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OPTIMAL CONTROL OVER MOVING SOURCES IN THE HEAT EQUATION

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We study the problem of optimal control over the processes described by the heat equation and a system of ordinary differential equations. For the problem of optimal control, we prove the existence and uniqueness of solutions, establish sufficient conditions for the Fréchet differentiability of the purpose functional, deduce the expression for its gradient, and obtain necessary conditions of optimality in the form of an integral maximum principle.

1. Introduction

Practical examples of moving sources are electron, laser, and ion beams, electric arcs, and inductive currents excited by moving inductors. These sources are used in numerous processes, e.g., in the course of melting and refinement of metals in metallurgy, in the processes of thermal processing, welding, microtreatment in machine building and instrument making, in the processes of production of semiconductors and resistors in microelectronics, etc.

As one of the main specific features of systems of optimal control over moving sources, we can mention their nonlinear dependence on the control specifying the law of motion of the source. This is especially clear if the problem of control is represented as the problem of moments, which becomes nonlinear. Hence, the method of moments widely used for the determination of optimal controls in linear systems with distributed and concentrated parameters becomes unsuitable for the systems of control over moving sources.

In [1, 2], one can find numerous examples of systems with moving sources of various types together with the analysis of their specific features that make the investigation of these systems by the already existing methods (e.g., by the method of moments) impossible. In [3, 5–9], the problems of optimal control of point sources were considered for a parabolic equation under the condition that the intensity of fixed sources is the sole controlled parameter. The problems of controllability of linear systems with generalized action were investigated in [4]. In [10, 11], the variational method for the solution of the problem of optimal control over moving sources is considered for systems described only by the heat equation.

In addition, the cited works deal only with the systems with distributed parameters. At the same time, in the construction of mathematical models of various dynamical systems, it is necessary to take into account auxiliary elements without which it is impossible to realize control over the analyzed process. As a rule, these elements have concentrated parameters. The behavior of these systems is described by a collection of ordinary and partial differential equations with initial and boundary conditions.

In the present paper, we consider the variational method for the solution of the problem of optimal control over moving sources in the form of the heat equation and a system of ordinary differential equations with initial and boundary conditions. For this problem, we prove the theorem on existence and uniqueness of the solution, establish sufficient conditions for the Fréchet differentiability of the objective functional, and deduce the expres-

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sion for its gradient. We also establish a necessary condition of optimality in the form of an integral maximum principle.

2. Statement of the Problem

Denote $\Omega_t = (0, l) \times (0, t)$, $\Omega = \Omega_T$, where $l > 0$ and $T > 0$ are numbers. Assume that a controlled process is described in the domain Ω by the initial boundary-value problem

$$u_t = a^2 u_{xx} + \sum_{i=1}^{n} p_i(t)\delta(x - s_i(t)), \quad (x, t) \in \Omega, \tag{1}$$

$$u_x \big|_{x=0} = 0, \quad u_x \big|_{x=l} = 0, \quad 0 < t \le T, \tag{2}$$

$$u(x, 0) = \phi(x), \quad 0 \le x \le l, \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta function and $a > 0$ is a given number.

Assume that the following condition is satisfied:

(A) $\phi(x) \in L_2(0, l)$ is an initial function, $p(t) \in L_2(0, T; R^n)$ is a vector function of the form $p(t) = (p_1(t), p_2(t), \dots, p_n(t))$, and a vector function $s(t) \in C([0, T], R^n)$ is the solution of the Cauchy problem [12, pp. 91, 92]

$$\dot{s} = f(s, q(t), t), \quad 0 < t \le T, \quad s(0) = s_0, \tag{4}$$

where $s_0 \in [0, l]$ is a given number, $q(t) \in L_2(0, T; R^m)$ is a continuously differentiable vector function such that the following restriction on the position of a moving action is satisfied: $0 \le s(t) \le l$; the vector function $f(s, q(t), t)$ is known and, for any s, belongs to the space $L_2(0, T; R^n)$.

A pair of functions $\vartheta = (p(t), q(t))$ is called control. For the sake of brevity, by

$$H = L_2(0, T; R^n) \times L_2(0, T; R^m)$$

we denote a Hilbert space of pairs $\vartheta = (p(t), q(t))$ with the scalar product

$$\left\langle \vartheta^1, \vartheta^2 \right\rangle_H = \int_0^T [(p^1(t), p^2(t)) + (q^1(t), q^2(t))]\, dt$$

and the norm

$$\|\vartheta\|_H = \sqrt{\langle \vartheta, \vartheta \rangle_H} = \sqrt{\|p\|_{L_2}^2 + \|q\|_{L_2}^2},$$

where $\vartheta^k = (p^k, q^k)$, $k = 1, 2$.

In what follows, to emphasize that the solutions of problems (1)–(3) and (4) depend on control, we sometimes use the notation $u(x, t, \vartheta)$ and $s(t, q)$.

We introduce the set of admissible controls as follows:

$$V = \left\{ \vartheta = (p, q) \in H: \ 0 \le p_i \le A_i, \left| q_j \right| \le B_j, \ i = \overline{1, n}, \ j = \overline{1, m} \right\}, \tag{5}$$

where $A_i > 0$, $i = \overline{1, n}$, and $B_j > 0$, $j = \overline{1, m}$, are given numbers.

Consider the following problem: To determine an admissible control $\vartheta = (p(t), q(t)) \in V$ and the corresponding solution $(u(x, t), s(t))$ of problem (1)–(4) for which the functional

$$J(\vartheta) = \int_0^l \int_0^T [u(x,t) - u_0(x,t)]^2 \, dx \, dt$$

$$+ \alpha_1 \sum_{i=1}^n \int_0^T [p_i(t) - \tilde{p}_i(t)]^2 \, dt + \alpha_2 \sum_{j=1}^m \int_0^T [q_j(t) - \tilde{q}_j(t)]^2 \, dt, \tag{6}$$

where $u_0(x, t) \in L_2(\Omega)$; $\omega = (\tilde{p}(t), \tilde{q}(t)) \in H$, $\tilde{p}(t) \in L_2(0, T; R^n)$, and $\tilde{q}(t) \in L_2(0, T; R^m)$ are given vector functions and $\alpha_1, \alpha_2 \ge 0$, $\alpha_1 + \alpha_2 > 0$, are given parameters, takes the minimum possible value.

In what follows, we use the functional spaces $W_2^{1,0}(\Omega)$, $W_2^{1,1}(\Omega)$, and $V_2^{1,0}(\Omega)$. For the definitions of these spaces, see, e.g., [13].

3. Existence and Uniqueness of the Solution

Definition 1. *A solution of problem (1)–(4) for a given control* $\vartheta = (p(t), q(t)) \in V$ *is defined as a pair of functions* $(u, s) = (u(x, t, \vartheta), s(t, q))$, *where the function* $u \in V_2^{1,0}(\Omega)$ *satisfies the integral identity*

$$\int_0^l \int_0^T [-u\eta_t + a^2 u_x \eta_x] \, dx \, dt = \int_0^l \phi(x) \eta(x, 0) \, dx + \sum_{i=1}^n \int_0^T p_i(t) \eta(s_i(t), t) \, dt \tag{7}$$

for all $\eta = \eta(x, t) \in W_2^{1,1}(\Omega)$, $\eta(x, T) = 0$, *and the function* $s \in C([0, T], R^n)$ *satisfies the integral equation*

$$s(t) = \int_0^t f(s, q(\tau), \tau) \, d\tau + s_0, \quad 0 \le t \le T. \tag{8}$$

Note that the existence of a unique generalized solution from $V_2^{1,0}(\Omega)$, for fixed control $\vartheta \in V$, of the boundary-value problem (1)–(4) follows from the results in [14, pp. 265–270]. In what follows, we use this fact.

The aim of the present paper is to study the problem of optimal control (1)–(6). In what follows, it is always assumed that the solution of problem (1)–(4) exists and is unique. As a corollary, we obtain that, under the condition (A), problem (1)–(6) has at least one solution. Note that, in the case where $\alpha_j = 0$, $j = \overline{1, 2}$, problem (1)–(6) is ill-posed in the classical sense [15].

The following theorem is true:

Theorem 1. *Under the condition (A), there exists a dense subset* K *of the space* H *such that, for any* $\omega \in K$, *for* $\alpha_i > 0$, $i = \overline{1,2}$, *the problem of optimal control (1)–(6) is uniquely solvable*.

Proof. We prove the continuity of the functional

$$J_0(\vartheta) = \|u(x,t) - u_0(x,t)\|^2_{L_2(\Omega)}.$$

Let $\delta\vartheta = (\delta p, \delta q) \in V$ be an increment of control on an element $\vartheta \in V$ satisfying the condition $\vartheta + \delta\vartheta \in V$. We denote

$$\delta u \equiv \delta u(x,t) = u(x,t;\vartheta + \delta\vartheta) - u(x,t;\vartheta),$$

$$\delta s_i \equiv \delta s_i(t) = s_i(t; q + \delta q) - s_i(t;q).$$

It follows from (1)–(4) that the function δu is a generalized solution of the boundary-value problem

$$\delta u_t = a^2 \delta u_{xx} + \sum_{i=1}^{n} [(p_i + \delta p_i)\delta(x - (s_i + \delta s_i)) - p_i \delta(x - s_i)], \quad (x,t) \in \Omega, \tag{9}$$

$$\delta u_x\big|_{x=0} = \delta u_x\big|_{x=l} = 0, \quad t \in [0,T], \tag{10}$$

$$\delta u\big|_{t=0} = 0, \quad x \in [0,l], \tag{11}$$

and the functions δs_i, $i = \overline{1,n}$, are the solutions of the Cauchy problem

$$\dot{\delta s}_i(t) = \delta f_i(s,q,t), \qquad \delta s_i(0) = 0, \quad i = \overline{1,n}, \tag{12}$$

where

$$\delta f_i(s,q,t) = f_i(s + \delta s, q + \delta q, t) - f_i(s,q,t).$$

We prove that the function $\delta u = \delta u(x,t)$ admits the estimate

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \le c_1 \|\delta\vartheta\|_H, \tag{13}$$

where $c_1 > 0$ is a constant.

Multiplying both sides of Eq. (9) by $\eta = \eta(x,t)$ and integrating over the domain Ω by parts, we arrive at the relation

$$\int_0^l \int_0^T \left(\delta u_t \eta + a^2 \delta u_x \eta_x \right) dx\, dt = \sum_{i=1}^{n} \int_0^T [\,(p_i + \delta p_i)\eta(s_i + \delta s_i, t) - p_i \eta(s_i, t)\,]\, dt. \tag{14}$$

Let $t_1, t_2 \in [0, T]$ be such that $t_1 \le t_2$. In identity (14), we set

$$\eta(x,t) = \begin{cases} \delta u(x,t), & t \in (t_1, t_2], \\ 0, & t \in [0, t_1] \cup (t_2, T]. \end{cases}$$

Applying the formulas of finite increments to the function $\delta u(s_i + \delta s_i, t)$, $i = \overline{1, n}$, in the form

$$\delta u(s_i + \delta s_i, t) = \delta u(s_i, t) + \delta u_x(\bar{s}_i, t)\delta s_i, \quad \text{where} \quad \bar{s}_i = s_i + \theta \delta s_i, \quad \theta \in [0, 1],$$

we obtain the equation of energy balance for problem (9)–(12):

$$\frac{1}{2} \|\delta u(x,t)\|^2_{L_2(0,l)} \Big|_{t=t_1}^{t=t_2} + a^2 \|\delta u_x(x,t)\|^2_{L_2(\Omega_t)} \Big|_{t=t_1}^{t=t_2}$$

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} [(p_i + \delta p_i)\delta s_i \, \delta u_x(\bar{s}_i, t) + \delta p_i \, \delta u(s_i, t)] dt. \quad (15)$$

By using the Cauchy–Buniakowski inequality on the right-hand side of this equation, we arrive at the inequality

$$\frac{1}{2} \|\delta u(x,t)\|^2_{L_2(0,l)} \Big|_{t=t_1}^{t=t_2} + a^2 \|\delta u_x(x,t)\|^2_{L_2(\Omega_t)} \Big|_{t=t_1}^{t=t_2}$$

$$\le \sum_{i=1}^{n} \Big[\Big(\|p_i\|_{L_2(t_1,t_2)} + \|\delta p_i\|_{L_2(t_1,t_2)} \Big) \max_{t_1 \le t \le t_2} |\delta s_i(t)| \, \|\delta u_x(\bar{s}_i, t)\|_{L_2(t_1,t_2)}$$

$$+ \|\delta p_i\|_{L_2(t_1,t_2)} \|\delta u(s_i, t)\|_{L_2(t_1,t_2)} \Big]. \quad (16)$$

The functions $\delta s_i(t)$, $i = \overline{1, n}$, as solutions of the Cauchy problem (12), satisfy the inequality [12, p. 94]

$$\|\delta s_i(t)\|_{C[t_1,t_2]} \le c_2 \|\delta q(t)\|_{L_2(t_1,t_2)}, \quad i = \overline{1, n},$$

for sufficiently small values of the quantity $\varepsilon = t_2 - t_1$,

It is easy to see that the following inequality is true for the function $u(x, t)$:

$$\|\delta u(s_i, t)\|_{L_2(t_1,t_2)} \le c_3 \|\delta u\|_{V_2^{1,0}(\Omega)}, \quad \|\delta u_x(\bar{s}_i, t)\|_{L_2(t_1,t_2)} \le c_4 \|\delta u\|_{V_2^{1,0}(\Omega)},$$

where $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$ are constants.

Hence, as

$$\|\delta\vartheta\|_{L_2(t_1,t_2)} \to 0,$$

inequality (16) yields the following inequality:

$$\frac{1}{2}\|\delta u(x,t)\|^2_{L_2(0,l)}\Big|^{t=t_2}_{t=t_1} + a^2\|\delta u_x(x,t)\|^2_{L_2(\Omega_t)}\Big|^{t=t_2}_{t=t_1} \le c_5\|\delta\vartheta\|_{L_2(t_1,t_2)}\|\delta u\|_{V_2^{1,0}(\Omega)}, \tag{17}$$

where $c_5 > 0$ is a constant.

By analogy with [16, pp. 166–168], for any $t \in [0, T]$, we split $[0, t]$ into finitely many intervals of sufficiently small length such that an inequality of the form (17) holds in each of these intervals. We now add these inequalities (for each interval). This yields

$$\frac{1}{2}\|\delta u(x,t)\|^2_{L_2(0,l)} + a^2\|\delta u_x(x,t)\|^2_{L_2(\Omega)} \le c_5\|\delta\vartheta\|_H\|\delta u\|_{V_2^{1,0}(\Omega)}$$

and, hence, we immediately arrive at inequality (13).

Thus, we get

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \to 0 \quad \text{as} \quad \|\delta\vartheta\|_H \to 0.$$

By using this result and the theorem on traces [17], we conclude that

$$\|\delta u(x,t)\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \|\delta\vartheta\|_H \to 0.$$

Since the increment of the functional $J_0(\vartheta)$ can be represented in the form

$$\delta J_0(\vartheta) = J_0(\vartheta + \delta\vartheta) - J_0(\vartheta)$$

$$= 2\int_0^l\int_0^T [u(x,t) - u_0(x,t)]\,\delta u(x,t)\,dx\,dt + \|\delta u(x,t)\|^2_{L_2(\Omega)},$$

the continuity of the functional $J_0(\vartheta)$ is proved.

Further, since the functional $J_0(\vartheta)$ is bounded below and continuous in V and H is a uniformly convex reflexive Banach space [18], by the Bidaut theorem [19], there exists a dense subset K of the space H such that, for any $\omega = (\tilde{p}(t), \tilde{q}(t)) \in H$ and $\alpha_i > 0$, $i = \overline{1, 2}$, the problem of optimal control (1)–(6) is uniquely solvable.

Theorem 1 is proved.

4. Necessary Condition of Optimality

For the problem of optimal control (1)–(6), we introduce the dual state $\psi = \psi(x,t)$ as a solution of the problem

$$\psi_t + a^2\psi_{xx} = -2[u(x,t) - \tilde{u}(x,t)], \quad (x,t) \in \Omega, \tag{18}$$

$$\psi_x\big|_{x=0} = \psi_x\big|_{x=l} = 0, \quad 0 \le t < T, \tag{19}$$

$$\psi(x,T) = 0, \quad 0 \le x \le l, \tag{20}$$

and functions $\xi_i(t)$, $i=\overline{1,n}$, as solutions of the dual system of equations

$$\dot{\xi}_i(t) = -\sum_{k=1}^{n} \frac{\partial f_k}{\partial s_i} \xi_k(t) + p_i(t)\psi_x(s_i(t),t), \quad 0 \le t < T, \quad \xi_i(T)=0, \quad i=\overline{1,n}. \tag{21}$$

The dual problems (18)–(20) and (21) are obtained according to the ordinary scheme [12, pp. 91–93, 128, 129].

Definition 2. *A generalized solution of problem (18)–(21) for the control* $\vartheta=(p(t),q(t))\in H$ *is defined as a pair of functions* $(\psi,\xi)=(\psi(x,t),\xi(t))$, *where the function* $\psi \in W_2^{1,1}(\Omega)$ *satisfies the integral identity*

$$\int_0^l \int_0^T \left(\psi\eta_{1t} + a^2\psi_x\eta_{1x} \right) dx\,dt = 2\int_0^l \int_0^T [\,u(x,t)-u_0(x,t)\,]\,\eta_1(x,t)\,dx\,dt \tag{22}$$

for all $\eta_1=\eta_1(x,t)\in W_2^{1,1}(\Omega)$, $\eta_1(x,0)=0$, *and the functions* $\xi_i \in C([0,T],R^n)$, $i=\overline{1,\,n}$, *satisfy the integral equation*

$$\xi_i(t) = \int_t^T \left[\sum_{k=1}^{n} \frac{\partial f_k}{\partial s_i}\xi_k(\tau) - p_i(\tau)\psi_x(s_i(\tau),\tau) \right] d\tau, \quad 0 \le t \le T, \quad i=\overline{1,\,n}. \tag{23}$$

The function

$$H(t,s,\psi,q,\vartheta) = -\left\{ \sum_{i=1}^{n} \Big[-f_i(s(t),q(t),t)\xi_i(t) + \psi(s_i(t),t)p_i(t) + \alpha_1\big(p_i(t)-\tilde{p}_i(t)\big)^2 \Big] + \alpha_2 \sum_{j=1}^{m} \big(q_j(t)-\tilde{q}_j(t)\big)^2 \right\} \tag{24}$$

is called the Hamilton–Pontryagin function of problem (1)–(6).

Theorem 2. *Assume that the following conditions are satisfied:*

(i) a function $f(s,q,t)$ *is defined and continuous in a collection of its arguments and has continuous bounded partial derivatives with respect to the variables* s *and* q *for* $(s,q,t)\in R^n \times R^m \times [0,T]$;

(ii) functions

$$f(s,q,t), \quad f_s = \frac{\partial f(s,q,t)}{\partial s}, \quad \text{and} \quad f_q = \frac{\partial f(s,q,t)}{\partial q}$$

satisfy the Lipschitz condition with respect to the variables s *and* q, *i.e.,*

$$|\,f(s+\delta s_s, q+\delta q, t) - f(s,q,t)\,| \le L\big(|\,\delta s\,| + |\,\delta q\,|\big),$$

$$|f_s(s+\delta s, q+\delta q, t) - f_s(s, q, t)| \le L(|\delta s| + |\delta q|),$$

$$\left|f_q(s+\delta s, q+\delta q, t) - f_q(s, q, t)\right| \le L(|\delta s| + |\delta q|)$$

for all $(s+\delta s, q+\delta q, t)$, $(s, q, t) \in R^n \times R^m \times [0, T]$, *where* $L = \text{const} \ge 0$ *and* δs *and* δq *are the increments of the variables* s *and* q, *respectively.*

If $(\psi(x,t), \xi(t))$ *is a solution of the adjoint problem (18)–(21), then functional (6) is Fréchet differentiable on the set* V *and its gradient satisfies the relation*

$$J'(\vartheta) = \left(\frac{\partial J(\vartheta)}{\partial p}, \frac{\partial J(\vartheta)}{\partial q}\right) = \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right). \tag{25}$$

Proof. Consider an increment of functional (6). We have

$$\begin{aligned}\delta J &\equiv J(\vartheta+\delta\vartheta) - J(\vartheta)\\ &= 2\int_0^l\int_0^T [u(x,t)-u_0(x,t)]\delta u(x,t)\,dx\,dt + \int_0^l\int_0^T |\delta u(x,t)|^2\,dx\,dt\\ &\quad+\sum_{i=1}^n\left\{2\alpha_1\int_0^T[\,p_i(t)-\tilde p_i(t)\,]\delta p_i(t)\,dt + \alpha_1\int_0^T|\delta p_i(t)|^2\,dt\right\}\\ &\quad+\sum_{j=1}^m\left\{2\alpha_2\int_0^T\left[\,q_j(t)-\tilde q_j(t)\,\right]\delta q_j(t)\,dt + \alpha_2\int_0^T\left|\delta q_j(t)\right|^2\,dt\right\},\end{aligned} \tag{26}$$

where $\vartheta = (p, q) \in V$, $\vartheta + \delta\vartheta \in V$ and δp_i and δq_j are the increments of the variables p_i and q_j, respectively.

Setting $\eta_1 = \delta u(x,t)$ in (22) and $\eta = \psi(x,t)$ in (14) and subtracting the relations obtained as a result, we find

$$2\int_0^l\int_0^T[\,u(x,t)-u_0(x,t)\,]\delta u(x,t)\,dx\,dt = \sum_{i=1}^n\int_0^T[\,(p_i+\delta p_i)\psi(s_i+\delta s_i,t) - p_i\psi(s_i,t)\,]dt. \tag{27}$$

Problems (12) and (21) can be rewritten in the form of equivalent integral relations

$$\int_0^T\left[\,\delta s_i(t)\dot\theta_i(t) + \delta f_i(s(t), q(t), t)\theta_i(t)\,\right]dt = 0 \tag{28}$$

for all $\theta_i(t) \in L_2(0,T)$, $\theta_i(T) = 0$, $i = \overline{1,n}$, and

$$\int_0^T \left[\xi_i(t)\dot{\theta}_{1i}(t) - \left(\sum_{k=1}^{n} \frac{\partial f_k}{\partial s_i} \xi_k(t) - p_i(t)\psi_x(s_i(t), t) \right) \theta_{1i}(t) \right] dt = 0 \tag{29}$$

for all $\theta_{1i}(t) \in L_2(0, T)$, $\theta_{1i}(0) = 0$, $i = \overline{1, n}$.

Setting

$$\theta_{1i}(t) = \delta s_i(t) \quad \text{and} \quad \theta_i(t) = \xi_i(t)$$

in these relations and finding the sum of these relations, we obtain

$$[\delta s_i(t)\xi_i(t)]\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{k=1}^{n} \frac{\partial f_k}{\partial s_i} \xi_k(t) - p_i(t)\psi_x(s_i(t), t) \right) \delta s_i(t) - \delta f_i \xi_i(t) \right] dt.$$

Since, according to the conditions of Theorem 2, the increment $\delta f_i = \delta f_i(s, q, t)$ can be represented in the form

$$\delta f_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial s_k} \delta s_k + \sum_{r=1}^{m} \frac{\partial f_i}{\partial q_r} \delta q_r + R_1,$$

where $R_1 = o\left(\sqrt{\|\delta s\|^2_{C[0,T]} + \|\delta q\|^2_{L_2(0,T)}} \right)$,

the last equality yields the following relation:

$$[\delta s_i(t)\xi_i(t)]\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{k=1}^{n} \frac{\partial f_k}{\partial s_i} \xi_k(t) - p_i(t)\psi_x(s_i(t), t)) \right) \delta s_i(t) \right.$$

$$\left. - \sum_{r=1}^{m} \frac{\partial f_i}{\partial q_r} \delta q_r(t)\xi_i(t) - \sum_{k=1}^{n} \frac{\partial f_i}{\partial s_k} \delta s_k(t)\xi_i(t) \right] dt + R_1.$$

In view of (12) and (21), this is equivalent to the equality

$$\int_0^T p_i(t)\psi_x(s_i(t), t)\delta s_i(t)\,dt = -\sum_{r=1}^{m} \int_0^T \frac{\partial f_i}{\partial q_r} \delta q_r(t)\xi_i(t)\,dt$$

$$- \sum_{k=1}^{n} \int_0^t \left[\frac{\partial f_i}{\partial s_k} \xi_i(t)\delta s_k(t) - \frac{\partial f_k}{\partial s_i} \xi_k(t)\delta s_i(t) \right] dt + R_1. \tag{30}$$

According to the Taylor formula, we can write

$$\psi(s_i + \delta s_i, t) = \psi(s_i, t) + \psi_x(s_i, t)\delta s_i + o(\|\delta s_i\|).$$

By using this relation and (27), we obtain

$$2\int_0^l\int_0^T [u(x,t)-u_0(x,t)]\delta u(x,t)\,dx\,dt = \sum_{i=1}^n \int_0^T \Big[\, (p_i(t)\psi_x(s_i(t),t)\delta s_i(t)$$

$$+\psi(s_i(t),t)\delta p_i(t)+\psi_x(s_i(t),t)\delta p_i(t)\delta s_i(t)+o\left(\|\delta s_i\|\right)\Big]\,dt.$$

Since

$$\sum_{i=1}^n \sum_{k=1}^n \left[\frac{\partial f_i}{\partial s_k}\,\xi_i(t)\delta s_k(t) - \frac{\partial f_k}{\partial s_i}\,\xi_k(t)\delta s_i(t) \right] = 0,$$

we get the following relation from the last equality and relation (30):

$$2\int_0^l\int_0^T [u(x,t)-u_0(x,t)]\delta u(x,t)\,dx\,dt = \sum_{i=1}^n \int_0^T \left[-\sum_{r=1}^m \frac{\partial f_i}{\partial q_r}\xi_i(t)\delta q_r(t) + \psi(s_i,t)\delta p_i \right] dt + R_2, \tag{31}$$

where

$$R_2 = \sum_{i=1}^n \int_0^T \Big[\, \psi_x(s_i(t),t)\delta p_i(t)\delta s_i(t) + o\left(\|\delta s_i\|\right)\Big]dt + R_1.$$

By using the standard scheme [12, p. 94], we can prove the estimate

$$\|\delta s\|_{C[0,T]} \le c_6\,\|\delta q\|_{L_2(0,T)}, \tag{32}$$

where $c_6 > 0$ is a constant.

This yields the relation

$$R_2 = o\left(\|\delta\vartheta\|_H\right).$$

On the other hand, inequality (13) implies the equality

$$\|\delta u(x,t)\|_{L_2(\Omega)} = O\left(\|\delta\vartheta\|_H\right).$$

Substituting the obtained relations in (26), we find

$$\delta J(\vartheta) = \sum_{i=1}^n \left(J_1(i) + \sum_{j=1}^m J_2(i,j) \right) + o\left(\|\delta\vartheta\|_H\right) \quad \text{as} \quad \|\delta\vartheta\|_H \to 0,$$

where

$$J_1(i) = \int_0^T \Big[\, \psi(s_i(t),t) + 2\alpha_1\left(p_i(t) - \tilde{p}_i(t) \right) \Big]\delta p_i(t)\,dt,$$

$$J_2(i,j) = \int_0^T \left[-\frac{\partial f_i(s(t), q(t), t)}{\partial q_j} \xi_i(t) + 2\alpha_2 \left(q_j(t) - \tilde{q}_j(t) \right) \right] \delta q_j(t) dt, \quad i = \overline{1,n}, \quad j = \overline{1,m}.$$

In view of (24), we can write

$$\delta J(\vartheta) = \left(-\frac{\partial H}{\partial \vartheta}, \delta\vartheta \right)_H + o\left(\|\delta\vartheta\|_H \right) \quad \text{as} \quad \|\delta\vartheta\|_H \to 0.$$

This proves the Fréchet differentiability of functional (6) and relation (25).

Theorem 2 is proved.

Theorem 3. *Assume that all conditions of Theorem 2 are satisfied. Then, for the optimality of control* $\vartheta^* = (p^*(t), q^*(t)) \in V$, *it is necessary that the conditions*

$$\langle J'(\vartheta^*), \vartheta - \vartheta^* \rangle_H = \int_0^T \sum_{i=1}^n \left[\left(\psi^*(s_i^*(t), t) + 2\alpha_1 (p_i^*(t) - \tilde{p}_i(t)), p_i(t) - p_i^*(t) \right) \right.$$
$$\left. + \sum_{j=1}^m \left(-\frac{\partial f_i(s^*(t), \vartheta^*(t), t)}{\partial q_j} \xi_i^*(t) + 2\alpha_2 (q_j^*(t) - \tilde{q}_j(t)), q_j(t) - q_j^*(t) \right) \right] dt \geq 0 \quad (33)$$

be satisfied for all $\vartheta = (p(t), q(t)) \in V$, *where* $\psi^*(s_i^*(t), t)$ *and* $\xi_i^*(t)$ *are the solutions of problems (18)–(20) and (21) for* $\vartheta = \vartheta^*$.

Proof. The proof of the theorem does not encounter any serious difficulties. For the optimality of control $\vartheta^* = (p^*(t), q^*(t)) \in V$, it is necessary [12, p. 28] to guarantee the validity of the inequality

$$\langle J'(\vartheta^*), \vartheta - \vartheta^* \rangle_H \geq 0 \quad \forall \vartheta \in V. \quad (34)$$

We determine the gradient of functional (6) and then consider inequality (34). In view of relation (25) and the explicit form of the Hamilton–Pontryagin function, this yields inequality (33).

Theorem 3 is proved.

5. Conclusions

For the problem of optimal control described by the heat equation and a system of ordinary differential equations, we prove the theorem on existence and uniqueness of solution, establish the sufficient conditions for the Fréchet differentiability of the objective functional, and deduce the explicit expression for its gradient. We also obtain the necessary condition of optimality in the form of the integral maximum principle.

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