the n-dimensional case were studied by Epstein and Kleiner in [7] and V. V. Volchkov in [8]. Later, the corresponding analogs were established for classical hyperbolic spaces [9, 10]. For compact symmetric spaces, which are also natural for the analyzed class of problems, similar investigations have not been performed yet. In the present paper, we determine the solution of the Globevnik problem for an n-dimensional sphere (see Theorem 1 in what follows). By using this result, we obtain a spherical analog of the Helgason theorem formulated above and new uniqueness theorems for functions with spherical means equal to zero (see Theorems 2–4).

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2. Statement of the Main Results

As usual, by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{C}$ we denote the sets of natural, integer, integer nonnegative, and complex numbers, respectively.

Let $n \geq 2$, let $\mathbb{S}^n$ be a unit sphere in $\mathbb{R}^{n+1}$ centered at zero, and let d be an interior measure on $\mathbb{S}^n$, i.e.,

$$d(\xi, \eta) = \arccos(\xi_1\eta_1 + \ldots + \xi_{n+1}\eta_{n+1}), \quad \xi, \eta \in \mathbb{S}^n,$$

where $\xi_1, \ldots, \xi_{n+1}$ and $\eta_1, \ldots, \eta_{n+1}$ are the Cartesian coordinates of the points ξ and η, respectively. For $0 < r \leq \pi$ and $0 \leq a < b \leq \pi$, we set

$$B_r(\eta) = \{\xi \in \mathbb{S}^n \colon d(\xi, \eta) < r\}, \quad B_r = B_r(o) \quad \big(o = (0, \ldots, 0, 1)\big),$$

$$S_r(\eta) = \{\xi \in \mathbb{S}^n \colon d(\xi, \eta) = r\}, \quad S_r = S_r(o),$$

$$B_{a,b} = \{\xi \in \mathbb{S}^n \colon a < d(o, \xi) < b\}.$$

We define the class $\mathcal{Z}(B_{a,b})$ by the equality

$$\mathcal{Z}(B_{a,b}) = \left\{ f \in C(B_{a,b}) \colon \int\limits_{S_r(\eta)} f(\xi) d\omega(\xi) = 0 \ \ \forall r \in (a, b), \ \ \eta \in B_{\min\{r-a, b-r\}} \right\} \tag{2}$$

[$d\omega$ is an $(n-1)$-dimensional Euclidean measure]. The integral condition in (2) can be rewritten in the form $(f \times \sigma_r)(\eta) = 0$, where σ_r is the surface delta-function concentrated on S_r, and "$\times$" is the sign of convolution on $\mathbb{S}^n$ (see [11], Introduction, Sec. 3). The class $\mathcal{Z}(B_{a,b})$ is a spherical analog of classes considered earlier by Globevnik, Helganson, and others (see Sec. 1). For the description of this class, we need Fourier series of special type.

Let $\mathcal{H}_k$, $k \in \mathbb{Z}_+$, be a space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$ (see [12], Chap. 4, Sec. 2). The dimension a_k of the space $\mathcal{H}_k$ is calculated by the relation

$$a_k = \begin{cases} \dfrac{(n+k-3)!(n+2k-2)}{k!(n-2)!}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

In what follows, we consider $\mathcal{H}_k$ as a subspace of $L^2(\mathbb{S}^{n-1}, d\omega)$. We introduce the spherical coordinates $\theta_1, \ldots, \theta_n$ on $\mathbb{S}^n$ as follows:

$$\xi_1 = \sin\theta_n \ldots \sin\theta_1, \qquad \xi_2 = \sin\theta_n \ldots \sin\theta_2 \cos\theta_1, \ldots, \qquad \xi_{n+1} = \cos\theta_n,$$

$$0 < \theta_1 < 2\pi, \quad 0 < \theta_k < \pi, \quad k \neq 1.$$

If $\xi' = (\xi_1, \ldots, \xi_n) \neq 0$, then the point $\sigma = \xi'/|\xi'|$ belongs to $\mathbb{S}^{n-1}$. We write the action of the function $f \in C(B_{a,b})$ in the form $f(\xi) = f(\sigma \sin\theta_n, \cos\theta_n)$ and associate it with the Fourier series

$$\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(\theta_n) Y_l^{(k)}(\sigma), \quad \theta_n \in (a, b), \tag{3}$$

where $\left\{Y_l^{(k)}\right\}_{l=1}^{a_k}$ is a fixed orthonormal basis in $\mathcal{H}_k$,

$$f_{k,l}(\theta_n) = \int\limits_{\mathbb{S}^{n-1}} f(\sigma \sin\theta_n, \cos\theta_n)\overline{Y_l^{(k)}(\sigma)}d\omega(\sigma).$$

In what follows, for $n = 2$, we assume that

$$Y_1^{(k)}(\sigma) = \frac{1}{\sqrt{2\pi}} i^k e^{-ik\theta_1}, \qquad Y_2^{(k)}(\sigma) = \frac{1}{\sqrt{2\pi}}(-i)^k e^{ik\theta_1}, \quad k \in \mathbb{N}. \tag{4}$$

For any $\theta_n \in (a, b)$, series (3) converges to $f(\sigma \sin\theta_n, \cos\theta_n)$ in the space $L^2(\mathbb{S}^{n-1}, d\omega)$ (see [12], Chap. 4, Sec. 2).

The following result gives the description of the class $\mathcal{Z}(B_{a,b})$ in terms of decompositions in series in spherical harmonics:

Theorem 1. *Let $0 \leq a < b \leq \pi$ and $f \in C(B_{a,b})$. In order that a function f belong to $Z(B_{a,b})$, it is necessary and sufficient that the Fourier coefficients of the function f have the form*

$$f_{0,1}(\theta_n) = 0, \tag{5}$$

$$f_{k,l}(\theta_n) = \sum_{m=0}^{k-1} c_{m,k,l} \frac{(\cos\theta_n)^m}{(\sin\theta_n)^{n+k-2}}, \qquad k \geq 1, \quad 1 \leq l \leq a_k, \tag{6}$$

where $a < \theta_n < b$ and $c_{m,k,l} \in \mathbb{C}$.

Equalities (5) and (6) show that it is impossible to find a nontrivial function $f \in \mathcal{Z}(B_{a,\pi})$ rapidly vanishing in approaching the pole $o^* = (0, \ldots, 0, -1)$. More precisely, the following theorem holds:

Theorem 2. *Let $0 < a < \pi$. Then the following assertions are true:*

1. *Let $f \in \mathcal{Z}(B_{a,\pi})$ and, for any $m \in \mathbb{Z}_+$, let*

$$\sup_{\xi \in B_{a,\pi}} (1 + \xi_{n+1})^{-m} |f(\xi)| < \infty. \tag{7}$$

   Then $f = 0$ in $B_{a,\pi}$.

2. *For any $m \in \mathbb{Z}_+$, there exists a nonzero function $f \in \mathcal{Z}(B_{a,\pi})$ with condition (7).*

Note that, for $b > \pi$, the set $B_{a,b}$ coincides with geodesic ball $B_{\pi-a}(o^*)$. In this case, any function $f \in C(B_{a,b})$ that satisfies the condition

$$\int\limits_{S_r(\eta)} f(\xi)d\omega(\xi) = 0 \quad \forall r \in (a, \pi), \quad \eta \in B_{r-a},$$

is identical zero.

One more application of Theorem 1 is new conditions of uniqueness for functions of the class $\mathcal{Z}(B_{a,b})$.

Theorem 3.

1. *Let E be an infinite set on the interval (a,b), let $f \in \mathcal{Z}(B_{a,b})$, and let $f(\xi) = 0$ for $d(o,\xi) \in E$. Then $f = 0$ in $B_{a,b}$.*
2. *For any finite set $E \subset (0,\pi)$, there exists a nonzero function $f \in \mathcal{Z}(B_{0,\pi})$ such that $f(\xi) = 0$ for $d(o,\xi) \in E$.*

Theorem 4.

1. *Let f belong to $\mathcal{Z}(B_{a,b})$ and to the class C^∞ in a certain neighborhood of a sphere $S_r \subset B_{a,b}$. Let all derivatives of the function f be equal to zero on S_r. Then $f = 0$ in $B_{a,b}$.*
2. *For any $s \in \mathbb{Z}_+$ and $r \in (0,\pi)$, there exists a nonzero function $f \in \mathcal{Z}^\infty(B_{0,\pi})$ such that all its derivatives on S_r are nonzero up to the order s, inclusively.*

For the other results connected with the injectivity of the operator of spherical mean, see [2, 13, 14] and the references therein.

3. Auxiliary Statements

We use the standard notation P_ν^μ and Q_ν^μ presented below for the Legendre functions of the first and second kind on $(-1,1)$, respectively. These functions are connected with the Gauss hypergeometric function by the equalities

$$\frac{(1-x^2)^{\frac{\mu}{2}} P_\nu^\mu(x)}{2^\mu \sqrt{\pi}} = \frac{F\left(-\dfrac{\nu+\mu}{2}, \dfrac{1+\nu-\mu}{2}; \dfrac{1}{2}; x^2\right)}{\Gamma\left(\dfrac{1-\nu-\mu}{2}\right)\Gamma\left(1+\dfrac{\nu-\mu}{2}\right)}$$

$$- \frac{2xF\left(\dfrac{1-\nu-\mu}{2}, 1+\dfrac{\nu-\mu}{2}; \dfrac{3}{2}; x^2\right)}{\Gamma\left(\dfrac{1+\nu-\mu}{2}\right)\Gamma\left(-\dfrac{\nu+\mu}{2}\right)}, \quad \nu,\mu \in \mathbb{C}, \tag{8}$$

$$\frac{(1-x^2)^{\frac{\mu}{2}} Q_\nu^\mu(x)}{2^\mu \pi^{3/2}} = \cot\left(\frac{\pi}{2}(\nu+\mu)\right) \frac{xF\left(\dfrac{1-\nu-\mu}{2}, \dfrac{\nu-\mu}{2}+1; \dfrac{3}{2}; x^2\right)}{\Gamma\left(\dfrac{1+\nu-\mu}{2}\right)\Gamma\left(-\dfrac{\nu+\mu}{2}\right)}$$

$$- \frac{1}{2}\tan\left(\frac{\pi}{2}(\nu+\mu)\right) \frac{F\left(-\dfrac{\nu+\mu}{2}, \dfrac{1+\nu-\mu}{2}; \dfrac{1}{2}; x^2\right)}{\Gamma\left(\dfrac{1-\nu-\mu}{2}\right)\Gamma\left(1+\dfrac{\nu-\mu}{2}\right)}, \quad -\nu-\mu \notin \mathbb{N}, \tag{9}$$

where Γ is the gamma function {see [15], Chap. 3, Sec. 3.4, relations (11) and (12)}. For $\theta \in (0,\pi)$, we set

$$\psi_{\nu,k}(\theta) = (\sin\theta)^{1-n/2} P_{\nu+n/2-1}^{-n/2-k+1}(\cos\theta), \quad \nu \in \mathbb{C},$$

$$\Psi_{\nu,k}(\theta) = \begin{cases} (\sin\theta)^{1-n/2} Q^{n/2+k-1}_{\nu+n/2-1}(\cos\theta) & \text{if} \quad n \quad \text{is even}, \quad 2-n-k-\nu \notin \mathbb{N}, \\ (\sin\theta)^{1-n/2} P^{n/2+k-1}_{\nu+n/2-1}(\cos\theta) & \text{if} \quad n \quad \text{is odd}, \quad \nu \in \mathbb{C}. \end{cases}$$

For fixed $r \in (0,\pi)$ and $k \in \mathbb{Z}_+$, the function $\psi_{\nu,k}(r)$ has infinitely many zeros ν. All zeros are real, simple and symmetric about the point $(1-n)/2$ (see [16], the proof of Lemma 3.4). In addition, $\psi_{\nu,k}(r) > 0$ for any $\nu \in [-k-n+1, k]$. Denote $\mathcal{N}(r) = \{\nu > 0 \colon \psi_{\nu,0}(r) = 0\}$.

Let $L = L_n$ be the Laplacian on $\mathbb{S}^n$, i.e.,

$$L = \frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n} + \frac{1}{\sin^2\theta_n\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}$$

$$+ \frac{1}{\sin^2\theta_n\sin^2\theta_{n-1}\sin^{n-3}\theta_{n-2}}\frac{\partial}{\partial\theta_{n-2}}\sin^{n-3}\theta_{n-2}\frac{\partial}{\partial\theta_{n-2}}$$

$$+ \ldots + \frac{1}{\sin^2\theta_n\sin^2\theta_{n-1}\ldots\sin^2\theta_3\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}.$$

For any $m \in \mathbb{Z}$, consider the differential operator D_m defined on the space $C^1(0,\pi)$ as follows:

$$(D_m u)(\theta) = (\sin\theta)^m \frac{d}{d\theta}\left(\frac{u(\theta)}{(\sin\theta)^m}\right), \quad u \in C^1(0,\pi).$$

The equality

$$L_{n-1}Y_l^{(k)} = -k(n+k-2)Y_l^{(k)}$$

(see [17], Chap. 9, Sec. 5.1) shows that if $f \in C^2(B_{a,b})$ has the form $f(\xi) = u(\theta_n)Y_l^{(k)}(\sigma)$, then

$$(Lf)(\xi) + k(n+k-1)f(\xi) = \Big(D_{1-k-n}D_k u\Big)(\theta_n)Y_l^{(k)}(\sigma). \tag{10}$$

We also note the following formulas:

$$D_k\,\psi_{\nu,k} = (k-\nu)(k+\nu+n-1)\psi_{\nu,k+1}, \qquad D_k\,\Psi_{\nu,k} = \Psi_{\nu,k+1}, \tag{11}$$

$$D_{1-k-n}\,\psi_{\nu,k+1} = \psi_{\nu,k}, \qquad D_{1-k-n}\,\Psi_{\nu,k+1} = (k-\nu)(k+\nu+n-1)\Psi_{\nu,k}, \tag{12}$$

$$(L+\nu(\nu+n-1)\mathrm{Id})\big(\psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma)\big) = (L+\nu(\nu+n-1)\mathrm{Id})\big(\Psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma)\big) = 0 \tag{13}$$

(Id is the identity operator). To prove (11) and (12), it suffices to use the definitions of $\psi_{\nu,k}$ and $\Psi_{\nu,k}$ and recurrence relations for the Legendre functions {see [15], Chap. 3, Sec. 3.8, relations (15), (17), and (19)}. Equality (13) follows from (10)–(12). For $-\nu \notin \mathbb{N}$, the functions $\psi_{\nu,k}$ and $\Psi_{\nu,k}$ form a fundamental system of solutions of the equation

$$\Big(D_{1-k-n}D_k u\Big)(\theta) = (k-\nu)(k+\nu+n-1)u(\theta), \quad \theta \in (0,\pi).$$

We set

$$\mathcal{S}_\nu^{k,l}(\xi) = \psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma), \quad \xi \in B_\pi.$$

Lemma 1. *Let* $0 \le r < \pi,\ t \in (0, \pi - r)$, *and* $\eta \in S_r$. *Then*

$$\int\limits_{S_t(\eta)} \mathcal{S}_\nu^{k,l}(\xi) d\omega(\xi) = (2\pi)^{n/2} (\sin t)^{n-1} \psi_{\nu,0}(t) \mathcal{S}_\nu^{k,l}(\eta).$$

Proof. In view of (13), by the Pizzetti formula (see, e.g., [18]), we get

$$\int\limits_{S_t(\eta)} \mathcal{S}_\nu^{k,l}(\xi) d\omega(\xi) = \frac{\omega(S_t)}{\left(\cos \frac{t}{2}\right)^{n-2}} \left(\mathcal{S}_\nu^{k,l}(\eta) + \Gamma\left(\frac{n}{2}\right) \sum_{m=1}^{\infty} \left(\sin \frac{t}{2}\right)^{2m} \right.$$

$$\left. \times \frac{\left(\left(L - \frac{(n-2)n}{4}\mathrm{Id}\right) \ldots \left(L - \frac{(n-2m)(n+2m-2)}{4}\mathrm{Id}\right) \mathcal{S}_\nu^{k,l}\right)(\eta)}{m!\Gamma\left(\frac{n}{2} + m\right)} \right)$$

$$= \frac{\omega(S_t)}{\left(\cos \frac{t}{2}\right)^{n-2}} \mathcal{S}_\nu^{k,l}(\eta) \left(1 + \Gamma\left(\frac{n}{2}\right) \sum_{m=1}^{\infty} \left(\sin \frac{t}{2}\right)^{2m} \right.$$

$$\left. \times \frac{\left(\nu(1-n-\nu) - \frac{(n-2)n}{4}\right) \ldots \left(\nu(1-n-\nu) - \frac{(n-2m)(n+2m-2)}{4}\right)}{m!\Gamma\left(\frac{n}{2} + m\right)} \right)$$

$$= \frac{\omega(S_t)}{\left(\cos \frac{t}{2}\right)^{n-2}} \mathcal{S}_\nu^{k,l}(\eta) F\left(-\nu - \frac{n}{2} + 1, \nu + \frac{n}{2}, \frac{n}{2}; \sin^2 \frac{t}{2}\right).$$

By using the equality

$$\psi_{\nu,0}(t) = \frac{2^{1-n/2}}{\Gamma\left(\frac{n}{2}\right)\left(\cos \frac{t}{2}\right)^{n-2}} F\left(-\nu - \frac{n}{2} + 1, \nu + \frac{n}{2}, \frac{n}{2}; \sin^2 \frac{t}{2}\right)$$

[see [15], Chap. 3, Sec. 3.5, relation (9)], we obtain the required result.

Let f be a continuous radial (i.e., depending only on θ_n) function in the ball B_R. By f_0 we denote a function defined on $[0, R)$ and satisfying the relation $f(\xi) = f_0(\theta_n),\ \xi \in B_R$.

Lemma 2. *Let* $0 < r < R \le \pi$ *and let* $f(\xi) = f_0(\theta_n) \in C^\infty(B_R)$. *Then the equality*

$$\Big(\big(D_{k-1} \ldots D_0\, f_0\big)(\theta_n) Y_l^{(k)}(\sigma) \times \sigma_r\Big)(\xi) = \Big(D_{k-1} \ldots D_0 (f \times \sigma_r)_0\Big)(\theta_n) Y_l^{(k)}(\sigma) \tag{14}$$

is true for $k \in \mathbb{N}$ *and* $\xi \in B_{R-r}$.

Proof. We fix $\eta \in B_{R-r}$ and $\varepsilon \in (0, R-r-d(o,\eta))$. Consider a function w_ε with the following properties:

(i) $\mathrm{w}_\varepsilon \in C^\infty[0,\pi]$;

(ii) $\mathrm{w}_\varepsilon = 1$ on $[0, R-\varepsilon]$ and $\mathrm{w}_\varepsilon = 0$ on $[R-\varepsilon/2, \pi]$.

For $\theta \in [0, R)$, we set $h(\theta) = f_0(\theta)\mathrm{w}_\varepsilon(\theta)$. Repeating the reasoning from the proofs of Lemma 4.3 and Theorem 2.1 in [16], we obtain

$$h(\theta) = \sum_{\nu \in \mathcal{N}(R-\varepsilon/3)} c_\nu \psi_{\nu,0}(\theta), \qquad c_\nu \in \mathbb{C}, \quad 0 \le \theta \le R - \frac{\varepsilon}{3},$$

in addition, $c_\nu = O(\nu^{-c}),\ \nu \to +\infty$, for any fixed $c > 0$. Then [see (11)]

$$\big(D_{k-1}\dots D_0\ f_0\big)(\theta_n)Y_l^{(k)}(\sigma) = \sum_{\nu \in \mathcal{N}(R-\varepsilon/3)} c_\nu b_{\nu,k} \mathcal{S}_\nu^{k,l}(\xi), \quad \xi \in B_{R-\varepsilon},$$

where

$$b_{\nu,k} = (-\nu)(\nu+n-1)\dots(k-1-\nu)(k+\nu+n-2). \tag{15}$$

By Lemma 1, we get

$$\Big(\big(D_{k-1}\dots D_0\ f_0\big)(\theta_n)Y_l^{(k)}(\sigma) \times \sigma_r\Big)(\eta) = \int\limits_{S_r(\eta)} \Big(D_{k-1}\dots D_0\ f_0\Big)(\theta_n)Y_l^{(k)}(\sigma)d\omega(\xi)$$

$$= (2\pi)^{n/2}(\sin r)^{n-1} \sum_{\nu \in \mathcal{N}(R-\varepsilon/3)} c_\nu b_{\nu,k} \psi_{\nu,0}(r) \mathcal{S}_\nu^{k,l}(\eta). \tag{16}$$

On the other hand, we similarly obtain

$$(f \times \sigma_r)(\eta) = \int\limits_{S_r(\eta)} f_0(\theta_n)d\omega(\xi) = (2\pi)^{n/2}(\sin r)^{n-1} \sum_{\nu \in \mathcal{N}(R-\varepsilon/3)} c_\nu \psi_{\nu,0}(r)\psi_{\nu,0}(\arccos \eta_{n+1})$$

and

$$\Big(D_{k-1}\dots D_0\ (f \times \sigma_r)_0\Big)(\theta)$$

$$= (2\pi)^{n/2}(\sin r)^{n-1} \sum_{\nu \in \mathcal{N}\big(R-\varepsilon/3\big)} c_\nu b_{\nu,k} \psi_{\nu,0}(r)\psi_{\nu,k}(\theta), \quad 0 \le \theta < R-r-\varepsilon. \tag{17}$$

Comparing (16) and (17), we arrive at (14).

Lemma 3. *Let* $k \in \mathbb{N},\ m \in \{0, \dots, k-1\}$,

$$f(\xi) = \frac{(\cos\theta_n)^m}{(\sin\theta_n)^{n+k-2}} Y_l^{(k)}(\sigma), \quad \xi \in B_{0,\pi}.$$

Then $f \times \sigma_r = 0$ *in* $B_{\min\{\pi-r,r\}}$ *for any* $r \in (0,\pi)$.

Proof. For $\varepsilon \in (0, r)$, we consider the function v_ε with the following properties:

(i) $\mathrm{v}_\varepsilon \in C^\infty[0, \pi]$; (ii)

(ii) $\mathrm{v}_\varepsilon = 0$ on $[0, \varepsilon/2]$ and $\mathrm{v}_\varepsilon = 1$ on $[\varepsilon, \pi]$.

We set

$$H(\eta) = \big(\Psi_{\nu,0} \cdot \mathrm{v}_\varepsilon\big)(\arccos \eta_{n+1}), \quad \eta \in B_\pi.$$

Let $\xi \in B_{\min\{\pi - r, r-\varepsilon\}}$. Since $(L + \nu(\nu + n - 1)\mathrm{Id})(H) = 0$ in $B_{\varepsilon,\pi}$ [see (13)] and $S_r(\xi) \subset B_{\varepsilon,\pi}$, we have

$$(L + \nu(\nu + n - 1)\mathrm{Id})(H \times \sigma_r)(\xi) = (L + \nu(\nu + n - 1)\mathrm{Id})(H) \times \sigma_r(\xi) = 0. \tag{18}$$

Taking into account that $H \times \sigma_r$ is a smooth radial function in $B_{\pi-r}$, we obtain $(H \times \sigma_r)(\xi) = c\, \psi_{\nu,0}(\theta_n)$, where

$$c = (2\pi)^{n/2}(\sin r)^{n-1}\Psi_{\nu,0}(r),$$

from (18), (13), and (10). Then, by Lemma 2, we get

$$\Big(\big(D_{k-1} \ldots D_0\ H_0\big)(\theta_n)Y_l^{(k)}(\sigma) \times \sigma_r\Big)(\xi) = c\Big(D_{k-1} \ldots D_0\ \psi_{\nu,0}\Big)(\theta_n)Y_l^{(k)}(\sigma).$$

With regard for (11) and (15), this equality takes the form

$$\big(\Psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma)\big) \times \sigma_r(\xi) = c\, b_{\nu,k}\mathcal{S}_\nu^{k,l}(\xi).$$

In particular,

$$\big(\Psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma)\big) \times \sigma_r(\xi) = 0 \qquad \text{for} \quad \nu = 0, 1, \ldots, k-1.$$

Therefore, since ε can be arbitrarily chosen on $(0, r)$, by using (8) and (9), we obtain the assertion of Lemma 3.

4. Properties of the Class $\mathcal{Z}(B_{a,b})$

For $s \in \mathbb{Z}_+ \cup \{\infty\}$, we set $\mathcal{Z}^s(B_{a,b}) = \mathcal{Z}(B_{a,b}) \cap C^s(B_{a,b})$.

Lemma 4. *Let $f \in \mathcal{Z}(B_{a,b})$, $k \in \mathbb{Z}_+$, $1 \le l$, and $p \le a_k$. Then:*

(i) $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in \mathcal{Z}(B_{a,b})$;

(ii) if $n \ge 3$, then $f_{k,l}(\theta_n)Y_p^{(k)}(\sigma) \in \mathcal{Z}(B_{a,b})$.

Similar assertions are also true for the class $\mathcal{Z}^s(B_{a,b})$.

Proof. The set $\{\tau \in SO(n+1)\colon \tau o = o\}$, where $SO(n+1)$ is a rotation group of $\mathbb{R}^{n+1}$, which is a subgroup in $SO(n+1)$ isomorphic to the group $SO(n)$. By $d\tau$ we denote a normed Haar measure on $SO(n)$. Let $T^k(\tau)$ be a contraction of a quasiregular representation of the group $SO(n)$ to the space $\mathcal{H}_k$ [17] (Chap. 9, Sec. 2.7) and let $\big\{t_{l,p}^k(\tau)\big\}$ be a matrix of the representation $T^k(\tau)$ in the basis $\big\{Y_l^{(k)}\big\}$, i.e.,

$$\big(T^k(\tau)Y_l^{(k)}\big)(\sigma) = Y_l^{(k)}(\tau^{-1}\sigma) = \sum_{p=1}^{a_k} t_{l,p}^k(\tau)Y_p^{(k)}(\sigma), \qquad \tau \in SO(n), \quad \sigma \in \mathbb{S}^{n-1}. \tag{19}$$

If $n = 2$ and τ is a rotation by angle θ in $\mathbb{R}^2$, then

$$t^k_{1,1}(\tau) = e^{-ik\theta}, \quad t^k_{2,2}(\tau) = e^{ik\theta}, \quad \text{and} \quad t^k_{1,2}(\tau) = t^k_{2,1}(\tau) = 0$$

[see (4)]. Moreover, for terms of series (3), we have the equality

$$f_{k,l}(\theta_2) Y^{(k)}_l(\sigma) = \int\limits_{SO(2)} f(\tau^{-1}\xi)\overline{t^k_{l,l}(\tau)} d\tau. \tag{20}$$

For $n \geq 3$, by using (19) and the irreducibility of the representations $T^k(\tau)$ [17] (Chap. 9, Sec. 2.10), we obtain

$$f_{k,l}(\theta_n) Y^{(k)}_p(\sigma) = a_k \int\limits_{SO(n)} f(\tau^{-1}\xi)\overline{t^k_{l,p}(\tau)} d\tau. \tag{21}$$

By using (20), (21) and the above-indicated imbedding $SO(n)$ in $SO(n+1)$, we obtain the required statements.

Lemma 5. *Let* $s \in \mathbb{N}$, $f \in \mathcal{Z}^s(B_{a,b})$, *and*

$$f^\circ(\theta_1, \ldots, \theta_n) = f(\sin\theta_n \ldots \sin\theta_1, \sin\theta_n \ldots \sin\theta_2 \cos\theta_1, \ldots, \cos\theta_n).$$

Then

$$-\sin\theta_{n-1} \cot\theta_n \frac{\partial f^\circ}{\partial \theta_{n-1}} + \cos\theta_{n-1} \frac{\partial f^\circ}{\partial \theta_n} \in \mathcal{Z}^{s-1}(B_{a,b}).$$

Proof. Let $r \in (a, b)$ and $\eta \in B_{\min\{r-a, b-r\}}$. By a_t we denote the motion of the sphere $\mathbb{S}^n$ defined by the equality

$$a_t\xi = (\xi_1, \ldots, \xi_{n-1},\ \xi_n \cos t + \xi_{n+1} \sin t,\ -\xi_n \sin t + \xi_{n+1} \cos t).$$

For sufficiently small $|t|$, we obtain the following relation from the condition of the lemma:

$$\int\limits_{S_r(\eta)} F(a_t\xi) d\omega(\xi) = 0,$$

where $F(x) = f\big(x/|x|\big)$. Differentiating with respect to t and setting $t = 0$, we obtain

$$\int\limits_{S_r(\eta)} h(\xi) d\omega(\xi) = 0,$$

where

$$h(\xi) = \xi_{n+1} \frac{\partial F}{\partial x_n}(\xi) - \xi_n \frac{\partial F}{\partial x_{n+1}}(\xi), \quad \xi \in B_{a,b}.$$

Hence, $h \in \mathcal{Z}^{s-1}(B_{a,b})$, which completes the proof of Lemma 5 because

$$h^\circ(\theta_1, \ldots, \theta_n) = -\sin\theta_{n-1} \cot\theta_n \frac{\partial f^\circ}{\partial \theta_{n-1}} + \cos\theta_{n-1} \frac{\partial f^\circ}{\partial \theta_n}.$$

Lemma 6.

1. *Let* $n \geq 3,\ s \in \mathbb{N},$ *and* $u(\theta_n)Y(\sigma) \in \mathcal{Z}^s(B_{a,b})$ *for some* $Y \in \mathcal{H}_k \setminus \{0\}$. *Then:*
   - *(a)* $\big(D_k\, u\big)(\theta_n)Y_l^{(k+1)}(\sigma) \in \mathcal{Z}^{s-1}(B_{a,b})$ *for all* $1 \leq l \leq a_{k+1}$;
   - *(b)* *if* $k \in \mathbb{N}$, *then* $\big(D_{2-k-n}\, u\big)(\theta_n)Y_l^{(k-1)}(\sigma) \in \mathcal{Z}^{s-1}(B_{a,b})$ *for all* $1 \leq l \leq a_{k-1}$.
2. *Let* $n = 2,\ s \in \mathbb{N},$ *and* $u(\theta_2)Y_l^{(k)}(\sigma) \in \mathcal{Z}^s(B_{a,b})$ *for some* $k \in \mathbb{Z}_+,\ l \in \{1, \dots, a_k\}$. *Then:*
   - *(a)* *if* $k \in \mathbb{N}$, *then* $\big(D_{\pm k}\, u\big)(\theta_2)Y_l^{(k\pm1)}(\sigma) \in \mathcal{Z}^{s-1}(B_{a,b})$;
   - *(b)* *if* $k = 0$, *then* $u'(\theta_2)Y_p^{(1)}(\sigma) \in \mathcal{Z}^{s-1}(B_{a,b})$ *for any* $p \in \{1, 2\}$.

Proof. 1. Since

$$\sin^k \theta_{n-1} \dots \sin^k \theta_2 e^{ik\theta_1} \in \mathcal{H}_k$$

(see [17], Chap. 9, Sec. 3.6), by using the condition and Lemma 4 (Sec. 2), we get

$$u(\theta_n) \sin^k \theta_{n-1} \dots \sin^k \theta_2 e^{ik\theta_1} \in \mathcal{Z}^s(B_{a,b}).$$

Then, by Lemma 5,

$$\big(D_k\, u\big)(\theta_n) \cos\theta_{n-1} \sin^k \theta_{n-1} \dots \sin^k \theta_2 e^{ik\theta_1} \in \mathcal{Z}^{s-1}(B_{a,b}).$$

By using

$$\cos\theta_{n-1} \sin^k \theta_{n-1} \dots \sin^k \theta_2 e^{ik\theta_1} \in \mathcal{H}_{k+1}$$

(see [17], Chap. 9, Sec. 3.6) and Lemma 4 (Sec. 2), we obtain assertion (a). We prove assertion (b). As above,

$$u(\theta_n) C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{Z}^s(B_{a,b}),$$

where $C_k^{\frac{n-2}{2}}$ is the Gegenbauer polynomial of degree k with index $\dfrac{n-2}{2}$. Applying Lemma 5 to this function and using the relations

$$\frac{d}{dt} C_m^{\alpha}(t) = 2\alpha C_{m-1}^{\alpha+1}(t),$$

$$(m+1)C_{m+1}^{\alpha}(t) = (2\alpha + m)tC_m^{\alpha}(t) - 2\alpha(1-t^2)C_{m-1}^{\alpha+1}(t)$$

(see [17], Chap. 9, Sec. 3.2), we obtain

$$\big(D_{2-k-n}\, u\big)(\theta_n) \cos\theta_{n-1} C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) - (k+1)u(\theta_n) \cot\theta_n C_{k+1}^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{Z}^{s-1}(B_{a,b}).$$

Since

$$C_{k+1}^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{H}_{k+1}, \quad \cos\theta_{n-1} C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{H}_{k-1} \setminus \{0\} + \mathcal{H}_{k+1}$$

{see [17], Chap. 9, Sec. 2.3, relation (5)}, by using this relation and Lemma 4 (Sec. 2), we obtain assertion (b).

2. Setting $h(\theta_1,\theta_2) = u(\theta_2)e^{im\theta_1}$, $m \in \mathbb{Z}_+$, we get

$$\cos\theta_1 \frac{\partial h}{\partial \theta_2} - \sin\theta_1 \cot\theta_2 \frac{\partial h}{\partial \theta_1} = \frac{1}{2}\big(D_m\, u\big)(\theta_2)e^{i(m+1)\theta_1} + \frac{1}{2}\big(D_{-m}\, u\big)(\theta_2)e^{i(m-1)\theta_1}.$$

Now assertion 2 of the lemma follows from Lemmas 5 and 4 (Sec. 1) and relations (4).

5. Proofs of the Main Results

Proof of Theorem 1.

Necessity. Let $f \in \mathcal{Z}(B_{a,b})$. By using Lemma 4 for $k = 0$, we obtain $f_{0,1}(\theta_n) \in \mathcal{Z}(B_{a,b})$. Then, by the definition of the class $\mathcal{Z}(B_{a,b})$,

$$\int\limits_{S_r} f_{0,1}(\theta_n)d\omega(\xi) = 0 \qquad \text{for any} \quad r \in (a,b),$$

which is equivalent to equality (5). By using Lemmas 4 and 6, we easily obtain representation (6) for smooth f. The general case is reduced to the considered case by smoothing the function f with the help of the convolutions $f \times \varphi_\varepsilon$, where φ_ε is a radial function of the class $C^\infty(\mathbb{S}^n)$ with support in the ball B_ε.

Sufficiency. Let $f \in C(B_{a,b})$ and let the Fourier coefficients of f have the form (5), (6). Then, by Lemma 3,

$$f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in \mathcal{Z}(B_{a,b}) \qquad \text{for all} \quad k \in \mathbb{Z}_+, \quad 1 \le l \le a_k. \tag{22}$$

For $r \in (a,b)$, we set

$$I(\eta) = \int\limits_{S_r(\eta)} f(\xi)d\omega(\xi), \quad \eta \in B_{\min\{r-a,b-r\}}.$$

By using (20)–(22), we obtain

$$\int\limits_{SO(n)} I(\tau^{-1}\eta)\overline{t^k_{l,l}(\tau)}d\tau = \int\limits_{SO(n)}\int\limits_{S_r(\eta)} f(\tau^{-1}\xi)d\omega(\xi)\overline{t^k_{l,l}(\tau)}d\tau$$
$$= \int\limits_{S_r(\eta)}\int\limits_{SO(n)} f(\tau^{-1}\xi)\overline{t^k_{l,l}(\tau)}d\tau d\omega(\xi) = 0.$$

In view of the completeness of the system $\left\{Y_l^{(k)}\right\}$ in $L^2(\mathbb{S}^{n-1})$ (see [12], Chap. 4, Sec. 2), this yields $I \equiv 0$. Hence, $f \in \mathcal{Z}(B_{a,b})$.

Proof of Theorem 2. Assume that f satisfies the condition of assertion 1 of the theorem. Then each term of series (3) has this property because, for any $\xi \in B_{a,\pi}$,

$$(1+\xi_{n+1})^{-m}\Big|f_{k,l}(\theta_n)Y_l^{(k)}(\sigma)\Big| \le a_k \int\limits_{SO(n)} (1+\xi_{n+1})^{-m}\big|f(\tau^{-1}\xi)\big|d\tau$$

$$= a_k \int\limits_{SO(n)} (1 + (\tau^{-1}\xi)_{n+1})^{-m} \big| f(\tau^{-1}\xi) \big| d\tau$$

$$\leq a_k \sup_{\eta \in B_{a,\pi}} (1 + \eta_{n+1})^{-m} \big| f(\eta) \big|$$

$\{$see relations (20) and (21) and [17], Chap. 1, Sec. 1.5, relation (3)$\}$. In particular,

$$\sup_{\theta_n \in (a,\pi)} (1 + \cos\theta_n)^{-m} \big| f_{k,l}(\theta_n) \big| < \infty \qquad \text{for any} \quad m \in \mathbb{Z}_+.$$

In addition, by Theorem 1, $f_{k,l}$ have the form (5), (6). This implies that all $f_{k,l}$ are equal to zero on (a,π) and, hence, f is a zero function.

Finally, by Lemma 3, the function

$$f(\xi) = \frac{(1 + \cos\theta_n)^{2m+n-1}}{(\sin\theta_n)^{2m+2n-2}} Y_1^{(2m+n)}(\sigma), \quad \xi \in B_{a,\pi},$$

satisfies all requirements of the second assertion of the theorem.

Proof of Theorems 3 and 4. The first assertions of these theorems are established in the same way as in Theorem 2. The following functions:

$$f(\xi) = \frac{(\cos\theta_n - \cos r_1)\ldots(\cos\theta_n - \cos r_p)}{(\sin\theta_n)^{n+p-1}} Y_1^{(p+1)}(\sigma), \qquad \text{where} \qquad E = \{r_1, \ldots, r_p\},$$

$$f(\xi) = \frac{(\cos\theta_n - \cos r)^{s+1}}{(\sin\theta_n)^{n+s}} Y_1^{(s+2)}(\sigma),$$

(see Lemma 3) satisfy all requirements of the second assertion.

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