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MODULES WITH UNIQUE CLOSURE RELATIVE TO A TORSION THEORY. III

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We continue the study of modules over a general ring R whose submodules have a unique closure relative to a hereditary torsion theory on Mod-R. It is proved that, for a given ring R and a hereditary torsion theory τ on Mod-R, every submodule of every right R-module has a unique closure with respect to τ if and only if τ is generated by projective simple right R-modules. In particular, a ring R is a right Kasch ring if and only if every submodule of every right R-module has a unique closure with respect to the Lambek torsion theory.

1. Introduction

In the present paper, all rings are associative with identity and all modules are unitary right modules. This paper is a continuation of [1] and [2], and any unexplained terms can be found in [1, 3, 4]. Let R be a ring. *All torsion theories on* Mod-R, *the category of all right R-modules, are hereditary.* For the Goldie torsion theory and terminology for torsion theories in general and any unexplained terminology, see also [5, 6, 9].

Let R be any ring and let τ be a torsion theory on Mod-R. Given an R-module M, $\tau(M)$ denotes the τ-torsion submodule of M. A submodule L of M is called *τ-essential* provided L is an essential submodule of M and M/L is a τ-torsion module. In addition, a submodule K of M is called *τ-closed in M* provided that K has no proper τ-essential extension in M, i.e., if K is a τ-essential submodule in a submodule L of M, then $K = L$. Note that if K is a submodule of M such that either M/K is τ-torsion-free or K is a closed submodule of M, then K is a τ-closed submodule of M. Given a submodule N of M, by a *τ-closure* of N in M, we mean a τ-closed submodule K of M containing N such that N is τ-essential in K. The module M is called a *τ-UC-module* provided every submodule has a unique τ-closure in M.

In [1, 2] we investigate, for a general torsion theory τ, when a submodule of a general R-module M has a unique τ-closure and when the module M is τ-UC. In the present paper, we are interested when a ring R has the property that every (right) R-module is τ-UC for a given torsion theory τ. One consequence of [10] (Theorem (1) $\Leftrightarrow$ (16)) is that the ring R is semiprime Artinian if and only if every (right) R-module is UC. The Goldie torsion theory is denoted by τ_G. In [1, p. 232], it is pointed out that every τ_G-UC-module is UC, and conversely. Thus, R is semiprime Artinian if and only if every module is τ_G-UC. Recall that if τ and ρ are torsion theories on Mod-R, then $\tau \le \rho$ provided that every τ-torsion module is ρ-torsion. It is proved in [1] (Proposition 3.6) that if τ and ρ are torsion theories on Mod-R such that $\tau \le \rho$, then every ρ-UC-module is τ-UC and, in particular, every UC-module is a τ-UC-module. Thus, we have proved our first result.

Proposition 1.1. *Let R be a ring and let τ be any torsion theory on Mod-R. Then the fact that R is semiprime Artinian implies that every R-module is τ-UC. Moreover, the converse holds for $\tau_G \le \tau$.*

Let M be an R-module. For any nonempty subset X of M, $\mathrm{ann}_R(X)$ denotes $\{r \in R\colon xr = 0$ for all $x \in X\}$ and, for any nonempty subset Y of R, $\mathrm{ann}_M(Y)$ denotes $\{m \in M\colon my = 0$ for all $y \in Y\}$.

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In the case $X = \{x\}$ and $Y = \{y\}$, we write $\operatorname{ann}_R(x)$ and $\operatorname{ann}_M(y)$ for $\operatorname{ann}_R(X)$ and $\operatorname{ann}_M(Y)$, respectively. We now prove the following result:

Theorem 1.1. *Let R be any ring. Then the following statements are equivalent for a torsion theory τ on* $\operatorname{Mod}\text{-}R$:

(1) Every right R-module is τ-UC.

(2) $Y = X$ for every right R-module X and τ-essential submodule Y of X.

(3) Every τ-torsion right R-module is projective.

(4) If E is an essential right ideal of R, then the R-module R/E is τ-torsion-free.

(5) If a right ideal E of R is τ-essential in R_R, then $E = R$.

(6) Every singular right R-module is τ-torsion-free.

(7) Every proper submodule of any module is τ-closed.

Proof. (1) $\Rightarrow$ (2). Let Y be a τ-essential submodule of X. Then $X \oplus (X/Y)$ is not τ-UC if $Y \neq X$ by [1] (Lemma 3.2). By hypothesis (1), $Y = X$.

(2) $\Rightarrow$ (3). Let M be any τ-torsion R-module. There exists a free R-module F and a submodule K of F such that $M \cong F/K$. Let L be a submodule of F maximal with respect to the property $K \cap L = 0$ (Zorn's lemma). Then $K \subseteq K \oplus L$ and $K \oplus L$ is an essential submodule of F. Since F/K is τ-torsion, we conclude that $K \oplus L$ is a τ-essential submodule of F and, hence, $F = K \oplus L$ and $M \cong L$ is projective.

(3) $\Rightarrow$ (4). Let E be an essential right ideal of R. There exists a submodule E' of R_R with $E \subseteq E'$ such that E'/E is τ-torsion and R/E' τ-torsion-free. By (3), E'/E is projective and, hence, $E' = E$. Thus, R/E is τ-torsion-free.

(4) $\Rightarrow$ (5). Let E be a τ-essential submodule of R_R. Then E is essential in R and R/E is τ-torsion. By (4), $E = R$.

(5) $\Rightarrow$ (1). Let M be an R-module. Suppose M is not τ-UC. Then there exists a submodule X of M and a proper τ-essential submodule Y of X such that the R-module $X \oplus (X/Y)$ embeds in M (see [1], Lemma 3.2). Let $x \in X \setminus Y$. Then $xR \oplus (xR + Y)/Y$ can be embedded in M, i.e., $xR \oplus (xR/(xR \cap Y))$ embeds in M and $xR \cap Y$ is τ-essential in xR. If $A = \operatorname{ann}_R(x)$, then $xR \cong R/A$ and $xR \cap Y \cong B/A$, where B/A is τ-essential in R/A. Then B is τ-essential in R_R. By (5), $B = R$; a contradiction. Therefore, M is a τ-UC-module.

(4) $\Rightarrow$ (6). Let M be a singular R-module. Let $m \in M$. Then $mR \cong R/E$ for some essential right ideal E of R. By (4), mR is a τ-torsion-free module. Hence, M is τ-torsion-free.

(6) $\Rightarrow$ (7). Let N be a proper submodule of an R-module M and let K be a τ-closure of N in M. Then N is τ-essential in K and, hence, K/N is singular and, thus, τ-torsion. But by the hypothesis, K/N is τ-torsion-free. Thus, $N = K$ and, hence, N is τ-closed in M.

(7) $\Rightarrow$ (6). Let N be a singular R-module. Let $u \in \tau(N)$. Then $uF = 0$ for some essential right ideal F of R. Note that $R/F \cong uR \subseteq \tau(N)$ and hence F is τ-essential in R. By (5) $F = R$ and, hence, $u = 0$. Therefore, N is τ-torsion-free.

(6) $\Rightarrow$ (4). Clear.

Theorem 1.1 is proved.

Corollary 1.1. *If every R-module is τ-UC, then every τ-torsion R-module is semisimple.*

Proof. Assume that every R-module is τ-UC. Then every singular right R-module is τ-torsion-free by Theorem 1.1 (6). Let M be a torsion R-module and let N be a submodule of M. By Zorn's lemma, there exists a submodule K of M such that $N \oplus K$ is essential in M. Then $M/(N \oplus K)$ is singular. By the hypothesis, $M/(N \oplus K)$ is τ-torsion-free and it is also τ-torsion. Thus, $M = N \oplus K$ and hence, M is semisimple.

Corollary 1.1 is proved.

The converse of Corollary 1.1 need not be true in general.

Example 1.1. Let R be a commutative ring and let P be a maximal ideal of R such that $P = P^2$ and $\mathrm{ann}_R(P) = 0$. Let τ be the hereditary torsion theory generated by R/P. Then every τ-torsion module is semisimple but not every R-module is τ-UC.

Proof. If R/P is a projective R-module, then $R = P \oplus I$ for some ideal I of R. Thus, $IP = 0$ and, hence, $I = 0$; a contradiction. Therefore, R/P is not projective. It follows that P is an essential ideal of R and, hence, R/P is a nonzero singular module which is τ-torsion and, hence, not τ-torsion-free. Thus, not every R-module is τ-UC by Theorem 1.1.

Let X be a τ-torsion module. Let $S = \{x \in X \colon Px = 0\}$. Suppose $X \neq S$. Then there exists an R-submodule T of X such that $S \subset T \subseteq X$ and $T/S \cong R/P$. Then $PT \subseteq S$ and, therefore,

$$PT = P^2T = P(PT) \subseteq PS = 0.$$

Thus, $T \subseteq S$; a contradiction. It follows that $X = S$ and, hence, X is semisimple.

Example 1.2. Let p be any prime integer, let F be a field of characteristic $p > 0$, and let G be the Prüfer p-group. Then the group algebra $R = F[G]$ has an augmentation ideal

$$P = \sum_{x \in G} R(x-1),$$

which satisfies $P = P^2$ and $\mathrm{ann}_R(P) = 0$.

Proof. By [8] (Lemma 3.1.2), $\mathrm{ann}_R(P) = 0$. Let $x \in G$. Then $x = y^p$ for some $y \in G$ and, hence, $x - 1 = y^p - 1 = (y-1)^p \in P^2$. It follows that $P = P^2$.

Corollary 1.2. *Let R be a ring such that every singular right R-module is τ-torsion-free for some torsion theory τ. Then $M/\operatorname{Soc}(M_R)$ is τ-torsion- free and $\tau(M) \subseteq \operatorname{Soc}(M_R)$ for every R-module M.*

Proof. Let M be any R-module and let $N_i\,(i \in I)$ denote the collection of essential submodules of M. For each $i \in I$, M/N_i is τ-torsion-free. Thus, $\prod_{i\in I}(M/N_i)$ is τ-torsion-free. Since $M/\operatorname{Soc}(M)$ is isomorphic to a submodule of $\prod_{i\in I}(M/N_i)$, it is τ-torsion free. The last part clearly follows.

Corollary 1.2 is proved.

Let R be any ring. Let

$$\tau(R_R) = \bigoplus_{i\in I} U_i,$$

where U_i is a simple module for each $i \in I$. We choose $J \subseteq I$ such that $U_j \ncong U_k$ if $j \neq k$ in J and for each $i \in I$ there exists $j \in J$ such that $U_i \cong U_j$. Then the torsion class of a torsion theory τ is generated by $\{U_j : j \in J\}$, i.e., X belongs to the torsion class of $\tau \Leftrightarrow X = \bigoplus_{\lambda\in\Lambda} V_\lambda$ where for every $\lambda \in \Lambda$, $V_\lambda \cong U_j$ for some $j \in J$.

We now characterize the torsion theories satisfying the equivalent conditions in Theorem 1.1.

Theorem 1.2. *Let τ be a torsion theory on* Mod-R. *Then every right R-module is τ-UC if and only if τ is generated by a collection $\mathcal{S}$ of projective simple R-modules.*

Proof. Assume that every R-module is τ-UC. By Corollary 1.1, every τ-torsion module is semisimple. Let $\mathcal{S}$ denote the set of representatives of τ-torsion simple modules. Let $V \in \mathcal{S}$. Then $V \cong R/P$ for some maximal right ideal P of R. Hence P is not essential. Otherwise, R/P is singular, so it is τ-torsion-free by Theorem 1.1. But, since R/P is τ-torsion, this is a contradiction. Therefore, $R = P \oplus X$ for some projective simple τ-torsion right ideal X. Then $V \cong X$ and, thus, $\mathcal{S}$ consists of all projective simple τ-torsion R-modules.

Conversely, let $\mathcal{T}$ be the collection of modules of the form $\bigoplus_{i\in I} W_i$ where for each $i \in I$ there exists $V_i \in \mathcal{S}$ such that $W_i \cong V_i$. Then $\mathcal{T}$ is closed under submodules, homomorphic images and direct sums. To complete the proof, we show that $\mathcal{T}$ is closed under extensions by short exact sequences. Let X be an R-module and let Y be a submodule of X with $Y \in \mathcal{T}$ and $X/Y \in \mathcal{T}$. Then X/Y is projective, being a direct sum of projective simple modules. Hence Y is direct summand of X. Therefore,

$$X \cong Y \oplus (X/Y) \in \mathcal{T}.$$

Theorem 1.2 is proved.

Let I be an idempotent ideal of a ring R. Then τ_I denotes the torsion theory whose torsion modules are the R-modules X such that $XI = 0$. As an application of Theorem 1.2, we now characterize the idempotent ideals I such that every R-module is a τ_I-UC-module.

Corollary 1.3. *Let I be an idempotent ideal of a ring R. Then every R-module is τ_I-UC if and only if $I = eR$ for some idempotent element e of R and the ring R/I is semiprime Artinian.*

Proof. Suppose first that every R-module is τ_I-UC. Let A be a right ideal of R maximal with respect to the property that $I \cap A = 0$. It is well known that $I \oplus A$ is an essential right ideal of R and, hence, $I \oplus A$ is a τ_I-essential right ideal of R because $(R/(I \oplus A))I = 0$. By Theorem 1.1 (5), $R = I \oplus A$. It follows that $I = eR$ for some $e = e^2 \in R$. Let E be a right ideal of R containing I such that E/I is an essential right ideal of R/I. Then E is an essential right ideal of R and clearly E is τ_I-essential in R. Again, using Theorem 1.1, $E = R$. Thus, the ring R/I does not contain a proper essential right ideal and, hence, R/I is a semiprime Artinian ring. Conversely, suppose that $I = eR$ for some idempotent $e \in R$ such that the ring R/I is semiprime Artinian. Let F be a right ideal of R such that F is τ_I-essential in R_R. Because R/F is τ_I-torsion, we have $(R/F)I = 0$ and, hence, $I \subseteq F$. Furthermore, F is essential in R_R and I is closed in R_R so that F/I is an essential right ideal of R/I by [4, p. 6]. But R/I is semiprime Artinian implies that $F/I = R/I$ and, hence, $F = R$. By Theorem 1.1, every R-module is τ_I-UC.

Corollary 1.3 is proved.

2. Further Results

Let R be a ring and let M be an R-module. Given a submodule L of M and an element $m \in M$, by $(L\colon m)$, we denote the set of elements $r \in R$ such that $mr \in L$. Note that $(L\colon m)$ is a right ideal of R.

Lemma 2.1. *Let R be any ring. The following conditions are equivalent for the torsion theories ρ and τ on* Mod-R.

(1) Every ρ-essential right ideal of R is a τ-essential right ideal of R.

(2) For every R-module M, every ρ-essential submodule of M is a τ-essential submodule of M.

Proof. (2) $\Rightarrow$ (1). Apply (2) in the case $M = R$.

(1) $\Rightarrow$ (2). Let M be an R-module and let L be a ρ-essential submodule of M. Then L is an essential submodule of M and M/L is ρ-torsion. Let $m \in M$ and let $A = (L\colon m)$. Then A is an essential right ideal of R such that $R/A \cong (mR + L)/L$ so that R/A is ρ-torsion. Thus, A is a ρ-essential right ideal of R. In particular, R/A is τ-torsion and, hence, $(mR + L)/L$ is τ-torsion, as well. Therefore, M/L is τ-torsion and L is a τ-essential submodule of M.
Lemma 2.1 is proved.

Note the following result:

Proposition 2.1. *Let R be any ring. Consider the following conditions for the torsion theories ρ and τ on* Mod-R.

(1) $\rho \leq \tau$.

(2) Every ρ-essential right ideal of R is a τ-essential right ideal of R.

(3) Every τ-UC R-module is ρ-UC.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Let M be an R-module which is not ρ-UC. By [1] (Theorem 3.4), there exist an R-module X and a proper ρ-essential submodule Y of X such that $X \oplus (X/Y)$ embeds in M. Now (2) and Lemma 2.1 show that Y is a τ-essential submodule of X. Applying [1] (Theorem 3.4) again, we see that M is not a τ-UC-module.
Proposition 2.1 is proved.

Corollary 2.1. *Let R be any ring and let τ be a torsion theory on Mod-R. Consider the following conditions:*

(1) $\tau_G \leq \tau$.

(2) Every essential right ideal of R is a τ-essential right ideal of R.

(3) Every τ-UC R-module is a UC-module.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Proposition 2.1.

(2) $\Rightarrow$ (1). Let M be a nonzero singular R-module. Let $m \in M$. Then $\mathrm{ann}_R(m)$ is an essential and, hence, τ-essential, right ideal of R. Then $mR \cong R/\mathrm{ann}_R(m)$ is τ-torsion. Therefore, M is τ-torsion. Thus, $\tau_G \leq \tau$.
Corollary 2.1 is proved.

We now present criteria for $\tau_G \leq \tau$.

Proposition 2.2. *Let R be any ring and let τ be a torsion theory on Mod-R. Then the following statements are equivalent:*

(1) $\tau_G \le \tau$.

(2) For any R-module M and any submodule N of M, there exists a submodule K of M such that $N \cap K = 0$ and $N \oplus K$ is τ-essential in M.

(3) For any right ideal A of R, there exists a right ideal B of R such that $A \cap B = 0$ and $A \oplus B$ is τ-essential in R_R.

(4) Every singular R-module is τ-torsion.

Proof. (1) $\Rightarrow$ (2). Assume that $\tau_G \le \tau$. Let N be a submodule of M. By Zorn's Lemma, there exists a submodule K of M maximal with respect to the property $N \cap K = 0$. Then $N \oplus K$ is essential in M. Hence $M/(N \oplus K)$ is singular. By assumption, $M/(N \oplus K)$ is τ-torsion and, therefore, $N \oplus K$ is τ-essential in M.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Let M be any singular module. For any $x \in M$, we have $xR \cong R/\operatorname{ann}_R(x)$ where $\operatorname{ann}_R(x)$ is an essential right ideal of R. Now $R/\operatorname{ann}_R(x)$ and, hence, xR is τ-torsion by (3). Therefore, M is τ-torsion. Thus, every singular module is τ-torsion.

(4) $\Rightarrow$ (1). Let X be any τ_G-torsion module. There exists a submodule Y of X such that both Y and X/Y are singular. By (4) both Y and X/Y are τ-torsion. It follows that X is τ-torsion. Thus, $\tau_G \le \tau$.

Proposition 2.2 is proved.

Further, we note that, in Proposition 2.1, statements (2) and (3), in general, do not imply (1) as shown by the following example:

Example 2.1. There are rings R and torsion theories ρ and τ on Mod-R such that every ρ-essential right ideal of R is a τ-essential right ideal of R (and, hence, every τ-UC-module is ρ-UC) but this is not the case $\rho \le \tau$.

Proof. Let

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

be the ring of 2×2 upper triangular matrices over a field F. The right ideals of R are

$$0, \quad I_1 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}, \quad \text{and} \quad I_{(x,y)} = \left\{ \begin{bmatrix} 0 & xc \\ 0 & yc \end{bmatrix} \;\middle|\; c \in F \right\}$$

for some $x, y \in F$ and R. Let $\tau_{I_1} = \{N \in \mathrm{Mod}\text{-}R \mid NI_1 = 0\}$ and $\tau_{I_2} = \{N \in \mathrm{Mod}\text{-}R \mid NI_2 = 0\}$ denote the torsion theories determined by the idempotent ideals I_1 and I_2, respectively. The ring R has I_2 as its only essential right ideal. Since $(R/I_2)I_2 = 0$, I_2 is a τ_{I_2}-essential right ideal of R. On the other hand, R does not have any τ_{I_1}-essential right ideal except R itself. Hence, every τ_{I_1}-essential right ideal of R is a τ_{I_2}-essential right ideal of R. Since $I_{(0,1)}I_1 = 0$ and $I_{(0,1)}I_2 = I_{(0,1)} \neq 0$, $I_{(0,1)}$ is a τ_{I_1}-torsion but not τ_{I_2}-torsion module. Hence $\tau_{I_1} \not\le \tau_{I_2}$.

Example 2.1 is completed.

Combining Proposition 2.1 and Example 2.1, we see that, in general, (3) does not imply (1) in Proposition 2.1. We do not know whether (3) implies (2) in Corollary 2.1. In some situations, (2) does imply (1) in Proposition 2.1. Thus, we have the following result:

Proposition 2.3. *Let R be a ring and let ρ and τ be torsion theories on* $\text{Mod-}R$ *such that $\rho \leq \tau_G$. Then $\rho \leq \tau$ if and only if every ρ-essential right ideal of R is a τ-essential right ideal of R.*

Proof. The necessity follows by Proposition 2.1. Conversely, suppose that every ρ-essential right ideal of R is a τ-essential right ideal of R. Let M be a ρ-torsion module. There exists a submodule N of M such that both N and M/N are singular. Let $m \in N$. Since N is singular, $\operatorname{ann}_R(m)$ is an essential right ideal of R. Since ρ is a hereditary torsion theory, mR is ρ-torsion and, thus, $R/\operatorname{ann}_R(m)$ is also ρ-torsion because $mR \cong R/\operatorname{ann}_R(m)$. This implies that $\operatorname{ann}_R(m)$ is a ρ-essential right ideal of R and, therefore, a τ-essential right ideal of R. Hence, $R/\operatorname{ann}_R(m)$ is τ-torsion, i.e., mR is τ-torsion for all $m \in N$ and, thus, N is τ-torsion. Similarly, M/N is τ-torsion. Therefore, M is τ-torsion. It follows that $\rho \leq \tau$.

Proposition 2.3 is proved.

Let τ be any torsion theory on $\text{Mod-}R$. For any R-module M, $Z_\tau(M)$ denotes the set of elements m in M such that $mE = 0$ for some τ-essential right ideal E of R. Note that $Z_\tau(M)$ is a submodule of the singular submodule $Z(M)$ of M.

Theorem 2.1. *Let R be a ring and let τ be a hereditary torsion theory on* $\text{Mod-}R$ *such that $Z_\tau(R_R) = 0$. Then $\tau_G \leq \tau$ if and only if every τ-UC-module is UC.*

Proof. The necessity follows by Proposition 2.1. Conversely, suppose that every τ-UC-module is a UC-module. Let E be any essential right ideal of R. Suppose that R/E is not τ-torsion. Then there exists a proper right ideal F of R such that $E \subseteq F$ and R/F is a τ-torsion-free R-module. Let M denote the R-module $R \oplus (R/F)$. Note that $Z_\tau(R_R) = 0$ and $Z_\tau(R/F) = 0$ and, hence, $Z_\tau(M) = 0$. By [1] (Corollary 3.5), M is a τ-UC-module. However, since F is a proper essential right ideal of R, M is not a UC-module by [10] (Theorem); a contradiction. Thus, the R-module R/E is τ-torsion for every essential right ideal E of R. This means that every singular module is τ-torsion. By Proposition 2.2, $\tau_G \leq \tau$.

Theorem 2.1 is proved.

3. The Lambek Torsion Theory

Let R be a ring. For any R-module M, $E(M)$ denotes the injective hull of M. Let τ_L denote the Lambek torsion theory on $\text{Mod-}R$. Recall that τ_L is the (hereditary) torsion theory on $\text{Mod-}R$ whose torsion class consists of all R-modules M such that $\text{Hom}(M, E(R_R)) = 0$. For basic facts about the Lambek torsion theory, see [5, 9]. Recall that the ring R is called *right nonsingular* provided that R_R is τ_G-torsion-free.

Lemma 3.1. *Let R be any ring. Then:*

(1) $\tau_L \leq \tau_G$;

(2) $\tau_L = \tau_G$ *if and only if R is right nonsingular.*

Proof. (1) By [9] (Ch. VI, Corollary 6.5).

(2) By [9] (Ch. VI, Proposition 6.7 and Corollary 6.8).

Lemma 3.1 is proved.

Lemma 3.2. *Let $E = E(R_R)$. Then an R-module M is τ_L-torsion if and only if $\operatorname{ann}_E(\operatorname{ann}_R(m)) = 0$ for all $m \in M$.*

Proof. Let $m \in M$ and let $e \in \operatorname{ann}_E(\operatorname{ann}_R(m))$. We define a mapping $\varphi\colon mR \to E$ by $\varphi(mr) = er$ $(r \in R)$. It is easy to check that φ is well defined and that it is an R-homomorphism. Moreover, for every homomorphism $\theta\colon mR \to E$, we have $\theta(m) \in \operatorname{ann}_E(\operatorname{ann}_R(m))$. Since E is injective, the result follows.

Lemma 3.2 is proved.

Corollary 3.1. *A submodule N of an R-module M is a τ_L-essential submodule of M if and only if*

(a) N is an essential submodule of M, and

(b) $\operatorname{ann}_E(N\colon m) = 0$ for all $m \in M$, where $E = E(R_R)$.

Proof. The submodule N is a τ_L-essential submodule of M if and only if (a) holds and M/N is τ_L-torsion. It is necessary to apply Lemma 3.2.

Corollary 3.1 is proved.

Theorem 3.1. *A ring R is right nonsingular if and only if every τ_L-UC-module is UC.*

Proof. The necessity follows by Lemma 3.1 (2). Conversely, suppose that every τ_L-UC-module is UC. We know that $\tau_L \leq \tau_G$. Further, we show that $Z_{\tau_L}(R_R) = 0$. Suppose that $x \in Z_{\tau_L}(R_R)$. Then there exists a τ_L-essential right ideal I of R such that $xI = 0$. Thus, I is essential in R_R and R/I is τ_L-torsion. Therefore, $\operatorname{Hom}_R(R/I, E(R_R)) = 0$. We define a mapping $\theta\colon R/I \to E(R_R)$ by $\theta(r + I) = xr$ for all $r \in R$. Since $xI = 0$, θ is well defined and is clearly an R-homomorphism. Thus, $x = 0$. It follows that $Z_{\tau_L}(R_R) = 0$. By Theorem 2.1, $\tau_G \leq \tau_L$. We have proved that $\tau_G = \tau_L$. Finally, Lemma 3.1 gives that R is right nonsingular.

Theorem 3.1 is proved.

Let R be a ring. Recall that R is called a *right Kasch ring* if every simple right module is embedded in R. Quasi-Frobenius rings are among examples of right Kasch rings (see, e.g., [7], Theorem 3.4). More generally, right pseudo-Frobenius rings are also examples of right Kasch rings (see, e.g., [7], Lemma 1.42 and Theorem 1.57). This brings us to our final theorem.

Theorem 3.2. *The following statements are equivalent for a ring R :*

(1) R is a right Kasch ring.

(2) Every τ_L-torsion module is zero.

(3) Every module is a τ_L-UC-module.

Proof. (1) $\Rightarrow$ (2). Let R be a right Kasch ring and let $M_R \neq 0$. For $0 \neq m \in M$, there exists a maximal submodule K of mR. Then there is an embedding $\varphi\colon mR/K \to R$ and an embedding $\iota\colon R \to E(R_R)$. By the injectivity of $E(R_R)$, the homomorphism $\iota\varphi$ extends to a nonzero homomorphism $\theta\colon M/K \to E(R_R)$. If $\pi\colon M \to M/K$ denotes the natural epimorphism, then $\theta\pi$ is a nonzero homomorphism from M to $E(R_R)$. Hence, $\operatorname{Hom}_R(M, E(R_R)) \neq 0$.

(2) $\Rightarrow$ (3). By Theorem 1.1.

(3) $\Rightarrow$ (1). Let V be any simple R-module. Then $V \cong R/P$ for some maximal right ideal P of R. If P is not an essential right ideal of R, then there exists a nonzero right ideal U of R such that $P \cap U = 0$. In this case, $R = P \oplus U$ and, hence, $V \cong R/P \cong U$. We now suppose that P is an essential right ideal of R. Since $P \neq R$, statement (3) combined with Theorem 1.1 implies that R/P is not τ_L-torsion. Thus, there exists a nonzero homomorphism $\alpha\colon V \to E(R_R)$. It follows that $V \cong \varphi(V) \subseteq R$ because R is essential in $E(R_R)$. We have proved that R is a right Kasch ring.

Theorem 3.2 is proved.

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