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ALMOST PERIODIC SOLUTIONS OF NONLINEAR EQUATIONS THAT ARE NOT NECESSARILY ALMOST PERIODIC IN BOCHNER'S SENSE

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We introduce a new class of almost periodic operators and establish conditions for the existence of almost periodic solutions of nonlinear equations that are not necessarily almost periodic in Bochner's sense.

1. Main Notation and Definitions

Let E be a Banach space with the norm $\|\cdot\|_E$ and let $L(X,X)$ be a Banach space of linear continuous operators A acting in a Banach space X with the norm

$$\|A\|_{L(X,X)} = \sup_{\|x\|_X=1} \|Ax\|_X.$$

Also let C^0 be a Banach space of functions $x = x(t)$ bounded and continuous on $\mathbb{R}$ with values in E and the norm

$$\|x\|_{C^0} = \sup_{t\in\mathbb{R}} \|x(t)\|_E$$

and let $R(x)$ be the set of values of the function $x \in C^0$, i.e., the set $\{x(t)\colon t \in \mathbb{R}\}$.

In the space C^0, we define a translation operator $S_h,\ h \in \mathbb{R}$, by the formula

$$(S_h x)(t) = x(t+h), \quad t \in \mathbb{R}. \tag{1}$$

Definition 1. *An element $y \in C^0$ is called almost periodic (in Bochner's sense; see [1, 2]) if the closure of the set $\{S_h y\colon h \in \mathbb{R}\}$ in the space C^0 is a compact subset of the space.*

The set B^0 of almost periodic elements of the space C^0 is a subspace of this space with the norm

$$\|x\|_{B^0} = \|x\|_{C^0}.$$

Definition 2. *The operator $A \in L(C^0, C^0)$ is called almost periodic (in Bochner's sense) if the closure of the set $\{S_h A S_{-h}\colon h \in \mathbb{R}\}$ in the space $L(C^0, C^0)$ is compact in $L(C^0, C^0)$.*

In what follows, to study nonlinear equations, we use a new class of almost periodic operators that are not necessarily almost periodic in Bochner's sense.

This class of operators is defined as follows: We fix an arbitrary open set $D \subset E$, which may coincide with E. By $\mathcal{K}_D$ we denote the set of all nonempty compact subsets $K \subset D$. For a set $D_1 \subset D$, let $\mathfrak{D}_{D_1}$ be the set of all elements $x \in C^0$ for each of which $R(x) \subset D_1$.

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Definition 3. *A mapping* $F\colon \mathfrak{D}_D \to C^0$ *is called almost periodic if, for any set* $K \in \mathcal{K}_D$ *and any sequence* $(h_k)_{k\geqslant 1}$ *of real numbers, there exists a subsequence* $(h_{k_l})_{l\geqslant 1}$ *such that*

$$\lim_{l_1\to\infty,\ l_2\to\infty}\ \sup_{x\in\mathfrak{D}_K}\left\|S_{h_{l_1}}FS_{-h_{l_1}}x - S_{h_{l_2}}FS_{-h_{l_2}}x\right\|_{C^0} = 0.$$

It is clear that every Bochner almost periodic operator $A \in L(C^0, C^0)$ is almost periodic in a sense of Definition 3. It is also obvious that, in the case $D = E$, where the space E is finite-dimensional and the operator $H\colon \mathfrak{D}_D \to C^0$ is linear, Definitions 2 and 3 are equivalent. However, if the space E is infinite-dimensional, then the operator H almost periodic in the sense of Definition 3 is not necessarily almost periodic in the sense of Definition 2 (an example of operator of this kind is presented in the next section).

2. Example of an Operator Almost Periodic According to Definition 3 but Not Almost Periodic in Bochner's Sense

Assume that the set D (see Sec. 1) coincides with the Banach space E and this space is real and infinite-dimensional. By S^0 we denote the set of all elements of the space C^0 for each of which the closure of the set of values in the space E is a compact set. It is clear that $B^0 \subset S^0$ and $x+y, \alpha x \in S^0$ if $x, y \in S^0$ and $\alpha \in \mathbb{R}$. Hence, S^0 is a vector space.

We now show that

$$\overline{S^0} = S^0. \tag{2}$$

This implies that the vector space S^0 is a subspace of the space C^0.

Let x be an arbitrary element of the set $\overline{S^0}$. There exists a sequence $(x_m)_{m\geq 1}$ of elements of the set S^0 for which

$$\lim_{m\to\infty}\|x_m - x\|_{C^0} = 0. \tag{3}$$

We fix an arbitrary number $\varepsilon > 0$. In view of (3), for a certain number $m_0 \in \mathbb{N}$, we get

$$\|x_{m_0} - x\|_{C^0} < \varepsilon. \tag{4}$$

Since $x_{m_0} \in S^0$, the set $\overline{R(x_{m_0})}$ is compact in E. Hence, for this set, there exists a finite ε-grid M. According to (4), the set M is a (2ε)-grid for $\overline{R(x)}$. Thus, in view of the arbitrariness of the choice of the number $\varepsilon > 0$ and the Hausdorff theorem (see [3, p. 47]), the set $\overline{R(x)}$ is compact.

Hence, equality (2) is true and the vector space S^0 is a subspace of the space C^0.

Further, we consider a set $X = \{x_1, x_2, \ldots, x_k, \ldots\} \subset E$ whose elements satisfy the relation

$$\left\|\sum_{l=1}^{p}\beta_l x_{k_l}\right\|_E = \sum_{l=1}^{p}|\beta_l| \tag{5}$$

for any $p \in \mathbb{N}$, real numbers $\beta_1, \ldots, \beta_p$, and different natural numbers $k_1, \ldots, k_p$. The set X with this property exists if, e.g., E is a Banach space of functions $x = x(t)$ bounded and continuous on $\mathbb{R}$ with values in $\mathbb{R}$ and the norm $\|x\|_{C^0} = \sup_{t\in\mathbb{R}}|x(t)|$. As elements $x_1, x_2, \ldots, x_k, \ldots,$ we can take the functions $\sin\lambda_1 t,\ \sin\lambda_2 t,\ \ldots,$ $\sin\lambda_k t,\ \ldots,$ respectively, where the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$ are linearly independent, i.e., for each $m \in \mathbb{N}$, the equality

$$n_1\lambda_1 + n_2\lambda_2 + \ldots + n_m\lambda_m = 0,$$

where $n_1, n_2, \ldots, n_m$ are integers, implies that $n_1 = n_2 = \ldots = n_m = 0$ [2]. It is obvious that the closure of the set X in the space E is not compact in E.

Consider the sums

$$S_m = \sum_{k=1}^{m} \frac{1}{k}, \quad m \in \mathbb{N}.$$

Since

$$\lim_{m\to\infty} S_m = +\infty,$$

we get

$$\mathbb{R} = \bigcup_{k=1}^{\infty} I_k,$$

where $I_1 = (-\infty, 1)$ and $I_k = [S_{k-1}, S_k),\ k \geqslant 2$. We define an element $y = y(t)$ of the space C^0 by the equality

$$y = \sum_{k=1}^{\infty} \chi_{I_k}(t) f_k(t) x_k, \tag{6}$$

where

$$f_1(t) \equiv 1,$$

$$f_k(t) = \begin{cases} 1 & \text{for} \quad \cos\left(k2^{k+2}\pi(t - S_{k-1})\right) \geqslant 4^{-k}, \\ 4^k \cos\left(k2^{k+2}\pi(t - S_{k-1})\right) & \text{for} \quad \left|\cos\left(k2^{k+2}\pi(t - S_{k-1})\right)\right| < 4^{-k}, \quad k \geqslant 2, \\ -1 & \text{for} \quad \cos\left(k2^{k+2}\pi(t - S_{k-1})\right) \leqslant -4^{-k}, \end{cases}$$

and

$$\chi_{I_k}(t) = \begin{cases} 1 & \text{for} \quad t \in I_k, \\ 0 & \text{for} \quad t \in \mathbb{R} \setminus I_k, \end{cases} \qquad k \geq 1.$$

Consider a set

$$Y = \{S_h y \colon h \in \mathbb{R}\},$$

where S_h is the translation operator given by relation (1) and the linear span $\operatorname{span}(Y)$ of this set, i.e., the minimal vector subspace of the space C^0 containing the set Y.

In what follows, we use the following statement:

Lemma. *Let*

$$u = \sum_{l=1}^{p} \beta_l S_{h_l} y$$

be an arbitrary nonzero element of the vector space $\operatorname{span}(Y)$, *where* p *is a natural number,* $\beta_1, \ldots, \beta_p$ *are real nonzero numbers, and* $h_1, \ldots, h_p$ *are real numbers such that* $h_i \neq h_j$ *for* $i \neq j$. *Then:*

(i) for any number $\varepsilon > 0$, *there exist a number* t_ε *and a set* $M_\varepsilon \subset [t_\varepsilon, +\infty)$ *with Lebesgue measure* $\mu(M_\varepsilon)$ *smaller than* ε *such that the relation*

$$\left\| \left(\sum_{l=1}^{p} \beta_l S_{h_l} y \right)(t) \right\|_E = \sum_{l=1}^{p} |\beta_l| \tag{7}$$

is true for all $t \in [t_\varepsilon, +\infty) \setminus M_\varepsilon$;

(ii) the closure of the set of values of the element $u = u(t)$ *in the space* E *is not compact in this space.*

Proof. Note that, for all $t \in \mathbb{R}$,

$$u(t) = \left(\sum_{l=1}^{p} \beta_l S_{h_l} y \right)(t) = \sum_{l=1}^{p} \beta_l y(t + h_l). \tag{8}$$

It follows from the definitions of the functions $y = y(t)$ and $f_k(t)$, $k \geq 1$, and the conditions imposed on the numbers $h_1, \ldots, h_p$ that, for any number $\varepsilon > 0$, there exist a sufficiently large number t_ε and a set $M_\varepsilon \subset [t_\varepsilon, +\infty)$ whose Lebesgue measure $\mu(M_\varepsilon)$ is smaller than ε such that, for every $t \in [t_\varepsilon, +\infty) \setminus M_\varepsilon$, there are different elements $x_{1,t}, \ldots, x_{p,t} \in \{x_1, x_2, \ldots, x_k, \ldots\}$ (that do not coincide with each other) such that

$$y(t + h_l) = \varphi(l, t) x_{l,t}, \quad l = \overline{1, p}, \tag{9}$$

where the scalar product $\varphi(l, t)$ takes values from the set $\{-1, 1\}$ (depending on the values of l and t). In view of (5), (8), and (9), relation (7) is true for all $t \in [t_\varepsilon, +\infty) \setminus M_\varepsilon$, i.e., the first part of the lemma is proved.

To prove the second part of the lemma, we consider an arbitrary increasing sequence $(t_n)_{n \geq 1}$ of elements of the set $[t_\varepsilon, +\infty) \setminus M_\varepsilon$ (here, ε is the same number as in the proof of the first part of the lemma) such that

$$t_{n+1} - t_n > \max_{i \neq j} |h_i - h_j|, \quad n \geq 1.$$

Then the relation

$$\{x_{1,t_i}, \ldots, x_{p,t_i}\} \cap \{x_{1,t_j}, \ldots, x_{p,t_j}\} = \varnothing$$

is true for $i \neq j$. Hence, for any natural i and j $(i \neq j)$ and a nonzero element $u = u(t)$, the following relations are true:

$$\|u(t_i) - u(t_j)\|_E = \left\| \sum_{l=1}^{p} \beta_l y(t_i + h_l) - \sum_{l=1}^{p} \beta_l y(t_j + h_l) \right\|_E$$

$$= \left\| \sum_{l=1}^{p} \beta_l \varphi(l, t_i) x_{l,t_i} - \sum_{l=1}^{p} \beta_l \varphi(l, t_j) x_{l,t_j} \right\|_E = 2 \sum_{l=1}^{p} |\beta_l| > 0.$$

This means that the closure of the set of values of the element $u = u(t)$ in the space E is not a compact set.

The lemma is proved.

We continue the construction of the operator required for our example.

By the lemma, the set of values of each nonzero element

$$u = \sum_{l=1}^{p} \beta_l S_{h_l} y$$

of the vector subspace $\operatorname{span}(Y)$ is not a precompact set, i.e., $u \notin S^0$ for $u \neq 0$.

It is clear that there exists a limit

$$\lim_{t \to -\infty} u(t) = \left(\sum_{l=1}^{p} \beta_l \right) x_1.$$

We now show that nonzero elements of the closure $\overline{\operatorname{span}(Y)}$ of the vector subspace $\operatorname{span}(Y)$ in the space C^0 have similar properties.

Let z be an arbitrary element from $\overline{\operatorname{span}(Y)} \setminus \operatorname{span}(Y)$ and let $(z_k)_{k \geqslant 1}$ be a sequence of elements from $\operatorname{span}(Y)$ for which

$$\lim_{k \to \infty} \|z_k - z\|_{C^0} = 0. \tag{10}$$

Further, we show that the relation

$$\lim_{t \to -\infty} z(t) = \alpha x_1 \tag{11}$$

is true for some number $\alpha \in \mathbb{R}$. Indeed, let

$$\lim_{t \to -\infty} z_k(t) = \alpha_k x_1, \tag{12}$$

where $\alpha_k \in \mathbb{R},\ k \geqslant 1$, and the sequence $(\alpha_k)_{k \geqslant 1}$ is convergent (this requirement does not decrease the generality of our considerations), i.e., for a certain number $\alpha \in \mathbb{R}$,

$$\lim_{k \to \infty} \alpha_k = \alpha. \tag{13}$$

It is clear that, for all $t \in \mathbb{R}$ and $k \geqslant 1$,

$$z(t) = (z(t) - z_k(t)) + (z_k(t) - \alpha_k x_1) + (\alpha_k a - \alpha x_1) + \alpha x_1.$$

Hence,

$$\|z(t) - \alpha x_1\|_E \leqslant \|z(t) - z_k(t)\|_E + \|z_k(t) - \alpha_k x_1\|_E + \|\alpha_k x_1 - \alpha x_1\|_E, \qquad t \in \mathbb{R}, \quad k \geqslant 1.$$

In view of (12), this yields

$$0 \leqslant \limsup_{t\to-\infty} \|z(t) - \alpha x_1\|_E$$

$$\leqslant \limsup_{t\to-\infty} \|z(t) - z_k(t)\|_E + \limsup_{t\to-\infty} \|z_k(t) - \alpha_k x_1\|_E + \limsup_{t\to-\infty} \|\alpha_k x_1 - \alpha x_1\|_E$$

$$\leqslant \|z - z_k\|_{C^0} + \|(\alpha_k - \alpha)x_1\|_E.$$

Since these relations hold for all $k \geqslant 1,$ by virtue of (10) and (13), relation (11) is true.

Further, we show that, for an element

$$z \in \overline{\text{span}(Y)} \setminus \text{span}(Y),$$

the set $\overline{R(z)}$ is not compact in $E.$ Let $(z_k)_{k\geqslant 1}$ be a sequence of elements from $\text{span}(Y)$ for which relation (10) is true. Note that, for any $k \geqslant 1,$ there exist numbers $p_k \in \mathbb{N},$ $\delta_{1,k}, \ldots, \delta_{p_k,k} \in \mathbb{R},$ and $h_{1,k}, \ldots, h_{p_k,k} \in \mathbb{R}$ (the numbers $h_{1,k}, \ldots, h_{p_k,k}$ are pairwise different) such that the element $z_k = z_k(t)$ can be rewritten in the form

$$z_k(t) = \sum_{l=1}^{p_k} \delta_{l,k} y(t + h_{l,k}).$$

Hence, by virtue of the lemma,

$$\|z_k\|_{C^0} = \sum_{l=1}^{p_k} |\delta_{l,k}|, \quad k \geqslant 1.$$

As in the proof of the second part of the lemma, for any $k \geqslant 1,$ there exists an increasing sequence $(t_{k,n})_{n\geqslant 1}$ for which $\lim_{n\to\infty} t_{k,n} = +\infty$ such that

$$\big\|z_k(t_{k,i}) - z_k(t_{k,j})\big\|_E \geqslant 2\|z_k\|_{C^0}$$

for all natural different natural numbers i and $j.$ It follows from these inequalities that

$$\|z(t_{k,i}) - z(t_{k,j})\|_E \geqslant \|z_k(t_{k,i}) - z_k(t_{k,j})\|_E - \|(z(t_{k,i}) - z_k(t_{k,i})) - (z(t_{k,j}) - z_k(t_{k,j}))\|_E$$

$$\geqslant \|z_k(t_{k,i}) - z_k(t_{k,j})\|_E - \|z(t_{k,i}) - z_k(t_{k,i})\|_E - \|z(t_{k,j}) - z_k(t_{k,j})\|_E$$

$$\geqslant 2\|z_k\|_{C^0} - 2\|z - z_k\|_{C^0}, \qquad k \geqslant 1, \quad i \neq j.$$

Hence, these inequalities, the inclusion

$$z \in \overline{\text{span}(Y)} \setminus \text{span}(Y),$$

and relation (10), imply that, for some number $\gamma > 0$ and a sufficiently large natural number $k_0,$ the following inequality is true:

$$\|z(t_{k_0,i}) - z(t_{k_0,j})\|_E \geqslant \gamma,$$

which means that the set $\overline{R(z)}$ is not compact in the space $E.$

Hence, $\overline{\mathrm{span}(Y)}$ is a subspace of the space C^0.

Further, we consider a subspace $L = \mathfrak{S} \oplus \overline{\mathrm{span}(Y)}$ of the space C^0. Note that every element $x \in L$ can be uniquely represented in the form $x = u + v$, where $u \in S^0$ and $v \in \overline{\mathrm{span}(Y)}$. Indeed, if there exist two mappings such that

$$x = u_1 + v_1,$$

$$x = u_2 + v_2 \qquad \left(u_1, u_2 \in S^0,\ v_1, v_2 \in \overline{\mathrm{span}(Y)}\right),$$

then $u_1 + v_1 = u_2 + v_2$ and, hence, for $u_1 = u_2$, we obtain $v_1 = v_2$ and, for $u_1 \neq u_2$, we get the equality $u_1 - u_2 = v_2 - v_1$, which contradicts the inclusion $u_2 - u_1 \in \mathfrak{S}$ because

$$v_2 - v_1 \in \overline{\mathrm{span}(Y)} \setminus \{0\}$$

and the set $S^0 \cap \left(\overline{\mathrm{span}(Y)} \setminus \{0\}\right)$ is empty.

We consider a linear continuous functional

$$\psi\colon \overline{\mathrm{span}\left(\{x_k\colon k \in \mathbb{N}\}\right)} \to \mathbb{R}$$

for which $\psi(x_1) = 1$ and $\|\psi\| = 1$. This functional exists (see, e.g., [4, pp. 176, 177]).

We define a linear functional $\varphi\colon L \to \mathbb{R}$ as follows: every element $x = u + v \in L$, where $u \in S^0$ and $v \in \overline{\mathrm{span}(Y)}$, is associated with the number

$$\varphi(x) = \varphi(u) + \varphi(v),$$

where

$$\varphi(u) = 0$$

and

$$\varphi(u) = \psi\left(\lim_{t \to -\infty} u(t)\right).$$

This functional is continuous in view of the continuity of the functional ψ.

By the Hahn–Banach theorem on the extension of linear continuous functional [4], there exists a linear continuous functional $l\colon C^0 \to \mathbb{R}$ such that $l(x) = \varphi(x)$ for all $x \in L$ and $\|l\| = \|\varphi\|$.

We fix an arbitrary element $s \in C^0 \setminus B^0$ and define a linear continuous operator $C\colon C^0 \to C^0$ by the formula

$$Cx = l(x)s, \quad x \in C^0. \tag{14}$$

We now show that this operator is almost periodic in the sense of Definition 3 and is not almost periodic in the sense of Definition 2.

In view of (14),

$$S_h C S_{-h} x = l(S_{-h} x) S_h s, \quad h \in \mathbb{R}, \tag{15}$$

for every $x \in C^0$ and

$$l(S_{-h}x) = 0, \quad h \in \mathbb{R},$$

for every $x \in S^0$. Therefore, for any compact set $K \subset E$, the closure of the set $\{S_h C S_{-h} x \colon h \in \mathbb{R},\ x \in \mathfrak{D}_K\}$ in the space C^0 is compact in C^0 because this set converges to $\{0\}$. This implies that the operator C is almost periodic in the sense of Definition 3. However, the closure of the set $\{S_h C S_{-h} \colon h \in \mathbb{R}\}$ in the space $L(C^0, C^0)$ is not compact in $L(C^0, C^0)$. Indeed, by virtue of (14) and (15), the element y defined by (6) satisfies the relation

$$S_h C S_{-h} y = S_h s, \quad h \in \mathbb{R},$$

and, hence,

$$\{S_h C S_{-h} \colon h \in \mathbb{R}\} y = \{S_h s \colon h \in \mathbb{R}\}. \tag{16}$$

If the operator C is almost periodic in the sense of Definition 2, i.e., $\{S_h C S_{-h} \colon h \in \mathbb{R}\}$ is a precompact set in the space $L(C^0, C^0)$, then the set $\{S_h C S_{-h} \colon h \in \mathbb{R}\} y$ is precompact in the space C^0. By virtue of equality (16), the set $\{S_h s \colon h \in \mathbb{R}\}$ is also precompact in the space C^0. However, this is not true for $\{S_h s \colon h \in \mathbb{R}\}$ because the element s is not almost periodic (see Definition 1).

Hence, we constructed the operator that is almost periodic according to Definition 3 but not almost periodic in Bochner's sense.

Remark 1. Assume that a Banach space E coincides with the space $l_1 = l_1(\mathbb{N}, \mathbb{R})$ of sequences $a = (a_1, a_2, \ldots, a_k, \ldots)$ for each of which

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

with the norm

$$\|a\|_{l_1} = \sum_{k=1}^{\infty} |a_k|. \tag{17}$$

As the set $X = \{x_1, x_2, \ldots, x_k, \ldots\} \subset E$ used in the construction of the presented example, we can take the set $\tilde{X}$ of sequences

$$x_k = (\delta_{k1}, \delta_{k2}, \delta_{k3}, \ldots), \quad k \in \mathbb{N},$$

where δ_{kl} is the Kronecker delta: $\delta_{kl} = 1$ for $k = l$ and $\delta_{kl} = 0$ for $k \neq l$.

It is obvious that, according to (17), the elements of the set $\tilde{X}$ satisfy relation (5).

3. Main Object of Investigations

Let Ω be an arbitrary domain in the space E. Consider a mapping $F \colon \mathfrak{D}_\Omega \to C^0$ such that, *for any* $K \in \mathcal{K}_\Omega$, *the closure of the set* $\{S_h F S_{-h} x \colon h \in \mathbb{R},\ x \in \mathfrak{D}_K\}$ *in the space* C^0 *is compact in* C^0, i.e., the mapping F is almost periodic in the sense of Definition 3.

It is clear that, for each $K \in \mathcal{K}$ and a sequence $(h_k)_{k \geqslant 1}$ of elements of the set $\mathbb{R}$, there exists a subsequence $(h_{k_l})_{l \geqslant 1}$ such that the subsequence $\left(S_{h_{k_l}} F S_{-h_{k_l}} x\right)_{l \geqslant 1}$ uniformly converges on $\mathfrak{D}_K$.

The aim of the present paper is to establish conditions for the existence of periodic solutions of the equation

$$Fx = 0. \tag{18}$$

To study this equation, we use a functional defined on the set of solutions of the equation with precompact sets of values.

Note that the following nonlinear equations are special cases of Eq. (18):

$$x(t+1) = f(t, x(t)), \quad t \in \mathbb{R}, \tag{19}$$

$$f(t, x(t)) = 0, \quad t \in \mathbb{R}. \tag{20}$$

In [5, 6], for these equations, we establish the existence of solutions continuous and almost periodic on $\mathbb{R}$.

4. Functional δ. Isolated and Strongly Isolated Solutions of Eq. (18)

We fix an arbitrary set $K \in \mathcal{K}$. By $\mathcal{N}(F, K)$ we denote the set of all solutions x of Eq. (18) for each of which $R(x) \subset K$ and $\overline{R(x)} \neq K$.

We fix an arbitrary element $x^* \in \mathcal{N}(F, K)$ (under the assumption that $\mathcal{N}(F, K) \neq \varnothing$) and set

$$r(x^*, K) = \sup \Big\{ \|u - v\|_E \colon u \in \overline{R(x^*)},\ v \in K \Big\}. \tag{21}$$

By virtue of the inequality $\overline{R(x)} \neq K$, we get

$$r(x^*, K) > 0.$$

We also fix an arbitrary number $\varepsilon \in [0, r(x^*, K)]$. By $\Omega(x^*, K, \varepsilon)$ we denote the set of all elements $y \in C^0$ for each of which

$$R(x^* + y) \subset K \tag{22}$$

and

$$\|y\|_{C^0} \geqslant \varepsilon. \tag{23}$$

Similarly, we can define the set $\Omega(z, K, \varepsilon)$ for any other element $z \in C^0$ for which $R(z) \subset K$.

Consider a functional

$$\delta(x^*, K, \varepsilon) = \inf_{y \in \Omega(x^*, K, \varepsilon)} \|F(x^* + y)\|_{C^0}. \tag{24}$$

Definition 4. *A solution $z \in \mathcal{N}(F, K)$ of Eq. (18) is called isolated in the set $\mathbb{R} \times K$ if either this solution is unique in the set $\mathbb{R} \times K$ or, for any other solution $u = u(t)$ with values in K, the following inequality is true:*

$$\inf_{t \in \mathbb{R}} \|z(t) - u(t)\|_E \geqslant \rho,$$

where ρ is a positive constant that depends only on z.

Definition 5. *A solution $z \in \mathcal{N}(F,K)$ of Eq. (18) is called strongly isolated in the set $\mathbb{R} \times K$ if*

$$\delta(z,K,\varepsilon) > 0$$

for each $\varepsilon \in (0, r(z,K))$.

It is clear that each solution $z \in \mathcal{N}(F,K)$ of Eq. (18) strongly isolated in the set $\mathbb{R} \times K$ is a solution of this equation isolated in the set $\mathbb{R} \times K$. However, the solution $z \in \mathcal{N}(F,K)$ of Eq. (18) isolated in the set $\mathbb{R} \times K$ can be not strongly isolated in the set $\mathbb{R} \times K$ (the corresponding example for difference equations with discrete argument is constructed in [7]).

In the next sections, we apply the functional δ to the investigation of the nonlinear equation (18) and a similar linear equation.

In [5, 6, 8], we use similar functionals for the investigation of the nonlinear equations (19), (20), and

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in \mathbb{R}, \tag{25}$$

with continuous mapping $f\colon \mathbb{R} \times \Omega \to E$, where Ω is an arbitrary domain of the space E.

5. Main Result

We now present conditions for the existence of almost periodic solutions of Eq. (18), which, unlike the well-known Amerio theorem on almost periodic solutions of nonlinear differential equations (see [9, 10]), do not use the $\mathcal{H}$-class of Eq. (18).

Let Λ be a bounded subset of the space E. We define the diameter $\operatorname{diam} \Lambda$ of the set Λ by the equality

$$\operatorname{diam} \Lambda = \sup \big\{ \|x - y\|_E \colon x, y \in \Lambda \big\}.$$

Theorem 1. *If, for a solution $z \in \mathcal{N}(F,K)$ of Eq. (18), where $K \in \mathcal{K}$, $\operatorname{diam} R(z) \neq 0$, and*

$$\delta(z,K,\varepsilon) > 0 \tag{26}$$

for every $\varepsilon \in (0, r(z,K))$, then this solution is almost periodic.

Remark 2. A solution $z \in \mathcal{N}(F,K)$ of Eq. (18) for which $\operatorname{diam} R(z) = 0$ is constant and, hence, almost periodic.

Proof. Assume that a solution $z \in \mathcal{N}(F,K)$ of Eq. (18) is not an element of the space B^0. Then there exists a sequence $\big(S_{h_p} z\big)_{p \geqslant 1}$ each subsequence of which $\big(S_{k_p} z\big)_{p \geqslant 1}$ is divergent. Hence, for some sequences $(p_r)_{r \geqslant 1}$ and $(q_r)_{r \geqslant 1}$ of natural numbers and a number $\gamma \in (0, \operatorname{diam} R(z))$, we find

$$\big\| S_{k_{p_r}} z - S_{k_{q_r}} z \big\|_{C^0} \geqslant \gamma, \quad r \geqslant 1. \tag{27}$$

Note that $\operatorname{diam} R(z) \leqslant r(z,K)$. Without loss of generality, we can assume that the sequence $\big(S_{k_p} F S_{-k_p} x\big)_{p \geqslant 1}$ uniformly converges on $\mathfrak{D}_K$. Then

$$\lim_{p,q \to \infty} \sup_{x \in \mathfrak{D}_K} \big\| S_{k_p} F S_{-k_p} x - S_{k_q} F S_{-k_q} x \big\|_{C^0} = 0. \tag{28}$$

We now consider the elements

$$y_r = S_{k_{p_r}} z - S_{k_{q_r}} z, \quad r \geqslant 1,$$

of the space C^0. Clearly,

$$y_r \in \Omega(S_{k_{q_r}} z, K, \gamma), \quad r \geqslant 1. \tag{29}$$

We show that

$$\delta(z, K, \gamma) = 0. \tag{30}$$

By virtue of (24), (29), and the fact that

$$S_{k_{p_r}} F z = 0, \quad r \geqslant 1,$$

for any $r \geqslant 1,$ the following relations are true:

$$\begin{aligned}
\delta(z, K, \gamma) &= \inf_{y \in \Omega(z,K,\gamma)} \|F(z+y)\|_{C^0} = \inf_{y \in \Omega(S_{k_{q_r}} z, K, \gamma)} \|S_{k_{q_r}} F(z + S_{-k_{q_r}} y)\|_{C^0} \\
&= \inf_{y \in \Omega(S_{k_{q_r}} z, K, \gamma)} \|S_{k_{q_r}} F S_{-k_{q_r}} (S_{k_{q_r}} z + y)\|_{C^0} \leqslant \|S_{k_{q_r}} F S_{-k_{q_r}} (S_{k_{q_r}} z + y_r)\|_{C^0} \\
&= \|S_{k_{q_r}} F S_{-k_{q_r}} (S_{k_{q_r}} z + (S_{k_{p_r}} z - S_{k_{q_r}} z))\|_{C^0} = \|S_{k_{q_r}} F S_{-k_{q_r}} S_{k_{p_r}} z\|_{C^0} \\
&\leqslant \|S_{k_{p_r}} F S_{-k_{p_r}} S_{k_{p_r}} z\|_{C^0} + \|S_{k_{q_r}} F S_{-k_{q_r}} S_{k_{p_r}} z - S_{k_{p_r}} F S_{-k_{p_r}} S_{k_{p_r}} z\|_{C^0} \\
&= \|S_{k_{p_r}} F z\|_{C^0} + \|S_{k_{q_r}} F S_{-k_{q_r}} S_{k_{p_r}} z - S_{k_{p_r}} F S_{-k_{p_r}} S_{k_{p_r}} z\|_{C^0} \\
&= \|S_{k_{q_r}} F S_{-k_{q_r}} S_{k_{p_r}} z - S_{k_{p_r}} F S_{-k_{p_r}} S_{k_{p_r}} z\|_{C^0} \\
&\leqslant \sup_{x \in \mathfrak{D}_K} \left\| S_{k_{q_r}} F S_{-k_{q_r}} x - S_{k_{p_r}} F S_{-k_{p_r}} x \right\|_{C^0}.
\end{aligned}$$

In view of (28), these estimates imply relation (30), which contradicts (26).
Hence, the assumption that the solution $z \in \mathcal{N}(F, K)$ of Eq. (18) is not almost periodic is not true.
Theorem 1 is proved.

Note that relation (26) means that the solution $z \in \mathcal{N}(F, K)$ of Eq. (18) is strongly isolated in the set $\mathbb{R} \times K.$ Hence, Theorem 1 can be reformulated as follows:

Theorem 2. *Let $K \in \mathcal{K}.$ If a solution $z \in \mathcal{N}(F, K)$ of Eq. (18) is strongly isolated in the set $\mathbb{R} \times K,$ then this solution is almost periodic.*

Remark 3. The condition of strong isolation of a bounded solution of Eq. (18) is a sufficient but not necessary condition for this solution to belong to the space $B^0.$ The solution of Eq. (18) can be almost periodic but not isolated in the set $\mathbb{R} \times K,$ which is confirmed by the difference equations

$$x(t+1) = x(t), \quad t \in \mathbb{R}.$$

6. The Case of the Linear Equation (18)

Consider an equation

$$Ax = h, \tag{31}$$

where $A\colon C^0 \to C^0$ is a linear continuous and almost periodic (in the sense of Definition 3) operator (this operator can be not almost periodic in Bochner's sense) and $h \in B^0$.

Since Eq. (31) is a special case of Eq. (18) $\big($the operator F is given by the formula $Fx = Ax - h,\ x \in C^0\big)$, Theorem 2 implies the following assertion:

Theorem 3. *Let $K \in \mathcal{K}$. A solution z of Eq. (31) strongly isolated in the set $\mathbb{R} \times K$ is almost periodic.*

We now present conditions for the strong isolation of the solution z of Eq. (31) on $\mathbb{R} \times K$.

Consider a linear homogeneous equation

$$Ax = 0 \tag{32}$$

for Eq. (31).

Theorem 4. *Let $K \in \mathcal{K}$. A solution z of Eq. (31) with values in K is strongly isolated in $\mathbb{R} \times K$ if and only if the trivial solution of Eq. (32) is strongly isolated in $\mathbb{R} \times K$.*

Proof. Since z is a solution of Eq. (31), each element u of the space C^0 for which

$$A(z+u) = h$$

is a solution of Eq. (32), i.e.,

$$Au = 0,$$

and vice versa. By the definition of the set $\Omega(x^*, K, \varepsilon)$ [see (22) and (23)] and the definition of the functional $\delta(x^*, K, \varepsilon)$ [see (24)], for the linear equations, we conclude that [see (21)],

$$\inf_{y \in \Omega(z,K,\varepsilon)} \|A(z+y) - h\|_{C^0} = \inf_{y \in \Omega(0,K,\varepsilon)} \|Ay\|_{C^0} > 0,$$

for every $\varepsilon \in (0, r(z,K))$, i.e.,

$$\delta(z,K,\varepsilon) = \delta(0,K,\varepsilon) > 0$$

for all $\varepsilon \in (0, r(z,K))$.

This yields the assertion of the theorem.

Theorem 4 is proved.

Theorem 5. *If*

$$\inf_{x \in S^0, \|x\|_{C^0} = 1} \|Ax\|_{C^0} > 0, \tag{33}$$

then each solution $z \in S^0$ of Eq. (31) is almost periodic.

Proof. Since $z \in S^0$, we conclude that $\overline{R(z)} \subset K$ for some $K \in \mathcal{K}$. In view of (33) and linearity of the operator A, we get

$$\inf_{x \in S^0,\ \overline{R(x)} \subset K,\ \|x\|_{C^0} = \varepsilon} \|Ax\|_{C^0} > 0$$

for any $\varepsilon \in (\mu(z,K), r(z,K)]$, where

$$\mu(z,K) = \inf\left\{\|x-y\|_E \colon x \in \overline{R(x^*)},\ y \in K\right\}$$

and, hence,

$$\inf_{x \in S^0,\ \overline{R(x)} \subset K,\ \|x\|_{C^0} \geqslant \varepsilon} \|Ax\|_{C^0} > 0$$

for any $\varepsilon \in (0, r(z,K)]$. By virtue of the last relation, the trivial solution of Eq. (32) is strongly isolated in $\mathbb{R} \times K$. Hence, by Theorem 4, the solution $z \in S^0$ of Eq. (31) is also strongly isolated in $\mathbb{R} \times K$.

Thus, by Theorem 3, the solution $z \in S^0$ of Eq. (31) is almost periodic.

Theorem 5 is proved.

Remark 4. The set of equations almost periodic in the sense of Definition 3 for which it is possible to apply the main theorems of Secs. 5 and 6 is nonempty. An element of this set is, e.g., the equation

$$x + Cx = h, \tag{34}$$

where $C \colon C^0 \to C^0$ is a linear continuous operator given by relation (14) and h is an almost periodic element of the space C^0.

It is clear that the operator $I + C$, where $I \colon C^0 \to C^0$ is the identity operator, is almost periodic in the sense of Definition 3 but not almost periodic in Bochner's sense.

Since $h \in S^0$ and $Cy = 0$ for any $y \in S^0$, Eq. (34) possesses a unique solution x in the space S^0 that coincides with h. By virtue of Definition 5 and the definition of the functional Δ [see (24)], this solution is strongly isolated in each set $\mathbb{R} \times K$, where K is an arbitrary compact set in E for which $\overline{R(h)} \subset K$.

7. Application of Theorem 2

Note that the difference equations with continuous argument and some classes of functional equations are special cases of Eq. (18). The same is true for the ordinary differential equations. Thus, the results presented above can be used for the investigation of specific features of the solutions of these equations.

7.1. Difference Equations. We specify the mapping $F \colon C^0 \to C^0$ appearing in Eq. (18) as follows:

$$(Fx)(t) = G(t, x(t), x(t-\Delta_1), \ldots, x(t-\Delta_m)), \qquad t \in \mathbb{R}, \quad x \in C^0, \tag{35}$$

where $m \in \mathbb{N}$, $\Delta_1, \ldots, \Delta_m$ are arbitrary real numbers and $G \colon \mathbb{R} \times \Omega^{m+1} \to E$ is a continuous mapping (here, Ω is the same domain of the space E as in Sec. 3). Assume that the *mapping* F given by relation (35) is *almost periodic in the sense of Definition* 3.

The mapping F is associated with the difference equation

$$G(t, x(t), x(t-\Delta_1), \ldots, x(t-\Delta_m)) = 0, \quad t \in \mathbb{R}. \tag{36}$$

Since this equation is a special case of the general equation (18), Theorem 2 yields the following assertion:

Theorem 6. *Let $K \in \mathcal{K}$. If a function z with values in K is a solution of the difference equation (36) strongly isolated in the set $\mathbb{R} \times K$, then this solution is almost periodic.*

7.2. Functional Equations. As in the previous section, we consider the mapping $F\colon C^0 \to C^0$ used earlier in Eq. (18). We specify this mapping by the formula

$$(Fx)(t) = G(t, x(t), x(\varphi_1(t)), \ldots, x(\varphi_m(t))), \qquad t \in \mathbb{R}, \quad x \in C^0, \tag{37}$$

where $m \in \mathbb{N}$, $\varphi_1\colon \mathbb{R} \to \mathbb{R}, \ldots, \varphi_m\colon \mathbb{R} \to \mathbb{R}$ are continuous functions and $G : \mathbb{R} \times \Omega^{m+1} \to E$ is a continuous mapping (here, Ω is the same domain of the space E as in Sec. 3). Assume that the *mapping* F defined by relation (37) is *almost periodic in the sense of Definition* 3 $\big($the necessary condition for this assertion is the almost periodicity of the functions $\varphi_1(t) - t, \ldots, \varphi_m(t) - t\big)$.

The mapping F is associated with the functional equation

$$G\Big(t, x(t), x\big(\varphi_1(t)\big), \ldots, x\big(\varphi_m(t)\big)\Big) = 0, \quad t \in \mathbb{R}. \tag{38}$$

Clearly this equation is a special case of the general equation (18).

Theorem 2 now yields the following statement:

Theorem 7. *Let $K \in \mathcal{K}$. If the function z with values in K is a solution of the functional equation (38) strongly isolated in the set $\mathbb{R} \times K$, then this solution is almost periodic.*

7.3. Differential Equation. Consider a differential equation

$$\frac{dx}{dt} = h(t, x), \tag{39}$$

where $h\colon \mathbb{R} \times E \to E$ is a continuous mapping.

Assume that, for any number $t_0 \in \mathbb{R}$ and vector $x_0 \in E$, the differential equation (39) possesses a unique solution $x = x(t)$ satisfying the initial condition

$$x(t_0) = x_0. \tag{40}$$

The conditions under which this requirement is satisfied can be found in [11].

Let $x = x(t, t_0, x_0)$ denote a solution of problem (39), (40).

Further, we define the mapping $U\colon \mathbb{R} \times E \to E$ by the formula

$$U(t, y) = x(t+1, t, y), \quad (t, y) \in \mathbb{R} \times E. \tag{41}$$

Clearly, each solution $y = y(t)$ of the differential equation (39) defined on $\mathbb{R}$ satisfies the relation

$$y(t+1) = x(t+1, t, y(t)), \quad t \in \mathbb{R},$$

i.e., in view of (41), it is also a solution of the difference equation

$$x(t+1) = U(t, x(t)), \quad t \in \mathbb{R}, \tag{42}$$

which is a special case of Eq. (18). Hence, Eq. (42) can be used for the investigation of bounded solutions of the differential equation (39).

Theorem 2 also yields the following assertion:

Theorem 8. *Assume that the differential equation (39) possesses a solution $z \in C^0$ with values in the compact set $K \in \mathcal{K}$ and that z is a solution of the difference equation (42) strongly isolated in $\mathbb{R} \times K$.*

If the mapping $U : \mathbb{R} \times E \to E$ is almost periodic in the sense of Definition 3, then the solution z of Eq. (39) is almost periodic.

In conclusion, we note that the established conditions for the existence of almost periodic solutions of Eqs. (18) and (31) are new. Unlike the above-mentioned Amerio theorem, in Theorems 1 and 2, the $\mathcal{H}$-class of Eq. (18) is not used and the Banach space E can be infinite-dimensional. Similarly, in Theorems 3 and 5, the $\mathcal{H}$-class of Eq. (31) is also not used and the operator A can be not almost periodic in Bochner's sense.

We also note that the property of almost periodicity of the solutions of equations is studied in numerous papers. Here, we mention only some of these works. The first theorems on almost periodic solutions of ordinary linear differential equations were proved by Favard in [12]. For nonlinear differential equations, the corresponding theorems were proved by Amerio in [9]. In these papers, the $\mathcal{H}$-classes of the investigated equations are essentially used. Moreover, an additional condition of isolation of the bounded solutions of equations is also used in [9]. The Favard results were later generalized by Mukhamadiev in [13, 14]. Numerous papers are devoted to the generalization of the Mukhamadiev theorems [15–17]. In this field, important results were also obtained by Levitan [2], Amerio [18], and Zhikov [19].

The conditions of almost periodicity of bounded solutions were obtained in [5–8] for the nonlinear difference and differential equations and Eq. (20) without using the $\mathcal{H}$-classes of these equations.

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