DOI 10.1007/s11253-015-1068-2
Ukrainian Mathematical Journal, Vol. 67, No. 1, June, 2015 (Ukrainian Original Vol. 67, No. 1, January, 2015)

ON THE SOLVABILITY OF ONE CLASS OF NONLINEAR INTEGRAL EQUATIONS WITH A NONCOMPACT HAMMERSTEIN–STIELTJES-TYPE OPERATOR ON THE SEMIAXIS

Kh. A. Khachatryan[1] **and A. S. Petrosyan**[2]

UDC 517.968.4+517.988.63

We study a class of nonlinear integral equations with a noncompact operator of the Hammerstein–Stieltjes-type on the semiaxis. The existence of positive solutions is proved in various function spaces by using the factorization methods and specially chosen successive approximations.

1. Introduction

Nonlinear integral equations of the form

$$\varphi(x) = -\int_0^\infty \mathcal{F}_0(t,\varphi(t))\,d_tF(x-t) + \mathcal{F}_1(x,\varphi(x)), \quad x \geq 0, \tag{1}$$

are used for the description of various problems in contemporary mathematical natural sciences. Thus, in particular, these equations are encountered in the theory of Markov processes, in the kinetic theory of gases, in economics, etc. (see [1–3]). An important specific feature of these equations is the noncompactness of the corresponding nonlinear Hammerstein–Stieltjes operator in natural Banach spaces. Hence, the classical methods of nonlinear analysis for the determination of fixed points of the operators either cannot be used at all or can be used under fairly severe restrictions imposed on the functions $\mathcal{F}_0$, $\mathcal{F}_1$, and F. The fact that the required solution $\varphi(x)$ is defined on the unbounded set $[0,+\infty)$ also makes the investigation of this class of equations much more complicated.

The function F in Eq. (1) is defined on the set $(-\infty,+\infty)$ and satisfies the following conditions:

(a) F is left-continuous on $(-\infty,+\infty)$,

(b) $F(-\infty)=0$ and $F(+\infty)=1$,

(c) F monotonically increases on the set $(-\infty,+\infty)$,

i.e., the function F can be regarded as a distribution function of a random variable ξ. By the Lebesgue theorem, the function F admits the representation

$$F = F_A + F_D + F_S, \tag{2}$$

where F_A is the absolutely continuous component of the function F, F_D is its discrete component, which is a jump function with finite sum of the moduli of jumps, and F_S is a continuous function of bounded variation whose

[1] Institute of Mathematics, Armenian National Academy of Sciences, Erevan, Armenia.
[2] Armenian National Agrarian University, Erevan, Armenia.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 67, No. 1, pp. 106–127, January, 2014. Original article submitted October 28, 2013.

 0041-5995/15/6701–0120

derivative is equal to zero almost everywhere. Since F is a monotonically nondecreasing (increasing) function, by the Lebesgue theorem, its components $F_A,\ F_D$, and F_S are also nondecreasing (increasing) functions.

The integral on the right-hand side of (1) is understood in the Lebesgue–Stieltjes sense.

The functions $\mathcal{F}_0$ and $\mathcal{F}_1$ defined on the set $(0,+\infty)\times(-\infty,+\infty)$ are measurable and satisfy the criticality condition (see [4])

$$\mathcal{F}_0(t,0)\equiv 0, \qquad \mathcal{F}_1(x,0)\equiv 0, \quad t,x\in(0,+\infty) \tag{3}$$

[i.e., the function identically equal to zero satisfies Eq. (1)].

In the case where $F=F_A$, $\mathcal{F}_0(t,z)=z$, and $\mathcal{F}_1(x,z)=g(x)\in L_1(0,+\infty)$, Eq. (1) turns into the integral Wiener–Hopf equation of the second kind

$$\varphi(x)=g(x)+\int\limits_0^\infty K(x-t)\varphi(t)dt, \quad x\geq 0, \tag{4}$$

where $K(x) = F'_A(x)$. Numerous works are devoted to the investigation of Eq. (4) (see, e.g., [5–9] and the references therein). Due to their specific features, various mathematical theories developed for the investigation of equations of the form (4) combine fine analytic constructions with efficient approximate methods. We also note that Eq. (4) is used in various branches of mathematical physics (see, e.g., [10–12]).

In a special case where $\mathcal{F}_1\equiv 0$ and $\mathcal{F}_0(t,z)=z$, Eq. (1) was investigated in detail in [13] in connection with its important application to the well-known problem of the probability theory: For a given random variable ξ, it is necessary to find an independently distributed random variable ζ such that the following relation is true:

$$\zeta \backsim (\xi+\zeta)_+ = \begin{cases} \xi+\zeta & \text{for} \quad \xi+\zeta\geq 0, \\ 0 & \text{for} \quad \xi+\zeta<0. \end{cases} \tag{5}$$

The equivalence of these random variables is understood in a sense of coincidence of their distribution functions. Relation (5) is reduced to the following special case of Eq. (1):

$$\varphi(x)=-\int\limits_0^\infty \varphi(t)d_tF(x-t), \quad x\geq 0,$$

where F plays the role of distribution function of the random variable ξ and φ is the required distribution function of ζ.

In the case where $\mathcal{F}_1\equiv 0$, $\mathcal{F}_0(t,z)=G(z)$, and G is a continuous function on a certain segment $[0,\eta]$ such that $G(z)\geq z,\ z\in[0,\eta],\ G(\eta)=\eta$, and $G\uparrow$ on $[0,\eta]$, Eq. (1) was investigated in [14] and the existence of a positive monotonically nondecreasing and bounded solution $\varphi(x)$ was proved. Moreover, the following limit relation was established:

$$\lim_{x\to+\infty}\varphi(x)=\eta.$$

In a recent paper of one of the authors (see [15]), Eq. (1) has been studied in a special case where

$$\mathcal{F}_1\equiv 0, \quad \mathcal{F}_0(t,z)=z-\omega(t,z), \quad \text{and} \quad F=F_A;$$

here, ω is a monotonically decreasing (starting from a certain number $\delta > 0$) function of z satisfying the Krasnosel'skii–Hankel condition: There exists a measurable function

$$\overset{\circ}{\omega}\colon\ \overset{\circ}{\omega} \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+), \qquad \overset{\circ}{\omega} \downarrow \ \text{with } z \quad \text{on} \quad [\delta, +\infty), \qquad m_1(\overset{\circ}{\omega}) \equiv \int_0^\infty x\,\overset{\circ}{\omega}(x)dx < +\infty,$$

such that

$$0 \le \omega(t,z) \le \overset{\circ}{\omega}(t+z), \quad (t,z) \in \mathbb{R}^+ \times [\delta, +\infty).$$

Another approach proposed in [15] enables one to construct a one-parameter family of positive and bounded solutions and also to find the limit of each solution from this family at infinity.

The present paper is devoted to the investigation of Eq. (1) and its special cases in which the function F is continuous and has a singular component:

$$F \equiv F_C = F_A + F_S.$$

In the last case where $\mathcal{F}_1 \equiv 0,$ we construct a one-parameter family of positive and bounded solutions of Eq. (1) under certain conditions imposed on the function $\mathcal{F}_0(t,z)$. In the general case (2), under absolutely different conditions imposed on $\mathcal{F}_0$ and $\mathcal{F}_1,$ we prove the existence of positive and integrable solutions. In conclusion, we present several examples of the functions $\mathcal{F}_j(t,z),\ j = 0,1.$

2. Definitions and Auxiliary Facts

Let E be one of the following Banach spaces: $C[0,+\infty),\ C_l[0,+\infty),$ or $V_c[0,+\infty),$ where $C_l[0,+\infty)$ is a Banach space of functions continuous on $[0,+\infty)$ with finite limit at ∞ and the norm

$$\|f\| = \sup_{x \ge 0} |f(x)|$$

and $V_c[0,+\infty)$ is a Banach space of continuous functions of bounded variation on $[0,+\infty)$ with the norm

$$\|f\| = |f(a)| + \bigvee_a^b f \tag{6}$$

[uniform convergence follows from the convergence in norm (6)].

Consider an integral operator

$$(\hat{F}f)(x) = -\int_0^\infty f(t)d_tF(x-t), \quad x \ge 0,$$

where F satisfies conditions (a)–(c) (see Sec. 1) and $f \in E.$

It is known that the operator $\hat{F}$ acts in each Banach space E. Moreover, for $F \equiv F_C = F_A + F_S,$ the operator $I - \hat{F}$ admits the following factorization (see [13]):

$$I - \hat{F} = (I - \hat{U}^-)(I - \hat{U}^+), \tag{7}$$

where I is the identity operator, $\hat{U}^{\pm}$ are Volterra operators:

$$(\hat{U}^{-}f)(x)=\int\limits_{x}^{\infty}f(t)d_t u_-(t-x)\qquad\text{and}\qquad(\hat{U}^{+}f)(x)=-\int\limits_{0}^{x}f(t)d_t u_+(x-t),$$

f belongs to E,

$$u_{\pm}(x)=w_{\pm}(0)-w_{\pm}(x),\tag{8}$$

$$w_{\pm}\in V_c[0,+\infty),\qquad w_{\pm}\downarrow\ \text{with}\ x,\qquad\lim_{x\to+\infty}w_{\pm}(x)=0,$$

and, in addition,

$$(1-w_-(0))(1-w_+(0))=0.$$

By

$$m(F)=\int\limits_{-\infty}^{+\infty}x\,dF(x),\tag{9}$$

we denote the first moment of the function F (i.e., the mathematical expectation of a random variable ξ). In what follows, unless otherwise specified, we assume that integral (9) is absolutely convergent in the Lebesgue–Stieltjes sense and

$$m(F)<0.\tag{10}$$

Under condition (10), it follows from the results obtained in [13] that

$$w_+(0)<1\qquad\text{and}\qquad w_-(0)=1.\tag{11}$$

In [13], by using conditions (a)–(c), inequality (10), and factorization (7), it is proved that, for $F\equiv F_C=F_A+F_S$, the equation

$$S(x)=-\int\limits_{0}^{\infty}S(t)d_tF(x-t),\quad x\ge 0,\tag{12}$$

with condition

$$S(0)=1$$

possesses a positive monotonically nondecreasing and bounded solution and, in addition,,

$$1=S(0)\le S(x)\le(1-w_+(0))^{-1},\quad x\ge 0,\tag{13}$$

$$\lim_{x\to+\infty}S(x)=(1-w_+(0))^{-1},\qquad\frac{1}{1-w_+(0)}-S\in L_1(\mathbb{R}^+).\tag{14}$$

We now consider the corresponding inhomogeneous equation

$$\rho(x) = g(x) - \int_0^\infty \rho(t) d_t F(x-t), \quad x \geq 0, \tag{15}$$

for the required measurable function $\rho(x)$, where $g(x) \geq 0,\ x \in \mathbb{R}^+$, and $g \in L_1(\mathbb{R}^+)$. By using factorization (7), we reduce the solution of Eq. (15) to the consecutive solution of the following coupled equations:

$$(I - U^-)Q = g, \tag{16}$$

$$(I - U^+)\rho = Q. \tag{17}$$

We now rewrite Eq. (16) in the open form as follows:

$$Q(x) = g(x) + \int_x^\infty Q(t) du_-(t-x), \quad x \geq 0. \tag{18}$$

In [14], it is shown that Eq. (18) has a nonnegative locally integrable solution with the asymptotics

$$\int_0^x Q(t)dt = o(x), \quad x \to +\infty,$$

obtained as the limit of the following successive approximations:

$$Q^{(n+1)}(x) = g(x) + \int_x^\infty Q^{(n)}(t) du_-(t-x), \qquad Q^{(0)}(x) = g(x), \quad n = 0, 1, 2, \ldots, \quad x \geq 0. \tag{19}$$

The following inequality for the solution Q was also established in [14]: For any $r > 0$, the following inequality is true:

$$\int_r^{r+1} Q(\tau)d\tau \leq \left(\int_1^\infty du_-(\tau) \right)^{-1} \int_r^\infty g(\tau)d\tau. \tag{20}$$

In what follows, under additional conditions imposed on the function g, we establish new properties of the function Q, namely, we prove the following lemma important for our subsequent presentation:

Lemma 1 (main). *Assume that a function g has the following properties:*

$$0 \leq g \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+), \quad g(x) \downarrow \quad \text{with} \quad x \quad \text{on} \quad \mathbb{R}^+, \qquad \text{and} \qquad m_1(g) \equiv \int_0^\infty x g(x) dx < +\infty.$$

Also let u_- be defined by (8) and (11), where $L_\infty(\mathbb{R}^+)$ is a space of functions almost everywhere bounded on $\mathbb{R}^+$. Then the solution $Q(x)$ of Eq. (18) has the following properties:

(i) $Q(x) \downarrow$ *with* x *on* $\mathbb{R}^+$,

(ii) $Q \in L_\infty^0(\mathbb{R}^+) \cap L_1(\mathbb{R}^+)$, *where* $L_\infty^0(\mathbb{R}^+)$ *is a space of functions almost everywhere bounded on* $\mathbb{R}^+$ *whose limit at infinity is equal to zero.*

Proof. First, we show that $Q^{(n)}(\tau) \downarrow$ with τ on $\mathbb{R}^+$, $n = 0, 1, 2, \ldots$. For $n = 0$, this is clear because $g(x) \downarrow$ with x on $\mathbb{R}^+$. Let $Q^{(n)}(\tau) \downarrow$ with τ for some $n \in \mathbb{N}$. Rewriting iterations (19) in the form

$$Q^{(n+1)}(\tau) = g(\tau) + \int_0^\infty Q^{(n)}(\tau + t)du_-(t), \tag{21}$$

$$Q^{(0)}(\tau) = g(\tau), \qquad n = 0, 1, 2, \ldots, \quad \tau \geq 0,$$

and taking into account the monotonicity of the function g, we conclude from (21) that $Q^{(n+1)}(\tau) \downarrow$ with τ on $\mathbb{R}^+$. Hence, the limit function

$$Q(\tau) = \lim_{n\to\infty} Q^{(n)}(\tau)$$

is also monotonically decreasing. In view of the monotonicity of Q, it follows from (20) that

$$0 \leq Q(r+1) \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_r^\infty g(t)dt \underset{r\to+\infty}{\longrightarrow} 0.$$

In turn, this implies that Q is bounded on the set $[1, +\infty)$ and $\lim_{\tau\to+\infty} Q(\tau) = 0$.

We now prove that Q belongs to $L_\infty^0(\mathbb{R}^+)$. In view of (18), we get

$$Q(x) = g(x) + \int_x^1 Q(t)d_t u_-(t-x) + \int_1^\infty Q(t)d_t u_-(t-x)$$

$$\leq \sup_{x\in\mathbb{R}^+} g(x) + Q(x)\int_x^1 d_t u_-(t-x) + \sup_{t\geq 1} Q(t)\int_1^\infty du_-(t-x)$$

$$\leq \sup_{x\in\mathbb{R}^+} g(x) + Q(x)u_-(1-x) + \sup_{t\geq 1} Q(t)$$

$$\leq \sup_{x\in\mathbb{R}^+} g(x) + \sup_{t\geq 1} Q(t) + Q(x)u_-(1) \quad \text{for} \quad x \in [0, 1]. \tag{22}$$

Note that $u_-(1) < 1$ because $u_-(0) = 0$, $u_-(+\infty) = 1$, and $u_-(x)$ is a strictly increasing function from $V_c[0, +\infty)$ (see [13, 14]). Hence, by using (22), we get

$$Q(x) \leq C(1 - u_-(1))^{-1},$$

where

$$C \equiv \sup_{x\in\mathbb{R}^+} g(x) + \sup_{t\geq 1} Q(t).$$

Thus, we have proved that Q belongs to $L_\infty^0(\mathbb{R}^+)$. To complete the proof, it remains to show that Q belongs to $L_1(\mathbb{R}^+)$.

Integrating both sides of (20) with respect to r from zero to a certain number $\delta > 0,$ we obtain

$$\int_0^\delta \int_r^{r+1} Q(\tau)d\tau dr \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_0^\delta \int_r^\infty g(\tau)d\tau dr$$

$$= \left(\int_1^\infty du_-(\tau)\right)^{-1} \left(\int_0^\delta \int_r^\delta g(\tau)d\tau dr + \int_0^\delta \int_\delta^\infty g(\tau)d\tau dr\right).$$

Changing the order of integration in the last two terms and using the Fubini theorem, we get

$$\int_0^\delta \int_r^{r+1} Q(\tau)d\tau dr \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \left(\int_0^\delta \tau g(\tau)d\tau + \delta\int_\delta^\infty g(\tau)d\tau\right)$$

$$\leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_0^\infty \tau g(\tau)d\tau \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} m_1(g) < +\infty.$$

In view of the monotonicity of $Q,$ we find

$$\int_0^\delta Q(r+1)dr \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} m_1(g) < +\infty$$

or

$$\int_1^{\delta+1} Q(z)dz \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} m_1(g).$$

Let δ tend to $+\infty$. This yields

$$\int_1^\infty Q(z)dz \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} m_1(g). \tag{23}$$

On the other hand, setting $r = 0$ in (20), we obtain

$$\int_0^1 Q(z)dz \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_0^\infty g(\tau)d\tau. \tag{24}$$

Finding the sum of (23) and (24), we arrive at the inequality

$$\int_0^\infty Q(z)dz \leq \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_0^\infty (\tau+1)g(\tau)d\tau.$$

Thus, the proof of the lemma is completed.

We now proceed to the solution of Eq. (17):

$$\rho(x) = Q(x) - \int_0^x \rho(t)d_t u_+(x-t), \quad x \geq 0. \tag{25}$$

Consider the iterations

$$\rho_{n+1}(x) = Q(x) - \int_0^x \rho_n(t)d_t u_+(x-t),$$

$$\rho_0 \equiv 0, \qquad n = 0, 1, 2, \ldots, \quad x \geq 0.$$

In view of the inequality $w_+(0) < 1$ and representation (8), by induction on $n,$ we can easily show that

(a) $\rho_n \uparrow$ with $n,$

(b) $\rho_n \in L_1(\mathbb{R}^+),\ n = 0, 1, 2, \ldots,$

(c) $\rho_n(\tau) \leq (1 - w_+(0))^{-1} \sup_{\tau \geq 0} Q(\tau),\ \tau \geq 0,$

(d) $\int_0^\infty \rho_n(\tau)d\tau \leq (1 - w_+(0))^{-1} \int_0^\infty Q(\tau)d\tau,\ n = 0, 1, 2, \ldots.$

Hence, the sequence of functions $\{\rho_n(\tau)\}_{n=0}^\infty$ has the pointwise limit as $n \to +\infty$:

$$\lim_{n\to\infty} \rho_n(\tau) = \rho(\tau).$$

Moreover, by the B. Levi theorem (see [16]), the limit function satisfies Eq. (25) and the inequalities

$$\rho(\tau) \geq Q(\tau), \quad \tau \geq 0,$$

$$\rho(\tau) \leq (1 - w_+(0))^{-1} \sup_{\tau \geq 0} Q(\tau) \equiv C_0, \tag{26}$$

$$\int_0^\infty \rho(\tau)d\tau \leq (1 - w_+(0))^{-1} \int_0^\infty Q(\tau)d\tau \leq (1 - w_+(0))^{-1} \left(\int_1^\infty du_-(\tau)\right)^{-1} \int_0^\infty (x+1)g(x)dx. \tag{27}$$

Since $g(x) \downarrow 0$ as $x \to +\infty$ and $w_+(0) < 1,$ by Lemma 3.3 in [13], we immediately conclude that

$$\lim_{x \to +\infty} \rho(x) = 0. \tag{28}$$

Relations (26)–(28) are used in what follows.

3. Solvability of Eq. (1) with Distribution Function F Containing Singular and Discrete Components

In this section, we consider Eq. (1) in the case where F admits representation (2).

Assume that a measurable function $G(x)$ defined on the set $\mathbb{R}$ satisfies the following conditions:

(i) there exists a positive number $\eta > 0$ such that $G \in C[0,\eta],\ \ G \uparrow [0,\eta],$

(ii) $G(0)=0$ and $G(\eta)=\eta;$ moreover, the number $\eta>0$ is the first positive root of the equation $G(x)=x,$

(iii) the function $G(x)$ satisfies the Lipschitz condition on the segment $[0,\eta]$ with constant $L > 0,$ i.e., there exists a number $L > 0$ such that

$$|G(x_1) - G(x_2)| \le L|x_1 - x_2|, \qquad x_1,\, x_2 \in [0,\eta].$$

The following theorem is true:

Theorem 1. *Let the distribution function F satisfy conditions (a)–(c), let $1 - F \in L_1(\mathbb{R}^+),$ and let there exist numbers $\eta_0 \in (0,\eta)$ and $\alpha \in (0, \min(1, 1/L))$ such that:*

(i)
$$\mathcal{F}_1(x, \mu_{\eta_0}(x)) \ge \mu_{\eta_0}(x), \qquad \mathcal{F}_1(x,\eta) \le \mu_\eta(x), \tag{29}$$

where

$$\mu_\delta(x) \equiv \delta(1 - F(x)), \quad x \in \mathbb{R}^+; \tag{30}$$

(ii) for any fixed $x \in \mathbb{R}^+,$ the functions $\mathcal{F}_0(x,z)$ and $\mathcal{F}_1(x,z),$ monotonically increase with the argument z on the segment $[0,\eta];$

(iii)
$$0 \le \mathcal{F}_0(x,z) \le \alpha G(z), \qquad x \in \mathbb{R}^+, \quad z \in [0,\eta]; \tag{31}$$

(iv) in the set $\mathbb{R}^+ \times [0,\eta],$ the functions $\mathcal{F}_0$ and $\mathcal{F}_1$ satisfy the Carathéodory condition with respect to the argument $z,$ i.e., for any fixed $z \in [0,\eta],$ the functions $\{\mathcal{F}_j(x,z)\}_{j=0,1}$ are measurable with respect to $x \in \mathbb{R}^+$ and continuous in z on the segment $[0,\eta]$ for almost all $x \in \mathbb{R}^+.$

Then Eq. (1) has a nonzero nonnegative solution in the space $L_1(\mathbb{R}^+) \cap L^0_\infty(\mathbb{R}^+),$ where

$$L^0_\infty(\mathbb{R}^+) = \big\{ f \in L_\infty(\mathbb{R}^+) \colon \lim_{x \to \infty} f(x) = 0 \big\}.$$

Proof. Parallel with Eq. (1), we consider the following auxiliary equation:

$$\tilde{\varphi}(x) = -\int_0^\infty G_\alpha(\tilde{\varphi}(t))d_tF(x-t), \quad x \geq 0, \tag{32}$$

for the required measurable and real function $\tilde{\varphi}(x)$, where

$$G_\alpha(z) = \eta - \alpha G(\eta - z). \tag{33}$$

We introduce the following iterations:

$$\tilde{\varphi}_{n+1}(x) = -\int_0^\infty G_\alpha(\tilde{\varphi}_n(t))d_tF(x-t),$$

$$\tilde{\varphi}_0(x) \equiv 0, \qquad n = 0, 1, 2, \ldots, \quad x \geq 0.$$

In view of the properties of the function G and conditions (a)–(c), by the induction on n, we can easily show that

(j$_1$) $\tilde{\varphi}_n(x) \uparrow$ with n,

(j$_2$) $\tilde{\varphi}_n(x) \not\equiv 0,\ n = 1, 2, 3, \ldots,$

(j$_3$) $\tilde{\varphi}_n(x) \uparrow$ with $x,\ n = 0, 1, 2, \ldots,$

(j$_4$) $\tilde{\varphi}_n(x) \leq \eta,\ n = 0, 1, 2, \ldots,\ x \in \mathbb{R}^+$.

Hence, the sequence of functions $\{\tilde{\varphi}_n(x)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$:

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \tilde{\varphi}_n(x).$$

Moreover, by the B. Levi theorem, the limit function satisfies Eq. (32), is monotonically increasing, and satisfies the inequalities

$$\eta(1-\alpha)\bigvee_{-\infty}^{x} F \leq \tilde{\varphi}(x) \leq \eta$$

for all $x \in \mathbb{R}^+$. In what follows, we show that

$$\eta - \tilde{\varphi} \in L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+).$$

To this end, we consider an auxiliary equation

$$\tilde{\rho}(x) = \eta(1 - F(x)) - \int_0^\infty (\eta - G_\alpha(\eta - \tilde{\rho}(t)))d_tF(x-t), \quad x \geq 0 \tag{34}$$

for the required function $\tilde{\rho}(x)$ and the following successive approximations for this equation:

$$\rho_{n+1}(x) = \eta(1 - F(x)) - \int_0^\infty \big(\eta - G_\alpha(\eta - \tilde{\rho}_n(t))\big) d_t F(x - t),$$
$$\tilde{\rho}_0(x) = \eta(1 - F(x)), \qquad n = 0, 1, 2, \dots, \quad x \geq 0. \tag{35}$$

By induction, we can easily show that

(i$_1$) $\tilde{\rho}_n(x) \uparrow$ with n;

(i$_2$) $\tilde{\rho}_n(x) \in L_1(\mathbb{R}^+),\ n = 0, 1, 2, \dots;$

(i$_3$) $\int_0^\infty \rho_n(x)dx \leq \eta(1 - \alpha L)^{-1}\|1 - F\|_{L_1(\mathbb{R}^+)},\ n = 0, 1, 2, \dots;$

(i$_4$) $\tilde{\rho}_n(x) \leq \eta,\ n = 0, 1, 2, \dots,\ x \geq 0.$

Thus, we prove property (i$_3$). The proofs of the remaining properties of the sequence $\{\tilde{\rho}_n(\tau)\}_{n=0}^\infty$ are simpler. For $n = 0$, the property (i$_3$) directly follows from (35) in view of the fact that $\alpha \in (0, \min(1, 1/L))$. Assume that (i$_3$) holds for a certain natural n. Thus, in view of the conditions imposed on F and G, it follows from relations (35) that

$$\int_0^\infty \tilde{\rho}_{n+1}(x)dx = \eta\int_0^\infty (1 - F(x))dx - \int_0^\infty\int_0^\infty (\eta - G_\alpha(\eta - \tilde{\rho}_n(t)))d_tF(x - t)dx$$

$$\leq \eta\|1 - F\|_{L_1(\mathbb{R}^+)} - \alpha L\int_0^\infty\int_0^\infty \tilde{\rho}_n(t)d_tF(x - t)dx$$

$$= \eta\|1 - F\|_{L_1(\mathbb{R}^+)} + \alpha L\int_0^\infty\int_{-\infty}^x \tilde{\rho}_n(x - t)d_tF(t)dx$$

$$= \eta\|1 - F\|_{L_1(\mathbb{R}^+)} + \alpha L\int_0^\infty\int_{-\infty}^0 \tilde{\rho}_n(x - t)d_tF(t)dx + \alpha L\int_0^\infty\int_0^x \tilde{\rho}_n(x - t)d_tF(t)dx$$

$$= \eta\|1 - F\|_{L_1(\mathbb{R}^+)} + \alpha L\int_{-\infty}^0\int_0^\infty \tilde{\rho}_n(x - t)dx d_tF(t) + \alpha L\int_0^\infty\int_t^{+\infty} \tilde{\rho}_n(x - t)dx d_tF(t)$$

$$= \eta\|1 - F\|_{L_1(\mathbb{R}^+)} + \alpha L\int_{-\infty}^0\int_{-t}^{+\infty} \tilde{\rho}_n(y)dy d_tF(t) + \alpha L\int_0^\infty\int_0^\infty \tilde{\rho}_n(y)dy d_tF(t)$$

$$\leq \eta \|1 - F\|_{L_1(\mathbb{R}^+)} + \alpha L \int_0^\infty \tilde{\rho}_n(y)dy \int_{-\infty}^{+\infty} d_t F(t)$$

$$\leq \eta \|1 - F\|_{L_1(\mathbb{R}^+)} \left(1 + \frac{\alpha L}{1 - \alpha L}\right)$$

$$= \eta (1 - \alpha L)^{-1} \|1 - F\|_{L_1(\mathbb{R}^+)}.$$

Hence, the sequence of functions $\{\tilde{\rho}_n(\tau)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to \infty$:

$$\tilde{\rho}(\tau) = \lim_{n \to \infty} \tilde{\rho}_n(\tau)$$

Moreover, by the Lebesgue theorem, this limit satisfies Eq. (34) and the relations

$$\eta(1 - F(x)) \leq \tilde{\rho}(x) \leq \eta, \quad x \in \mathbb{R}^+,$$

$$\int_0^\infty \tilde{\rho}(x)dx \leq \eta(1 - \alpha L)^{-1} \|1 - F\|_{L_1(\mathbb{R}^+)}.$$

By using the conditions imposed on the function G and the fact that $\alpha \in (0, \min(1, 1/L))$, we can easily show that the solution of Eq. (34) is unique in the following class of measurable functions:

$$\Omega_\eta \equiv \{\varphi \colon 0 \leq \varphi(x) \leq \eta,\, x \in \mathbb{R}^+\}.$$

On the other hand, we can directly show that the function

$$\tilde{\rho}^*(x) \equiv \eta - \tilde{\varphi}(x) \in \Omega_\eta$$

satisfies Eq. (34). This means that

$$\tilde{\rho}^*(x) \in L_1(\mathbb{R}^+).$$

We now prove that

$$\lim_{x \to +\infty} \tilde{\rho}^*(x) = 0.$$

Since $0 \leq \tilde{\varphi}(x) \leq \eta,\ x \in \mathbb{R}^+$, and $\tilde{\varphi}(x) \uparrow$ with x on $\mathbb{R}^+$, there exists a limit

$$\lim_{x \to +\infty} \tilde{\varphi}(x) = c_0 \leq \eta.$$

We now prove that $c_0 = \eta$. To this end, we first show that

$$\lim_{x \to +\infty} \left(- \int_0^\infty G_\alpha(\tilde{\varphi}(t)) d_t F(x - t) \right) = G_\alpha(c_0). \tag{36}$$

We find

$$\left|G_\alpha(c_0)+\int\limits_0^\infty G_\alpha(\tilde\varphi(t))d_tF(x-t)\right|\le G_\alpha(c_0)(1-F(x))+\int\limits_0^\infty |G_\alpha(c_0)-G_\alpha(\tilde\varphi(t))||d_tF(x-t)|$$

$$=I_1(x)+I_2(x).$$

It is clear that $I_1(x)\to 0$ as $x\to+\infty$. It is necessary to prove that $I_2(x)\to 0$ as $x\to+\infty$. By using relation (33), we obtain

$$I_2(x)\le \alpha L\int\limits_0^\infty |c_0-\tilde\varphi(t)||d_tF(x-t)|$$

$$=\alpha L\left(\int\limits_0^x |c_0-\tilde\varphi(t)||d_tF(x-t)|+\int\limits_x^\infty |c_0-\tilde\varphi(t)||d_tF(x-t)|\right)$$

$$=\alpha L(J_1(x)+J_2(x)).$$

Since $\tilde\varphi(t)\to c_0$ as $t\to+\infty$, for any $\varepsilon>0$, there exists $\delta>0$ such that if $t>\delta$, then

$$|\tilde\varphi(t)-c_0|<\varepsilon.$$

Let $x>2\delta$. Then we get the following relation for the second integral $J_2(x)$:

$$J_2(x)\le\varepsilon\int\limits_x^\infty |d_tF(x-t)|\le\varepsilon.$$

Moreover, after necessary transformations for the first integral $J_1(x)$, we obtain

$$J_1(x)=\int\limits_0^{x/2} |c_0-\tilde\varphi(t)||d_tF(x-t)|+\int\limits_{x/2}^x |c_0-\tilde\varphi(t)||d_tF(x-t)|$$

$$\le 2c_0\int\limits_{x/2}^x |d_tF(\tau)|+\varepsilon\int\limits_{x/2}^x |d_tF(x-t)|$$

$$\le 2c_0\int\limits_{x/2}^x |d_tF(\tau)|+\varepsilon.$$

Since

$$\int\limits_{x/2}^x |d_tF(t)|\to 0\quad\text{as}\quad x\to+\infty,$$

for any $\varepsilon > 0,$ there exists $\delta_1 > 0$ such that

$$\int\limits_{x/2}^{x} |d_t F(t)| < \varepsilon$$

for $x > \delta_1.$

Choosing $x > \max(2\delta, \delta_1),$ we obtain

$$J_1(x) \leq \varepsilon(1 + 2c_0).$$

Hence, for $x > \max(2\delta, \delta_1),$

$$I_2(x) \leq \alpha L(2\varepsilon + 2\varepsilon c_0).$$

Thus, relation (36) is proved.

Passing to the limit on both sides of Eq. (32) as $x \to +\infty,$ we obtain

$$c_0 = G_\alpha(c_0), \qquad 0 < c_0 \leq \eta.$$

We now check that η is the first positive root of the equation $G_\alpha(x) = x.$ Assume the contrary, i.e., that there exists a number $\eta_1 \in (0, \eta)$ such that

$$G_\alpha(\eta_1) = \eta_1, \quad \text{i.e.,} \quad \eta - \alpha G(\eta - \eta_1) = \eta_1.$$

Then we get

$$\eta - \eta_1 = G_\alpha(\eta) - G_\alpha(\eta_1) \leq \alpha L(\eta - \eta_1).$$

This directly implies that $\eta \leq \eta_1$ because $\alpha L < 1.$ In turn, this contradiction proves that $c_0 = \eta.$

Thus, we have shown that Eq. (32) possesses a nonzero nonnegative monotonically increasing and bounded solution $\tilde{\varphi}(x)$ and, in addition,

$$\lim_{x \to +\infty} \tilde{\varphi}(x) = \eta \quad \text{and} \quad \eta - \tilde{\varphi} \in L_1(\mathbb{R}^+).$$

We use these results to construct a positive solution of Eq. (1) in the space $L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+).$ Thus, we introduce the following iterations:

$$\varphi_{n+1}(x) = -\int\limits_0^\infty \mathcal{F}_0(t, \varphi_n(t)) d_t F(x - t) + \mathcal{F}_1(x, \varphi_n(x)), \tag{37}$$

$$\varphi_0(x) = \mu_{\eta_0}(x), \qquad n = 0, 1, 2, \ldots, \quad x \geq 0.$$

By induction, we can easily show that

$$\varphi_n(x) \uparrow \quad \text{with} \quad n, \tag{38}$$

$$\varphi_n(x) \leq \eta - \tilde{\varphi}(x), \quad n = 0, 1, 2, \ldots\ , \tag{39}$$

We now prove inequalities (39). Let $n=0$. Then

$$\varphi_0(x)=\mu_{\eta_0}(x)\leq\mu_\eta(x)=\eta(1-F(x))$$
$$\leq\eta(1-F(x))+\left(-\int\limits_0^\infty(\eta-G_\alpha(\eta-\tilde{\rho}^*(t)))d_tF(x-t)\right)$$
$$=\tilde{\rho}^*(x)\equiv\eta-\tilde{\varphi}(x).$$

Assume that inequalities (39) are true for some $n\in\mathbb{N}$. By using (37) and (29)–(31), we find

$$\varphi_{n+1}(x)\leq-\int\limits_0^\infty\mathcal{F}_0(t,\eta-\tilde{\varphi}(t))d_tF(x-t)+\mathcal{F}_1(x,\eta-\tilde{\varphi}(x))$$
$$\leq-\int\limits_0^\infty\mathcal{F}_0(t,\eta-\tilde{\varphi}(t))d_tF(x-t)+\mathcal{F}_1(x,\eta)$$
$$\leq-\int\limits_0^\infty(\eta-G_\alpha(\tilde{\varphi}(t)))d_tF(x-t)+\mu_\eta(x)$$
$$=\eta F(x)+\int\limits_0^\infty G_\alpha(\tilde{\varphi}(t))d_tF(x-t)+\mu_\eta(x)$$
$$=\eta-\tilde{\varphi}(x).$$

In view of the conditions (i) and (ii) of Theorem 1, we can prove the monotonicity of the sequence $\{\varphi_n(x)\}_{n=0}^\infty$ with respect to n. Hence, the sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ has a pointwise limit as $n\to\infty$:

$$\varphi(x)=\lim_{n\to\infty}\varphi_n(x).$$

Furthermore, $\varphi(x)$ satisfies Eq. (1). In view of the B. Levi theorem, this follows from the condition (iv) of Theorem 1. By using (39), we get

$$\mu_{\eta_0}(x)\leq\varphi(x)\leq\eta-\tilde{\varphi}(x),\quad x\in\mathbb{R}^+. \tag{40}$$

Since $\mu_{\eta_0}(x)\geq 0,\ x\in\mathbb{R}^+$, and

$$\eta-\tilde{\varphi}\in L_1(\mathbb{R}^+)\cap L_\infty^0(\mathbb{R}^+),$$

in view of (40), we conclude that $\varphi\in L_1(\mathbb{R}^+)\cap L_\infty^0(\mathbb{R}^+)$.

The theorem is proved.

In what follows, we give several examples of the functions $\mathcal{F}_0$ and $\mathcal{F}_1$ satisfying all conditions of Theorem 1.

Examples of the Function $\mathcal{F}_0(x,z)$:

$$(1^0) \qquad \mathcal{F}_0(x,z) = \alpha^p q_0(x,z)\frac{Q^p(z)}{\eta^{p-1}},\ x \in \mathbb{R}^+,\ z \in [0,\eta],$$

where $q_0(x,z)$ is a real function defined on $\mathbb{R}^+ \times \mathbb{R}$ and satisfying the following conditions:

(A) $0 \le q_0(x,z) \le 1,\ x \in \mathbb{R}^+,\ z \in [0,\eta]$,

(B) $q_0 \uparrow$ with z on the segment $[0,\eta]$ for any fixed $x \in \mathbb{R}^+$,

(C) $q_0(x,z)$ satisfies the Carathéodory condition with respect to the argument z on the set $\mathbb{R}^+ \times [0,\eta]$.

$$(2^0) \qquad \mathcal{F}(x,z) = q_0(x,z)\ln(\alpha Q(z)+1), \quad x \in \mathbb{R}^+, \quad z \in [0,\eta].$$

The following functions serve as examples of the functions $q_0(x,z)$ and $Q(z)$:

$$q_0(x,z) = e^{-\gamma x}(1-e^{-\beta z}), \quad \gamma, \beta > 0,$$

$$Q(z) = \frac{z^p}{\eta^{p-1}}, \qquad p > 1, \quad z \in [0,\eta].$$

Examples of the Function $\mathcal{F}_1(x,z)$:

(3^0) $\mathcal{F}_1(x,z) = \mu_{\eta_0+\eta_1}(x)\dfrac{z}{z+\mu_{\eta_1}(x)}$, where $x \in \mathbb{R}^+,\ z \in [0,\eta],\ \eta_0,\eta_1 > 0$, and $\eta \ge \eta_0 + \eta_1$,

(4^0) $\mathcal{F}_1(x,z) = \mu_{\eta_0+\eta_1}(x)\dfrac{\lambda z^2}{(z+\mu_{\eta_1}(x))^2}$, where $x \in \mathbb{R}^+,\ z \in [0,\eta],\ \lambda \ge 1 + \dfrac{\eta_1}{\eta_0},\ \eta \ge \lambda(\eta_0+\eta_1)$, and $\eta_0, \eta_1 > 0$.

4. One-Parameter Family of Positive and Bounded Solutions for Eq. (1) in the Case Where $\mathcal{F}_1 \equiv 0$ and $F \equiv F_C = F_A + F_S$

In the present section, we construct a one-parameter family of positive and bounded solutions of Eq. (1) in a special case where $\mathcal{F}_1(x,z) \equiv 0$ and the distribution function F satisfies the conditions (a)–(c) and has only absolutely continuous and singular components, i.e., admits the representation

$$F \equiv F_C = F_A + F_S.$$

We essentially use the main lemma from Sec. 2 in the proof of the following theorem:

Theorem 2. *Let $\mathcal{F}_1(x,z) \equiv 0$ and let the function F satisfy conditions (10) and (a)–(c). Furthermore, let $F \equiv F_C = F_A + F_S$. Assume that $\mathcal{F}_0(t,z)$ has the following structure:*

$$\mathcal{F}_0(t,z) = z - \omega(t,z),$$

where $\omega(t,z)$ is a measurable real function defined on the set $\mathbb{R}^+ \times \mathbb{R}$ and satisfying the following conditions:

(γ_1) *there exists a number* $A > 0$ *such that, for any fixed* $t \in \mathbb{R}^+$, *the function* $\omega(t, z) \downarrow$ *with* z *on the set* $[A, +\infty)$;

(γ_2) *there exists a measurable function* $\overset{\circ}{\omega}(z)$ *defined on* $\mathbb{R}$ *with the following properties:*

$$0 \le \overset{\circ}{\omega}(z) \downarrow \ \ \textit{on} \ \ [A, +\infty), \quad \overset{\circ}{\omega} \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+), \quad \textit{and} \quad m_1(\overset{\circ}{\omega}) \equiv \int_0^\infty x\, \overset{\circ}{\omega}(x) dx < +\infty;$$

furthermore,

$$0 \le \omega(t, z) \le \overset{\circ}{\omega}(t + z), \qquad t \in \mathbb{R}^+, \quad z \in [A, +\infty); \tag{41}$$

(γ_3) *the function* ω *satisfies the Carathéodory condition with respect to the argument* z *on the set*

$$\mathbb{R}^+ \times [A, +\infty).$$

Then Eq. (1) has a one-parameter family of positive and bounded solutions $\{\varphi_\beta(x)\}_{\beta \in \Delta}$ *and, moreover, each function from this family has the following properties:*

$$\lim_{x \to \infty} \varphi_\beta(x) = 2\beta(1 - w_+(0))^{-1}, \quad \beta \in \Delta,$$

if $\beta_1, \beta_2 \in \Delta,\ \beta_1 > \beta_2$, *then*

$$\varphi_{\beta_1}(x) - \varphi_{\beta_2}(x) \ge 2(\beta_1 - \beta_2), \quad x \in \mathbb{R}^+.$$

Here,

$$\Delta = [\max(\varkappa, \beta_0), +\infty),$$

where β_0 $(\beta_0 \ge A)$ *is a certain fixed number for which*

$$\overset{\circ}{\omega}(\beta_0) < \beta_0 \qquad \textit{and} \qquad \varkappa = \sup_{x \ge 0} Q(x),$$

$Q(x)$ *is a positive and bounded solution of the inhomogeneous integral equation*

$$Q(x) = 2\,\overset{\circ}{\omega}(x + A) - \int_0^\infty Q(t) d_t F(x - t), \quad x \ge 0. \tag{42}$$

Remark 1. The existence of a positive and bounded solution of Eq. (42) is not assumed and follows from the main lemma proved in Sec. 2. The existence of the number $\beta_0 \ge A$ directly follows from the properties of the function $\overset{\circ}{\omega}$.

We split the ***proof*** into several steps.

Step 1 (investigation of a single auxiliary linear integral equation). Parallel with Eq. (1), we consider an auxiliary equation

$$\tilde{Q}(x) = 2\overset{\circ}{\omega}(x + S_\beta(x)) - \lambda_\beta(x)\int\limits_0^\infty \tilde{Q}(t)d_tF(x-t), \quad x \geq 0, \tag{43}$$

for the required measurable function $\tilde{Q}(x)$, where

$$S_\beta(x) = \beta S(x), \quad \beta \in \Delta,$$
$$\lambda_\beta(x) = 1 - \frac{\overset{\circ}{\omega}(x + S_\beta(x))}{S_\beta(x)}, \tag{44}$$

and $S(x)$ is a solution of Eq. (12) with properties (13) and (14).

We introduce the following iterations:

$$\tilde{Q}_{n+1}(x) = 2\overset{\circ}{\omega}(x + S_\beta(x)) - \lambda_\beta(x)\int\limits_0^\infty \tilde{Q}_n(t)d_tF(x-t),$$
$$\tilde{Q}_0(x) = 2\overset{\circ}{\omega}(x + S_\beta(x)), \qquad n = 0, 1, 2, \dots, \quad x \geq 0. \tag{45}$$

By the induction on n, we can prove that

(θ_1) $\tilde{Q}_n(x) \uparrow$ with n,

(θ_2) $\tilde{Q}_n(x) \leq Q(x),\ n = 0, 1, 2, \dots,\ x \geq 0.$

Indeed, we prove, e.g., (θ_2). Since $S_\beta(x) \geq \beta \geq \beta_0 \geq A$, in view of the monotonicity of $\overset{\circ}{\omega}$ on $[A, +\infty)$, we get

$$\overset{\circ}{\omega}(x + S_\beta(x)) \leq \overset{\circ}{\omega}(x + A),$$

which implies that

$$Q_0(x) = 2\overset{\circ}{\omega}(x + S_\beta(x)) \leq 2\overset{\circ}{\omega}(x + A) \leq Q(x).$$

We assume that (θ_2) is true for some $n \in \mathbb{N}$, we prove this assertion for $n + 1$. We first note that the function $\lambda_\beta(x)$ has the following properties:

there exists a number $\delta = \delta_\beta > 0$ such that

$$0 < \delta \leq \lambda_\beta(x) \leq 1, \quad x \in \mathbb{R}^+, \tag{46}$$

$$1 - \lambda_\beta \in L_1(\mathbb{R}^+), \quad \beta \in \Delta, \tag{47}$$

$$\lim_{x\to\infty} \lambda_\beta(x) = 1 \qquad \text{for any} \quad \beta \in \Delta. \tag{48}$$

Indeed, as $\delta > 0,$ we choose a number

$$\delta = (\beta_0 - \overset{\circ}{\omega}(\beta_0))\frac{(1 - w_+(0))}{\beta} > 0$$

and take into account the monotonicity of the function $\overset{\circ}{\omega}$. This yields (46). Properties (47) and (48) directly follow from the inclusion $\overset{\circ}{\omega} \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$.

By using (45), (46), and the inequality

$$Q_0(x) \leq \overset{\circ}{\omega}(x + A),$$

we arrive at the following formula:

$$\tilde{Q}_{n+1}(x) \leq 2\overset{\circ}{\omega}(x + A) - \int_0^\infty \tilde{Q}_n(t) d_t F(x - t) \leq 2\overset{\circ}{\omega}(x + A) - \int_0^\infty Q(t) d_t F(x - t) = Q(x).$$

The proof of monotonicity of the sequence $\{\tilde{Q}_n\}_0^\infty$ is simpler. Then the sequence of functions $\{\tilde{Q}_n\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$:

$$\lim_{n\to\infty} \tilde{Q}_n(x) = \tilde{Q}_\beta(x).$$

Furthermore, the limit function satisfies Eq. (43) and the inequalities

$$2\overset{\circ}{\omega}(x + S_\beta(x)) \leq \tilde{Q}_\beta(x) \leq Q(x).$$

We now show that, for any $\beta \in \Delta$, the inequality

$$S_\beta(x) \geq \tilde{Q}_\beta(x), \quad x \in \mathbb{R}^+, \tag{49}$$

is true. Indeed, it follows from the definition of the set Δ that

$$S_\beta(x) \geq \beta \geq \varkappa \geq Q(x) \geq \tilde{Q}_\beta(x), \quad x \in \mathbb{R}^+. \tag{50}$$

Step 2 (construction of a nontrivial solution of the corresponding homogeneous equation; a priori estimates). We now consider the homogeneous equation corresponding to (43):

$$E_\beta(x) = -\lambda_\beta(x) \int_0^\infty E_\beta(t) d_t F(x - t), \quad x \geq 0, \tag{51}$$

for the required measurable function $E_\beta(x),\ \beta \in \Delta$. We can directly show that the function

$$\tilde{E}_\beta(x) \equiv 2S_\beta(x) - \tilde{Q}_\beta(x)$$

satisfies Eq. (51). Since $S_\beta(x) \geq \tilde{Q}_\beta(x)$ [see (49) and (50)], we find

$$\tilde{E}_\beta(x) \geq S_\beta(x), \qquad x \in \mathbb{R}^+, \quad \beta \in \Delta.$$

For Eq. (51), we consider the following iterations:

$$E_\beta^{(n+1)}(x) = -\lambda_\beta(x) \int\limits_0^\infty E_\beta^{(n)}(t) d_t F(x-t), \quad x \geq 0,$$

$$E_\beta^{(0)}(x) = 2S_\beta(x), \quad n = 0, 1, 2, \ldots.$$

By the induction on $n,$ we can easily show that the following results are true:

$$E_\beta^{(n)}(x) \downarrow \quad \text{with} \quad n, \qquad \beta \in \Delta, \quad x \in \mathbb{R}^+,$$

$$E_\beta^{(n)}(x) \leq 2\lambda_\beta(x) S_\beta(x), \qquad n = 1, 2, 3, \ldots, \quad x \geq 0,$$

$$E_\beta^{(n)}(x) \geq \tilde{E}_\beta(x), \quad n = 0, 1, 2, \ldots .$$

Hence, the sequence of functions $\left\{E_\beta^{(n)}(x)\right\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$:

$$\lim_{n \to \infty} E_\beta^{(n)}(x) = E_\beta(x)$$

and, moreover, this limit satisfies Eq. (51) and the chain of inequalities

$$2\lambda_\beta(x) S_\beta(x) \geq E_\beta(x) \geq \tilde{E}_\beta(x) \geq S_\beta(x), \quad x \in \mathbb{R}^+. \tag{52}$$

We now consider the “main” auxiliary homogeneous equation

$$P_\beta(x) = -\int\limits_0^\infty \lambda_\beta(t) P_\beta(t) d_t F(x-t), \quad x \geq 0, \tag{53}$$

for the required function $P_\beta(x),\ \beta \in \Delta.$

By virtue of (51) and (52), the function

$$P_\beta(x) = \frac{E_\beta(x)}{\lambda_\beta(x)}$$

satisfies Eq. (53) and the inequalities

$$S_\beta(x) \leq \tilde{E}_\beta(x) \leq E_\beta(x) \leq P_\beta(x) \leq 2S_\beta(x), \qquad x \geq 0, \quad \beta \in \Delta. \tag{54}$$

In the last step, by using the properties of the constructed solution $P_\beta(x)$ and a specially chosen iterative process, we prove the existence of a one-parameter family of positive solutions for Eq. (1) in the case where $\mathcal{F}_1 \equiv 0.$

Step 3 [special iterative process for the solution of Eq. (1)]. We now consider the following successive iterations:

$$\varphi_{n+1}^{\beta}(x) = -\int_{0}^{\infty}(\varphi_n^{\beta}(t) - \omega(t, \varphi_n^{\beta}(t)))d_t F(x-t),$$
$$\varphi_0^{\beta}(x) = 2S_{\beta}(x), \qquad n = 0, 1, 2, \ldots, \quad x \geq 0. \tag{55}$$

By the induction on $n,$ we can easily show that

$$\varphi_n^{\beta}(x) \downarrow \quad \text{with} \quad n. \tag{56}$$

$$\varphi_n^{\beta}(x) \geq P_{\beta}(x), \quad n = 0, 1, 2, \ldots, \tag{57}$$

If $\beta_1,\ \beta_2 \in \Delta,\ \beta_1 > \beta_2,$ are arbitrary numbers, then

$$\varphi_n^{\beta_1}(x) - \varphi_n^{\beta_2}(x) \geq 2(S_{\beta_1}(x) - S_{\beta_2}(x)) \geq 2(\beta_1 - \beta_2), \qquad n = 0, 1, 2, \ldots, \quad x \geq 0. \tag{58}$$

We prove, e.g., inequality (57). The proof of the other properties of the sequence $\{\varphi_n^{\beta}(x)\}_{n=0}^{\infty}$ is simpler. For $n = 0,$ inequality (57) directly follows from the chain of inequalities (54). Let (57) be true for some $n \in \mathbb{N}.$ Then, by virtue of (41), (44), and (54), it follows from (55) that

$$\varphi_{n+1}^{\beta}(x) \geq -\int_{0}^{\infty}(P_{\beta}(t) - \omega(t, P_{\beta}(t)))d_t F(x-t)$$
$$\geq -\int_{0}^{\infty}(P_{\beta}(t) - \omega(t, S_{\beta}(t)))d_t F(x-t)$$
$$\geq -\int_{0}^{\infty}(P_{\beta}(t) - \omega(t + S_{\beta}(t)))d_t F(x-t)$$
$$= -\int_{0}^{\infty}(P_{\beta}(t) - (1 - \lambda_{\beta}(t))S_{\beta}(t))d_t F(x-t)$$
$$\geq -\int_{0}^{\infty}\lambda_{\beta}(t)P_{\beta}(t)d_t F(x-t) = P_{\beta}(x).$$

Thus, it follows from (56), (57), and (58) that the sequence of functions $\{\varphi_n^{\beta}(x)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to \infty$:

$$\lim_{n\to\infty}\varphi_n^{\beta}(x) = \varphi_{\beta}(x).$$

Furthermore, by the limit B. Levi theorem, the limit function satisfies Eq. (1) and the relations

$$\tilde{E}_\beta(x) \le P_\beta(x) \le \varphi_\beta(x) \le 2S_\beta(x), \quad x \in \mathbb{R}^+, \tag{59}$$

$$\varphi_{\beta_1}(x) - \varphi_{\beta_2}(x) \ge 2(S_{\beta_1}(x) - S_{\beta_2}(x)) \ge 2(\beta_1 - \beta_2), \quad x \in \mathbb{R}^+. \tag{60}$$

It directly follows from (60) that $\varphi_\beta(x) \uparrow$ with β on Δ.

Since

$$\tilde{E}_\beta(x) = 2S_\beta(x) - \tilde{Q}_\beta(x) \quad \text{and} \quad 0 \le \tilde{Q}_\beta(x) \le Q(x),$$

we conclude that $\lim_{x\to\infty} Q(x) = 0$ (see the main lemma in Sec. 2) and

$$\lim_{x\to\infty} S_\beta(x) = \beta(1 - w_+(0))^{-1}.$$

Finally, by using (59), we establish the existence of the finite limit

$$\lim_{x\to\infty} \varphi_\beta(x) = 2\beta(1 - w_+(0))^{-1}.$$

The theorem is proved.

Remark 2. In a special case where $F = F_A$, this result was obtained in [15].

5. Construction of a Summable Solution of Eq. (1) with a Function $\mathcal{F}_0(t, z)$ Whose Majorant Has the Form $z + \omega(t, \xi)$

The following theorem is true:

Theorem 3 (main). *Assume that the distribution function F satisfies the conditions of Theorem 2 and that*

$$1 - F \in L_1(\mathbb{R}^+).$$

Let $\mathcal{F}_0(x, z)$ and $\mathcal{F}_1(x, z)$ be given measurable and real functions defined on $\mathbb{R}^+ \times \mathbb{R}$. Suppose that there exist numbers

$$\xi \ge \frac{2\max(\varkappa, \beta_0)}{1 - w_+(0)} \qquad \textit{and} \qquad \xi_0 \in (0, \xi)$$

such that $\mathcal{F}_0(x, z)$ and $\mathcal{F}_1(x, z)$ satisfy the conditions:

(A) $\mathcal{F}_1(x, \mu_{\xi_0}(x)) \ge \mu_{\xi_0}(x)$ *and* $\mathcal{F}_1(x, \xi) \le \mu_\xi(x),\ x \in \mathbb{R}^+$;

(B) $0 \le \mathcal{F}_0(t, z) \le z + \omega(t, \xi),\ t \in \mathbb{R}^+,\ z \in [0, \xi]$, *where ω satisfies the conditions of Theorem 2;*

(C) for any fixed $x \in \mathbb{R}^+$, the functions $\{\mathcal{F}_j(x, z)\}_{j=0,1}$, monotonically increase with z on the segment $[0, \xi]$;

(D) *the functions* $\{\mathcal{F}_j(x,z)\}_{j=0,1}$ *satisfy the Carathéodory condition with respect to the argument* z *on the set* $\mathbb{R}^+ \times [0,\xi]$.

Then Eq. (1) has a nonzero nonnegative solution in the space $L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+)$.

Proof. It follows from the definition of the set Δ that

$$\beta^* = \frac{\xi(1-w_+(0))}{2} \in \Delta.$$

Hence, by virtue of Theorem 2, we can state that the equation

$$\tilde{\varphi}(x) = -\int_0^\infty (\tilde{\varphi}(t) - \omega(t,\tilde{\varphi}(t)))d_tF(x-t), \quad x \geq 0,$$

has a positive and bounded solution $\tilde{\varphi}_{\beta^*}(x)$ with the following properties:

$$\lim_{x\to\infty} \tilde{\varphi}_{\beta^*}(x) = 2\beta^*(1-w_+(0))^{-1} = \xi \tag{61}$$

and

$$S_{\beta^*}(x) \leq 2S_{\beta^*}(x) - \tilde{Q}_{\beta^*}(x) \leq \tilde{\varphi}_{\beta^*}(x) \leq 2S_{\beta^*}(x). \tag{62}$$

We now show that

$$\xi - \tilde{\varphi}_{\beta^*}(x) \in L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+). \tag{63}$$

The inclusion

$$\xi - \tilde{\varphi}_{\beta^*} \in L_\infty^0(\mathbb{R}^+)$$

directly follows from (61) and (62). We prove that

$$\xi - \tilde{\varphi}_{\beta^*} \in L_1(\mathbb{R}^+).$$

By virtue of (62) and $\lim_{x\to\infty} \tilde{Q}_{\beta^*}(x) = 0$, we get

$$0 \leq \xi - \tilde{\varphi}_{\beta^*}(x) \leq \xi - 2S_{\beta^*}(x) + \tilde{Q}_{\beta^*}(x), \quad x \in \mathbb{R}^+. \tag{64}$$

Since

$$\xi - 2S_{\beta^*}(x) = 2\beta^*\left(\frac{1}{1-w_+(0)} - S(x)\right) \in L_1(\mathbb{R}^+)$$

[see relation (14)] and $\tilde{Q}_{\beta^*} \in L_1(\mathbb{R}^+)$, we derive inclusion (63) from (64).

On the other hand, we check the validity of inequality

$$\xi - \tilde{\varphi}_{\beta^*}(x) \geq \mu_{\xi_0}(x).$$

By using the chain of inequalities (62) and relations (13) and (14), we obtain

$$\xi - \tilde{\varphi}_{\beta^*}(x) \geq \xi - 2S_{\beta^*}(x) = 2\beta^* \left(\frac{1}{1 - w_+(0)} - S(x) \right)$$

$$= 2\beta^* \left(\frac{1}{1 - w_+(0)} \bigvee_x^\infty F + \frac{1}{1 - w_+(0)} \bigvee_{-\infty}^x F - S(x) \right)$$

$$= 2\beta^* \left(\frac{1}{1 - w_+(0)} \bigvee_x^\infty F + \int_{-\infty}^x \left(\frac{1}{1 - w_+(0)} - S(x-t) \right) dF(t) \right)$$

$$\geq \frac{2\beta^*}{1 - w_+(0)} (1 - F(x)) = \xi(1 - F(x)) \geq \xi_0(1 - F(x)) = \mu_{\xi_0}(x).$$

We now consider the following iterations:

$$\varphi_{n+1}(x) = -\int_0^\infty \mathcal{F}_0(t, \varphi_n(t)) d_t F(x-t) + \mathcal{F}_1(x, \varphi_n(x)),$$
$$\varphi_0(x) = \xi - \tilde{\varphi}_{\beta^*}(x), \qquad n = 0, 1, 2, \ldots, \quad x \geq 0. \tag{65}$$

By the induction on $n,$ we can show that

$$\varphi_n(x) \downarrow \quad \text{with} \quad n, \tag{66}$$

$$\varphi_n(x) \geq \mu_{\xi_0}(x), \quad n = 0, 1, 2, \ldots. \tag{67}$$

We first prove that

$$\varphi_1(x) \geq \mu_{\xi_0}(x), \qquad \varphi_1(x) \leq \varphi_0(x), \quad x \in \mathbb{R}^+.$$

In view of the inequality in condition (A) and the properties of the function $\mathcal{F}_0,$ we get the following inequalities from (65):

$$\varphi_1(x) \geq \mathcal{F}_1(x, \mu_{\xi_0}(x)) \geq \mu_{\xi_0}(x), \quad x \in \mathbb{R}^+,$$

$$\varphi_1(x) = \mathcal{F}_1(x, \xi - \tilde{\varphi}_{\beta^*}(x)) - \int_0^\infty \mathcal{F}_0(t, \xi - \tilde{\varphi}_{\beta^*}(t)) d_t F(x-t)$$

$$\leq \mathcal{F}_1(x, \xi) - \int_0^\infty (\xi - \tilde{\varphi}_{\beta^*}(t) + \omega(t, \xi)) d_t F(x-t)$$

$$\leq \mu_\xi(x) + \xi F(x) - \int_0^\infty (-\tilde{\varphi}_{\beta^*}(t) + \omega(t, \xi)) d_t F(x-t)$$

$$\leq \xi - \int_0^\infty (-\tilde{\varphi}_{\beta^*}(t) + \omega(t, \tilde{\varphi}_{\beta^*}(t)))d_t F(x-t) = \xi - \tilde{\varphi}_{\beta^*}(x) = \varphi_0(x)$$

because $\omega(t,\xi) \leq \omega(t, \tilde{\varphi}_{\beta^*}(t))$ by virtue of the inequalities $\tilde{\varphi}_{\beta^*}(t) \leq \xi$ and $\tilde{\varphi}_{\beta^*}(t) \geq A$.

We now assume that $\varphi_n(x) \leq \varphi_{n-1}(x)$ and $\varphi_n(x) \geq \mu_{\xi_0}(x)$ for some $n \in \mathbb{N}$ and take into account the monotonicity of the functions $\mathcal{F}_0$ and $\mathcal{F}_1$ in z. As a result, from (65), we obtain

$$\varphi_{n+1}(x) \leq \varphi_n(x) \qquad \text{and} \qquad \varphi_{n+1}(x) \geq \mu_{\xi_0}(x).$$

Hence, the sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$:

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x).$$

Moreover, by virtue of the condition (D) and the B. Levi theorem, this limit satisfies Eq. (1). Relations (66) and (67) also imply that

$$\mu_{\xi_0}(x) \leq \varphi(x) \leq \xi - \tilde{\varphi}_{\beta^*}(x), \quad x \in \mathbb{R}^+,$$

whence it follows that $\varphi \in L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+)$.

The theorem is proved.

Note that the role of the function $\mathcal{F}_1$ in Theorem 3, can be played by the examples presented in Sec. 3 for $\eta_0 = \xi_0$ and $\eta = \xi$.

We now present one more example of the function $\mathcal{F}_0$ satisfying all conditions of Theorem 3:

$$\mathcal{F}_0(t,z) = z + \omega(t,\xi) \sin \frac{z}{\omega(t,\xi)}, \qquad t \in \mathbb{R}^+, \quad z \in [0,\xi].$$

Remark 3. Unlike the assertion of Theorem 2, the solution $\varphi(x)$ constructed in Theorems 1 and 3 cannot be a distribution function because

$$\varphi \in L_1(\mathbb{R}^+) \cap L_\infty^0(\mathbb{R}^+).$$

The present work was financially supported by the State Committee of Science of the Ministry of Education and Science of the Republic of Armenia within the framework of the Scientific Project SCS 13YR-1A0003.

REFERENCES

1. C. Cercignani, *Theory and Application of the Boltzmann Equation*, Scottish Academic Press, Edinburgh (1975).
2. I. D. Sargan, " The distribution of wealth," *Econometrics*, **25**, No. 4, 568–590 (1957).
3. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, Wiley, New York (1966).
4. Kh. A. Khachatryan, *Problems of Solvability of Some Nonlinear Integral and Integrodifferential Equations with Noncompact Operators in the Critical Case* [in Russian], Doctoral-Degree Thesis (Physics and Mathematics), Erevan (2011).
5. D. V. Lindley, "The theory of queue with a single sever," *Proc. Cambridge Phil. Soc.*, No. 48, 277–289 (1952).
6. F. D. Gakhov and Yu. I. Cherskii, *Equations of the Convolution Type* [in Russian], Nauka, Moscow (1978).
7. F. Spitzer, "The Wiener–Hopf equation whose kernel is probability density," *Duke Math. J.*, **24**, No. 3, 323–343 (1957).
8. N. B. Engibaryan and A. A. Arutyunyan, "Integral equations on a half line with difference kernels and nonlinear functional equations," *Mat. Sb.*, **97**, No. 1, 35–58 (1975).

9. L. G. Arabadzhyan and N. B. Engibaryan, "Convolution equations and nonlinear functional equations," in: *VINITI Series in Mathematical Analysis* [in Russian], Vol. 22 (1984), pp. 175–240.
10. V. A. Ambartsumyan, *Scientific Works* [in Russian], Vol. 1, Erevan (1960).
11. M. V. Maslennikov, "Milne problem with anisotropic scattering," *Tr. Mat. Inst. Akad. Nauk SSSR*, **97** (1968).
12. N. B. Engibaryan and A. Kh. Khachatryan, "Problems of the nonlinear theory of the dynamics of rarefied gases," *Mat. Model.*, **16**, No. 1, 67–74 (2004).
13. N. B. Engibaryan, "Convolution equations containing singular probability distributions," *Izv. Ros. Akad. Nauk, Ser. Mat.*, **60**, No. 2, 21–48 (1996).
14. A. Kh. Khachatryan and Kh. A. Khachatryan, "On convolution-type nonlinear integral equations containing singular and discrete probability distributions," *Adv. Appl. Math. Sci. (India)*, **5**, No. 1, 1–15 (2010).
15. Kh. A. Khachatryan, "On a class of nonlinear integral equation with a noncompact operator," *J. Contemp. Math. Anal.*, **46**, No. 2, 89–100 (2011).
16. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* [in Russian], Nauka, Moscow (1981).