DOI 10.1007/s11253-015-1057-5
Ukrainian Mathematical Journal, Vol. 66, No. 12, May, 2015 (Ukrainian Original Vol. 66, No. 12, December, 2014)

CRITICAL POINTS APPROACHES TO ELLIPTIC PROBLEMS DRIVEN BY A $p(x)$-LAPLACIAN

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UDC 517.9

We establish the existence of at least three solutions for elliptic problems driven by a $p(x)$-Laplacian. The existence of at least one nontrivial solution is also proved. The approaches are based on the variational methods and critical-point theory.

1. Introduction

In the present paper, we study the following elliptic problem:

$$
\begin{gathered}
-\Delta_{p(x)}u = \lambda f(x,u) \qquad \text{in} \quad \Omega, \\
u = 0 \qquad \text{on} \quad \partial\Omega,
\end{gathered}
\tag{1}
$$

where

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$$

is the $p(x)$-Laplacian operator, $\Omega \subset R^N,\ N \geq 1,$ is a nonempty bounded open set with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ satisfies the condition

$$N < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) < +\infty,$$

$\lambda > 0,$ and $f : \Omega \times R \to R$ is an L^1-Carathéodory function.

In recent years, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. It arises from the nonlinear elasticity theory, the theory of electrorheological fluids, etc. (see [29, 31]). Problems of this kind also have extensive applications in various research fields, such as the image-processing model (see, e.g., [16, 24]), stationary thermorheological viscous flows (see [1]), and the mathematical description of the processes of filtration of ideal barotropic gases through porous media (see [2]).

Note that, for $p(x) = p =$ constant, there is a large literature dealing with the problems involving the p-Laplacian with Dirichlet boundary conditions, both in the scalar case and for elliptic systems in bounded or unbounded domains. It is not necessary to cite these works here because the reader can easily find them. Numerous authors investigated the existence and multiplicity of solutions for the problems with $p(x)$-Laplacian. In recent years, the interest to the study of variational problems and elliptic equations with variable exponent is increasing. We refer the reader to [19, 21, 27] for the theory of $L^{p(x)}$ and $W^{1,p(x)}(\Omega)$. The problem of $p(x)$-Laplacian with

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Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 66, No. 12, pp. 1676–1693, December, 2014. Original article submitted October 4, 2012; revision submitted June 26, 2014.

Dirichlet conditions in the scalar case was studied by Fan and Zhang [20]. In fact, in [20], Fan and Zhang, first, introduced some basic properties of the generalized Lebesgue–Sobolev spaces $W_0^{1,p(x)}(\Omega)$, which can be regarded as a special class of generalized Orlicz–Sobolev spaces, second, presented several important properties of the $p(x)$-Laplace operator, and finally, under certain appropriate conditions imposed on the nonlinear term, established some existence results for the weak solutions of problem (1). Bonanno and Chinnì [7] used a three-critical-point theorem for nondifferentiable functionals due to Bonanno and Marano [12] (Theorem 3.6) and established the existence of at least three weak solutions for the problem

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda(f(x,u)+\mu g(x,u)) && \text{in} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset R^N,\ N \geq 1$, is a nonempty bounded open set with smooth boundary $\partial\Omega,\ p \in C(\overline{\Omega}),\ \lambda$ and μ are two positive parameters, and $f, g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are two functions measurable with respect to each variable separately and possibly discontinuous with respect to u. In [8], Bonanno and Chinnì also studied the multiplicity of solutions for problem (1) on the basis of a convenient form of the recent three-critical-points theorem established by Bonanno and Marano [12].

In the present paper, motivated by [7, 8], we first apply two related three-critical-points theorems for differentiable functionals due to Bonanno and Candito [6] to prove the existence of at least three weak solutions to problem (1) (see Theorems 4 and 5) and then use a very recent local minimum theorem for differentiable functionals due to Bonanno [5] (under different assumptions imposed in Theorems 4 and 5) to establish the existence of at least one nontrivial weak solution to problem (1) (see Theorem 7). Theorems 4 and 5 extend the results of [8].

For a thorough account on the subject, we refer the reader to the papers [9, 10, 13, 14, 22, 23, 26].

2. Preliminaries and Basic Notation

In the present section, we introduce some definitions and results used in the next section. First, we introduce some theories of Lebesgue–Sobolev spaces with variable exponent. The details can be found in [17, 19, 21]. We set

$$L_+^\infty(\Omega) = \Big\{ p \in L^\infty(\Omega)\colon \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1 \Big\}.$$

For $p \in L_+^\infty(\Omega)$, we denote

$$p^- = p^-(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = p^+(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

For any $p(x) \in L_+^\infty(\Omega)$, we define the variable-exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u\colon u \ \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}.$$

We introduce a norm (the so-called *Luxemburg norm*) in this space by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space. We define the variable-exponent Sobolev space $W^{1,p(x)}(\Omega)$ as follows:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \colon |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

This space is equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

By $W_0^{1,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. In $W_0^{1,p(x)}(\Omega)$, we consider the norm

$$\|u\| \colon = \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

We now present some facts used in what follows.

Proposition 1 (see [20]). *(i) The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) There is a constant $c > 0$ such that

$$\|u\|_{L^{p(x)}(\Omega)} \leq c\|\nabla u\|_{L^{p(x)}(\Omega)}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proposition 2 (see [7]). *Let*

$$\rho_p(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

For $u \in W_0^{1,p(x)}(\Omega)$, the following assertions are true:

(i) $\|u\| < 1 (= 1; > 1) \Longleftrightarrow \rho_p(|\nabla u|) < 1 (= 1; > 1)$.

(ii) If $\|u\| > 1$, then $\dfrac{1}{p^+}\|u\|^{p^-} \leq \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)} dx \leq \frac{1}{p^-}\|u\|^{p^+}$.

(iii) If $\|u\| < 1$, then $\dfrac{1}{p^+}\|u\|^{p^+} \leq \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)} dx \leq \frac{1}{p^-}\|u\|^{p^-}$.

As shown in [20, 27], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and, since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, $W^{1,p(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Therefore, in particular, there exists a positive constant c_0 such that

$$\|u\|_{C^0(\overline{\Omega})} \leq c_0\|u\| \tag{2}$$

for each $u \in W_0^{1,p(x)}(\Omega)$.

By X, we denote the Sobolev space $W_0^{1,p(x)}(\Omega)$. Let

$$G(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx$$

for all $u \in X$. We denote $L = G' \colon X \to X^*$. Then

$$L(u)(v) = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x)\, dx$$

for all $u, v \in X$.

Proposition 3 (see [20]). *(i)* $L \colon X \to X^*$ *is a continuous, bounded, and strictly monotone operator.*

(ii) L *is a mapping of the type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}(L(u_n), u_n - u) \leq 0$, *then* $u_n \to u$ *in* X.

(iii) $L \colon X \to X^*$ *is a homeomorphism.*

We say that u is a weak solution to the problem (1) if $u \in X$ and

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx - \lambda \int_\Omega f(x, u(x)) v(x) dx = 0$$

for every $v \in X$.

We set

$$\delta(x) = \sup \left\{ \delta > 0 \colon S(x, \delta) \subset \Omega \right\}$$

where $S(x, \delta)$ denotes a ball of radius δ with center at x. For all $x \in \Omega$, we can prove that there exists $x_0 \in \Omega$ such that $S(x_0, D) \subset \Omega$, where $D = \sup_{x \in \Omega} \delta(x)$. We set

$$m := \frac{\pi^{N/2}}{\frac{N}{2} \Gamma\left(\frac{N}{2}\right)}$$

where Γ is the Euler function. Moreover, for each $r > 0$, we define

$$\gamma_r := \max \left\{ (p^+ r)^{1/p^-},\, (p^+ r)^{1/p^+} \right\}$$

and, in addition,

$$F(x, t) = \int_0^t f(x, \xi)\, d\xi$$

for all $(x, t) \in \Omega \times R$.

3. Existence of Three Solutions

In the present section, we establish the existence of at least three weak solutions of problem (1). As our main tools, we use two three-critical-points theorems. In the first of these theorems, the coercivity of the functional $\Phi - \lambda\Psi$ is required; in the second theorem, a suitable sign hypothesis is assumed. The first result was obtained in [4]. The second result was obtained in [3]. Here, we recall these results according to [6].

Theorem 1 ([6], Theorem 3.2). *Let X be a reflexive real Banach space, let $\Phi\colon X \longrightarrow R$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, let $\Psi\colon X \longrightarrow R$ be a continuously Gâteaux differentiable functional whose derivative is compact, and let*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there is a positive constant r and $\overline{v} \in X$ with $2r < \Phi(\overline{v})$ such that

(a_1) $$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

(a_2) for all

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points.

Theorem 2 ([6], Theorem 3.3). *Let X be a reflexive real Banach space, let $\Phi\colon X \longrightarrow R$ be a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, let $\Psi\colon X \longrightarrow R$ be a continuously Gâteaux differentiable functional whose derivative is compact, and let*

(i) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(ii) for any $\lambda > 0$ and for every u_1 and u_2 that are local minima for the functional $\Phi - \lambda\Psi$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, the following inequality be true:

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1 and r_2 and $\overline{v} \in X$ with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$ such that

(b_1) $$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

(b_2) $$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points lying in* $\Phi^{-1}(]-\infty, r_2[)$.

The following theorem is a special case of our main results:

Theorem 3. *Let* $\Omega \subseteq R^2$ *be a nonempty bounded open set with smooth boundary* $\partial\Omega$, *let* $f\colon R \to R$ *be a continuous function, and let*

$$F(t) = \int\limits_0^t f(\xi)\, d\xi$$

for all $t \in R$ *be such that* $F(h) > 0$ *for some* $h > 0$ *and* $F(\xi) \geq 0$ *in* $[0, h]$. *For fixed* $p(x) = p > 2$, *it is assumed that*

$$\liminf_{\xi\to 0}\frac{F(\xi)}{|\xi|^p} = \limsup_{|\xi|\to+\infty}\frac{F(\xi)}{|\xi|^p} = 0.$$

Then there is $\lambda^* > 0$ *such that, for each* $\lambda > \lambda^*$, *the problem*

$$-\Delta_p u = \lambda f(u) \qquad \text{in} \quad \Omega,$$

$$u = 0 \qquad \text{on} \quad \partial\Omega$$

admits at least three weak solutions.

Remark 1. The results similar to Theorem 3 were obtained in [11] (Theorem 0), where a class of Dirichlet quasilinear elliptic systems driven by a (p, q)-Laplacian operator was considered, and also in [25] (Theorem 1), where a quasilinear second-order differential equation was studied.

We formulate the existence results as follows:

Theorem 4. *Let* $f\colon \Omega \times R \to R$ *be an* L^1*-Carathéodory function such that*

$$\operatorname*{ess\,inf}_{x\in\Omega} F(x,\xi) \geq 0$$

for all $\xi \in R$. *Assume that there exist two positive constants* r *and* h *such that*

(A 1) $$\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N} > 2r;$$

(A 2) $$\frac{\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}{r} < \frac{2\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{\frac{3}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)};$$

$$(A_3) \qquad \limsup_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^-}/p^+} < \frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r}.$$

Then, for each

$$\lambda \in \left] \frac{3}{2} \frac{\frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} (2^N - 1)}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}, \frac{r}{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx} \right[,$$

problem (1) admits at least three weak solutions.

Proof. In order to apply Theorem 1 to our problem, we introduce the functionals Φ, $\Psi : X \to R$ for each $u \in X$, as follows:

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \left|\nabla u(x)\right|^{p(x)} dx$$

and

$$\Psi(u) = \int_\Omega F(x, u(x))dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u \in X$ are functionals $\Phi'(u), \Psi'(u) \in X^*$ given, for every $v \in X$, by the formulas

$$\Phi'(u)(v) = \int_\Omega \left|\nabla u(x)\right|^{p(x)-2} \nabla u(x) \nabla v(x) dx$$

and

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx,$$

respectively. At the same time, Ψ is sequentially weakly upper semicontinuous. Moreover, Φ is sequentially weakly lower semicontinuous and Φ' admits a continuous inverse on X^*. Furthermore, $\Psi' : X \to X^*$ is a compact operator. We set

$$w(x) = \begin{cases} 0 & \text{for} \quad x \in \Omega \setminus S(x_0, D), \\ h & \text{for} \quad x \in S\left(x_0, \frac{D}{2}\right), \\ \frac{2h}{D}\left(D - \sqrt{\sum_{i=1}^N (x_i - x_{0i})^2}\right) & \text{for} \quad x \in S(x_0, D) \setminus S\left(x_0, \frac{D}{2}\right). \end{cases} \tag{3}$$

It is easy to see that $w \in X$ and, in particular, we get

$$\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}mD^N\frac{2^N-1}{2^N} \leq \Phi(w)$$

$$\leq \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}mD^N\frac{2^N-1}{2^N} \tag{4}$$

and

$$\Psi(w) \geq \int\limits_{S(x_0,\,D/2)} F(x,w(x))dx \geq \operatorname*{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N. \tag{5}$$

In view of (4), it follows from (A_1) that $\Phi(w) > 2r$. The embedding $X \hookrightarrow C^0(\overline{\Omega})$ implies that

$$\Phi^{-1}(]-\infty,r[) = \{u \in X; \Phi(u) < r\} = \left\{u \in X; \int\limits_{\Omega}\frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx < r\right\}$$

$$\subseteq \Big\{u \in X; |u(x)| \leq c_0\gamma_r \text{ for all } x \in \Omega\Big\}.$$

This yields

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r[)}\int\limits_{\Omega}F(x,u(x))dx \leq \int\limits_{\Omega}\sup_{|t|\leq c_0\gamma_r}F(x,t)\,dx.$$

Therefore, in view of the assumption (A_2) and inequalities (4) and (5), we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\int_\Omega F(x,u(x))dx}{r}$$

$$\leq \frac{\int_\Omega \sup_{|t|\leq c_0\gamma_r}F(x,t)dx}{r}$$

$$< \frac{2}{3}\frac{\operatorname{ess\,inf}_{x\in\Omega}F(x,h)}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Furthermore, it follows from (A_3) that there exist two constants $\eta, \vartheta \in R$ with

$$\eta < \frac{\int_\Omega \sup_{|t|\leq c_0\gamma_r}F(x,t)dx}{r}$$

such that

$$|\Omega|c_0F(x,t) \leq \eta\frac{|t|^{p^-}}{p^+} + \vartheta \qquad \text{for all} \quad x \in \Omega \qquad \text{and all} \quad t \in R^n.$$

We fix $u \in X$. Then

$$F(x,u(x)) \leq \frac{1}{|\Omega|c_0}\left(\eta\frac{|u(x)|^{p^-}}{p^+} + \vartheta\right) \qquad \text{for all} \quad x \in \Omega. \tag{6}$$

Further, in order to prove the coercivity of the functional $\Phi-\lambda\Psi,$ we first assume that $\eta > 0.$ Thus, if $\|u\| \geq 1,$ then, for any fixed

$$\lambda \in \left]\frac{3}{2}\frac{\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}, \frac{r}{\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}\right[,$$

in view of inequalities (2) and (6) and Proposition 2, we find

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx - \lambda\int_\Omega F(x,u(x))dx \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda\eta}{|\Omega|c_0}\frac{\int_\Omega |u(x)|^{p^-}dx}{p^+} - \frac{\lambda\vartheta}{c_0} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda\eta}{|\Omega|c_0}\frac{|\Omega|c_0\|u\|^{p^-}}{p^+} - \frac{\lambda\vartheta}{c_0} \\
&\geq \left(1 - \eta\frac{r}{\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}\right)\frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda\vartheta}{c_0},
\end{aligned}$$

and, hence,

$$\lim_{\|u\|\to+\infty}\big(\Phi(u) - \lambda\Psi(u)\big) = +\infty.$$

On the other hand, for $\eta \leq 0,$ it is clear that

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

Both cases lead to the coercivity of the functional $\Phi - \lambda\Psi.$

Thus, the assumptions (a_1) and (a_2) in Theorem 1 are satisfied. Hence, by using Theorem 1, in view of the fact that the weak solutions of problem (1) are exactly the solutions of the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0,$$

we arrive at the required assertion.

Theorem 4 is proved.

Theorem 5. *Let $f : \Omega \times R \to R$ be an L^1-Carathéodory function such that*

$$\operatorname*{ess\,inf}_{x\in\Omega} F(x,\xi) \geq 0$$

for all $\xi \in R$ and satisfies the condition $f(x,t) \geq 0$ for all $(x,t) \in \Omega \times (R^+ \cup \{0\})$. Assume that there exist three positive constants r_1, r_2, and h with

$$2r_1 < \frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N-1}{2^N} < \frac{r_2}{2}$$

such that

(B₁) $$\frac{\displaystyle\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}{r_1} < \frac{2\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{3\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}(2^N-1)};$$

(B₂) $$\frac{\displaystyle\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}{r_2} < \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{3\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}(2^N-1)}.$$

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}(2^N-1)}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)},\right.$$

$$\left.\min\left\{\frac{r_1}{\displaystyle\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}, \frac{\dfrac{r^2}{2}}{\displaystyle\int_\Omega \sup_{|t|\leq c_0\gamma_r} F(x,t)dx}\right\}\right[$$

problem (1) admits at least three nonnegative weak solutions $v^1,\ v^2,\ v^3$ such that

$$\int\limits_\Omega \frac{1}{p(x)}|\nabla v^j(x)|^{p(x)}dx \le r_2 \qquad \textit{for each} \quad x\in\Omega, \quad j=1,2,3.$$

Proof. Let Φ and Ψ be as in the proof of Theorem 4. We apply Theorem 2 to our functionals. Obviously, Φ and Ψ satisfy Condition 1 of Theorem 2. We now check that the functional $\Phi-\lambda\Psi$ satisfies Condition 2 of Theorem 2. Let $u^\star$ and $u^{\star\star}$ be two local minima of $\Phi-\lambda\Psi$. Then $u^\star$ and $u^{\star\star}$ are critical points for $\Phi-\lambda\Psi$ and, hence, they are weak solutions of problem (1). Since $f(x,t)\ge 0$ for all $(x,t)\in\Omega\times(R^+\cup\{0\})$, it follows from the weak maximum principle (see, e.g., [15]) that $u^\star(x)\ge 0$ and $u^{\star\star}(x)\ge 0$ for every $x\in\Omega$. This means that $su^\star+(1-s)u^{\star\star}\ge 0$ for all $s\in[0,1]$, that $f(su^\star+(1-s)u^{\star\star},t)\ge 0$ and, consequently, that $\Psi(su^\star+(1-s)u^{\star\star})\ge 0$ for all $s\in[0,1]$. Moreover, the conditions

$$2r_1 < \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N} < \frac{r_2}{2}$$

enable us to conclude that

$$2r_1 < \Phi(w) < \frac{r_2}{2}.$$

Further, in view of the embedding $X\hookrightarrow C^0(\overline{\Omega})$, we get

$$\Phi^{-1}\big(]-\infty,r_1[\big)=\big\{u\in X;\Phi(u)<r_1\big\}=\left\{u\in X;\int\limits_\Omega\frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx<r_1\right\}$$
$$\subseteq\big\{u\in X;\ |u(x)|\le c_0\gamma_{r_1}\ \text{ for all }\ x\in\Omega\big\}$$

and, hence,

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)=\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int\limits_\Omega F(x,u(x))dx\le\int\limits_\Omega\sup_{|t|\le c_0\gamma_r}F(x,t)dx.$$

Therefore, in view of the assumption (B_1), we find

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1}=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int_\Omega F(x,u(x))dx}{r_1}$$
$$\le\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_1}}F(x,t)dx}{r_1}$$

$$< \frac{2}{3} \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

As above, by using the assumption (B_2), we obtain

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\int_\Omega F(x,u(x))dx}{r_2}\\ &\leq \frac{\int_\Omega \sup_{|t|\leq c_0\gamma_{r_2}} F(x,t)dx}{r_2}\\ &< \frac{1}{3} \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)} \leq \frac{1}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Thus, the assumptions (b_1) and (b_2) in Theorem 2 are satisfied. Hence, in view of Theorem 2 and the fact that the weak solutions of problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, problem (1) admits at least three distinct weak solutions in X.

Theorem 5 is proved.

At the end of this section, we prove Theorem 3.

Proof of Theorem 3. We fix

$$\lambda > \lambda^* := \frac{9}{2}\frac{\frac{1}{p}\left(\frac{2h}{D}\right)^p}{F(h)}.$$

Since $\liminf_{\xi\to 0}\frac{F(\xi)}{|\xi|^p} = 0$, there is $\{r_n\}_{n\in N} \subseteq]0,+\infty[$ such that $\lim_{n\to+\infty} r_n = 0$ and

$$\lim_{n\to+\infty}\frac{|\Omega|\max_{|t|\leq c_0(pr_n)^{1/p}} F(t)}{r_n} = 0.$$

Hence, there exists $\overline{r} > 0$ such that

$$\frac{|\Omega|\max_{|t|\leq c_0(p\overline{r})^{1/p}} F(t)}{\overline{r}} < \min\left\{\frac{2p}{9}\frac{F(h)}{\left(\frac{2h}{D}\right)^p}; \frac{1}{\lambda}\right\}$$

and

$$2\overline{r} < \frac{3D^2\pi}{4p}\left(\frac{2h}{D}\right)^p.$$

The required assertion follows from Theorem 4.

4. Existence of a Nontrivial Solution

First, we recall, for the reader's convenience, Theorem 2.5 from [28] presented in the form of Theorem 5.1 from [5] (see also Proposition 2.1 in [5] for the related results). This theorem is our main tool in proving the main result.

For a given nonempty set X and two functionals $\Phi, \Psi \colon X \to R,$ we define the following functions:

$$\vartheta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in R,\ r_1 < r_2.$

Theorem 6 ([5], Theorem 5.1)**.** *Let X be a reflexive real Banach space, let $\Phi \colon X \to R$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi \colon X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Also let $I_\lambda = \Phi - \lambda\Psi$. Assume that there are $r_1, r_2 \in R,\ r_1 < r_2,$ such that*

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each

$$\lambda \in \left] \frac{1}{\rho(r_1, r_2)}, \frac{1}{\vartheta(r_1, r_2)} \right[,$$

there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for any $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0.$

We formulate the main result of this section as follows:

Theorem 7. *Let $f : \Omega \times R \to R$ be an L^1-Carathéodory function such that*

$$\operatorname*{ess\,inf}_{x \in \Omega} F(x, \xi) \geq 0$$

for all $\xi \in R$. Assume that there exist a nonnegative constant r_1 and two positive constants r_2 and h with

$$r_1 < \frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} mD^N \frac{2^N - 1}{2^N}$$

and

$$\frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} mD^N \frac{2^N - 1}{2^N} < r_2$$

such that

$$(C_1)\qquad \frac{\int_\Omega \sup_{|t|\le c_0\gamma_{r_2}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N}{r_2 - \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}}$$

$$< \frac{\int_\Omega \sup_{|t|\le c_0\gamma_{r_1}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N}{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}}. \tag{7}$$

Then, for each

$$\lambda \in \left] \frac{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_1}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N}, \right.$$

$$\left. \frac{r_2 - \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_2}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N} \right[,$$

problem (1) admits at least one nontrivial weak solution $u_0 \in X$ such that

$$r_1 < \int_\Omega \frac{1}{p(x)}|\nabla u_0(x)|^{p(x)}dx < r_2.$$

Proof. In order to apply Theorem 6 to our problem, we assume that the functionals $\Phi, \Psi\colon X \to R$ are the same as in the proof of Theorem 4. As follows from the proof of Theorem 4, Φ and Ψ satisfy the regularity assumptions of Theorem 6. We choose w as indicated in (3). Taking into account relation (4), in view of the conditions

$$r_1 < \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N} < r_2,$$

we find

$$r_1 < \Phi(w) < r_2.$$

By virtue of the embedding $X \hookrightarrow C^0(\overline{\Omega})$, we get

$$\Phi^{-1}\big(]-\infty, r_2[\big) = \big\{u \in X; \Phi(u) < r_2\big\} = \left\{u \in X; \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx < r_2\right\}$$

$$\subseteq \big\{u \in X;\ |u(x)| \le c_0 \gamma_{r_2} \text{ for all } x \in \Omega\big\}.$$

This yields

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \int_\Omega F(x, u(x)) dx \le \int_\Omega \sup_{|t| \le c_0 \gamma_{r_2}} F(x,t) dx.$$

Therefore, we obtain

$$\vartheta(r_1, r_2) \le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)}$$

$$\le \frac{\displaystyle\int_\Omega \sup_{|t| \le c_0 \gamma_{r_2}} F(x,t) dx - \Psi(w)}{r_2 - \Phi(w)}$$

$$\le \frac{\displaystyle\int_\Omega \sup_{|t| \le c_0 \gamma_{r_2}} F(x,t) dx - \operatorname{ess\,inf}_{x \in \Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{r_2 - \dfrac{1}{p^-} \max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

On the other hand, arguing as earlier, we conclude that

$$\rho(r_1, r_2) \ge \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(w) - r_1}$$

$$\ge \frac{\Psi(w) - \displaystyle\int_\Omega \sup_{|t| \le c_0 \gamma_{r_1}} F(x,t) dx}{\Phi(w) - r_1}$$

$$\ge \frac{\displaystyle\int_\Omega \sup_{|t| \le c_0 \gamma_{r_1}} F(x,t) dx - \operatorname{ess\,inf}_{x \in \Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{r_1 - \dfrac{1}{p^+} \min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

Hence, it follows from the assumption (C_1) that

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Therefore, by applying Theorem 6 for each

$$\lambda \in \left] \frac{r_1 - \frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_1}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}, \right.$$

$$\left. \frac{r_2 - \frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_2}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N} \right[,$$

we conclude that the functional $\Phi - \lambda\Psi$ has at least one critical point $u_0 \in X$ such that $r_1 < \Phi(u_0) < r_2$, i.e.,

$$r_1 < \int_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

Thus, in view of the fact that the weak solutions of problem (1) are exactly the solutions of the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0,$$

we arrive at the required assertion.

Theorem 7 is proved.

We now establish the following corollary of Theorem 7:

Theorem 8. *Suppose that*

$$\frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} \le \frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}.$$

Let $f : \Omega \times R \to R$ be an L^1-Carathéodory function such that $\operatorname{ess\,inf}_{x\in\Omega} F(x,\xi) \ge 0$ for all $\xi \in R$. Assume that there exist two positive constants r and h with

$$\frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m D^N \frac{2^N - 1}{2^N} < r$$

such that

$$(C_2) \qquad \frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r} < \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{\frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}.$$

Then, for each

$$\lambda \in \left] \frac{\frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}, \frac{r}{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx} \right[,$$

problem (1) admits at least one nontrivial weak solution $u_0 \in X$ *such that*

$$r_1 < \int_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

Proof. The required assertion follows from Theorem 7 if we take $r_1 = 0$ and $r_2 = r$. Indeed, in view of our assumptions, we get

$$\frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{r - \frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}$$

$$< \frac{\left(1 - \frac{\frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}{r}\right) \int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r - \frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}$$

$$= \frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r} < \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{\frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}$$

$$\le \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{\frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}}.$$

In particular, we find

$$\frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{r - \frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\}} < \frac{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r}.$$

Hence, the application of Theorem 7 completes the proof.

Theorem 8 is proved.

Let $f\colon R \to R$ be a continuous function and let

$$F(t) = \int_0^t f(\xi)\, d\xi$$

for all $t \in R$. The following result is obtained as a direct consequence of Theorem 7.

Theorem 9. *Let $f\colon R \to R$ be a nonnegative continuous function. Assume that there exist a nonnegative constant r_1 and two positive constants r_2 and h with*

$$r_1 < \frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N-1}{2^N} < r_2$$

such that

$$\text{(C}_3\text{)} \qquad \frac{|\Omega| F(c_0\gamma_{r_2}) - F(h)m\left(\frac{D}{2}\right)^N}{r_2 - \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}} < \frac{|\Omega| F(c_0\gamma_{r_1}) - F(h)m\left(\frac{D}{2}\right)^N}{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}.$$

Then, for each

$$\lambda \in \left] \frac{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{|\Omega| F(c_0\gamma_{r_1}) - F(h)m\left(\frac{D}{2}\right)^N}, \frac{r_2 - \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{|\Omega| F(c_0\gamma_{r_2}) - F(h)m\left(\frac{D}{2}\right)^N} \right[,$$

the problem

$$-\Delta_{p(x)} u = \lambda f(u) \quad \text{in} \quad \Omega,$$

$$u = 0 \qquad \text{on} \quad \partial\Omega$$

admits at least one nontrivial weak solution $u_0 \in X$ such that

$$r_1 < \int_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

At the end of the paper, we present the following special case of our main result in this section:

Theorem 10. *Let $p(x) = p > N$, let $h\colon \Omega \to R$ be a positive and essentially bounded function, and let $g\colon R \to R$ be a nonnegative function such that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty.$$

Then, for each

$$\lambda \in \left]0, \left(\frac{1}{\displaystyle\int_\Omega h(x)dx}\right) \sup_{r>0} \frac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi)\, d\xi}\right[,$$

the problem

$$-\Delta_p u = \lambda h(x) g(u) \quad \text{in} \quad \Omega,$$

$$u = 0 \qquad\qquad \text{on} \quad \partial\Omega$$

admits at least one nontrivial weak solution in X.

Proof. For fixed

$$\lambda \in \left]0, \left(\frac{1}{\displaystyle\int_\Omega h(x)dx}\right) \sup_{r>0} \frac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi)\, d\xi}\right[,$$

there exists a positive constant r such that

$$\lambda < \left(\frac{1}{\displaystyle\int_\Omega h(x)dx}\right) \frac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi) d\xi}.$$

Moreover, the condition

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty$$

implies that

$$\lim_{t\to 0^+} \frac{\displaystyle\int_0^t g(\xi) d\xi}{t^p} = +\infty.$$

Therefore, we can choose a positive constant h satisfying the inequality

$$\frac{1}{p}\left(\frac{2h}{D}\right)^p m D^N \frac{2^N - 1}{2^N} < r$$

such that

$$\left(\frac{1}{\lambda}\right)\frac{2^p}{pD^p \operatorname{ess\,inf}_{x\in\Omega} h(x) m \left(\dfrac{D}{2}\right)^N} < \frac{\displaystyle\int_0^h g(\xi)d\xi}{h^p}.$$

Hence, by using Theorem 8, we arrive at the required result.

Remark 2. All proofs in the present paper are based on finding the values of the corresponding functionals on the function $w(x)$ introduced in (3). Note that this is the same function as in [7].

The research of S. Heidarkhani was partly supported by a grant from IPM (No. 91470046).

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