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TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN GENERALIZED LEBESGUE SPACES WITH VARIABLE EXPONENT

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We investigate the approximation properties of the trigonometric system in $L_{2\pi}^{p(\cdot)}$. We consider the moduli of smoothness of fractional order and obtain direct and inverse approximation theorems together with a constructive characterization of a Lipschitz-type class.

1. Introduction

Generalized Lebesgue spaces $L^{p(x)}$ with variable exponent and the corresponding Sobolev-type spaces are extensively applied in elasticity theory, fluid mechanics, differential operators [31, 10], nonlinear Dirichlet boundary-value problems [24], problems of nonstandard growth, and variational calculus [33].

These spaces appeared for the first time in [28] as an example of modular spaces [14, 26]. Sharapudinov [36] established the topological properties of $L^{p(x)}$. Furthermore, if

$$p^* := \operatorname*{ess\,sup}_{x \in \boldsymbol{T}} p(x) < \infty,$$

then $L^{p(x)}$ is a special case of the Musielak–Orlicz spaces [26]. Later, many mathematicians studied the principal properties of these spaces [36, 24, 32, 12]. There is a rich theory of boundedness of integral transforms of various types in $L^{p(x)}$ [22, 33, 9, 37].

For $p(x) := p$, $1 < p < \infty$, $L^{p(x)}$ coincides with the Lebesgue space L^p; the basic problems of trigonometric approximation in L^p were investigated by numerous mathematicians (among others, see [39, 19, 30, 40, 6, 4], etc.). The problems of approximation by algebraic polynomials and rational functions in Lebesgue spaces, Orlicz spaces, symmetric spaces, and their weighted versions on sufficiently smooth complex domains and curves were studied in [1–3, 15, 18, 16]. For a complete treatise on polynomial approximation, we refer the reader to the books [5, 8, 41, 29, 35, 23].

In the harmonic and Fourier analyses, some operators (e.g., the operator of partial sum of Fourier series, conjugate operator, operator of differentiation, and operator of shift $f \to f(\cdot + h)$, $h \in \mathbb{R}$) are extensively used to prove approximation inequalities of direct and inverse types. Unfortunately, the space $L^{p(x)}$ is not $p(\cdot)$-continuous and not translation invariant [24]. Under various assumptions (including translation invariance) imposed on the modular space, Musielak [27] established some approximation theorems in modular spaces with respect to the ordinary moduli of smoothness. Since $L^{p(x)}$ is not translation invariant, by using Butzer–Wehrens-type moduli of smoothness (see [7, 13]) Israfilov et al. [17] obtained direct and inverse trigonometric approximation theorems in $L^{p(x)}$.

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In the present paper, we study the approximation properties of the trigonometric system in $L_{2\pi}^{p(\cdot)}$. We consider the moduli of smoothness of fractional order and obtain direct and inverse approximation theorems together with a constructive characterization of a Lipschitz-type class.

Let $\boldsymbol{T} := [-\pi, \pi]$ and let $\mathcal{P}$ be the class of 2π-periodic Lebesgue measurable functions $p = p(x)\colon \boldsymbol{T} \to (1, \infty)$ such that $p^* < \infty$. We introduce the class $L_{2\pi}^{p(\cdot)} := L_{2\pi}^{p(\cdot)}(\boldsymbol{T})$ of 2π-periodic measurable functions f defined on $\boldsymbol{T}$ and satisfying the inequality

$$\int_{\boldsymbol{T}} |f(x)|^{p(x)}\, dx < \infty.$$

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space [24] with the norms

$$\|f(x)\|_{p,\pi} := \|f\,(x)\|_{p,\pi,\boldsymbol{T}} := \inf\left\{\alpha > 0\colon \int_{\boldsymbol{T}} \left|\frac{f(x)}{\alpha}\right|^{p(x)} |dx| \le 1\right\}$$

and

$$\|f(x)\|_{p,\pi}^{*} := \sup\left\{\int_{\boldsymbol{T}} |f(x)g(x)|\, dx\colon\ g \in L_{2\pi}^{p'(\cdot)},\ \int_{\boldsymbol{T}} |g(x)|^{p'(x)}\, dx \le 1\right\},$$

which possess the property[1]

$$\|f\|_{p,\pi} \asymp \|f\|_{p,\pi}^{*}, \tag{1}$$

where $p'(x) := p(x)/\left(p\,(x) - 1\right)$ is the exponent conjugate to $p(x)$.

We say that a variable exponent $p(x)$ defined on $\boldsymbol{T}$ possesses the *Dini–Lipschitz property* DL_γ *of order* γ on $\boldsymbol{T}$ if

$$\sup_{x_1,x_2\in\boldsymbol{T}} \left\{\left|p\,(x_1) - p\,(x_2)\right|\colon |x_1 - x_2| \le \delta\right\} \left(\ln\frac{1}{\delta}\right)^{\gamma} \le c, \quad 0 < \delta < 1.$$

Let $f \in L_{2\pi}^{p(\cdot)}$, let $p \in \mathcal{P}$ possess the property DL_1, let $0 < h \le 1$, and let

$$\sigma_h f(x) := \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t)dt, \quad x \in \boldsymbol{T},$$

be the Steklov mean operator. In this case, the operator σ_h is bounded [37] in $L_{2\pi}^{p(\cdot)}$. Using these facts and setting $x, t \in \boldsymbol{T}$, $0 \le \alpha < 1$, we define

$$\sigma_h^{\alpha} f(x) := (I - \sigma_h)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f\,(x + u_1 + \ldots + u_k)\, du_1 \ldots du_k, \tag{2}$$

[1] The relation $X \asymp Y$ means that there exist constants $C, c > 0$ such that $cY \le X \le CY$. Throughout the paper, $c, C, c_1, c_2, \ldots$ denote constants different in different cases. The relation $X_n = \mathcal{O}(Y_n)$, $n = 1, 2, \ldots,$ means that there exists a constant $C > 0$ such that $X_n \le CY_n$ for $n = 1, 2, \ldots$.

where $f \in L_{2\pi}^{p(\cdot)}$,

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad \text{for} \quad k>1, \qquad \binom{\alpha}{1} := \alpha, \qquad \binom{\alpha}{0} := 1,$$

and I is the identity operator.

Since the binomial coefficients $\binom{\alpha}{k}$ satisfy the relation [34, p. 14]

$$\left|\binom{\alpha}{k}\right| \leq \frac{c(\alpha)}{k^{\alpha+1}}, \quad k \in \mathbb{Z}^+,$$

we get

$$C(\alpha) := \sum_{k=0}^{\infty} \left|\binom{\alpha}{k}\right| < \infty$$

and, therefore,

$$\left\|\sigma_h^{\alpha} f\right\|_{p,\pi} \leq c \left\|f\right\|_{p,\pi} < \infty, \tag{3}$$

provided that $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ possesses the property DL_1, and $0 < h \leq 1$.

For $0 \leq \alpha < 1$ and $r = 1, 2, 3, \dots$, we define the *fractional modulus of smoothness of index* $r+\alpha$ for $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ possessing the property DL_1, and $0 < h \leq 1$ as follows:

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} := \sup_{0 \leq h_i, h \leq \delta} \left\| \prod_{i=1}^{r} \left(I - \sigma_{h_i}\right) \sigma_h^{\alpha} f \right\|_{p,\pi}$$

and

$$\Omega_{\alpha}(f,\delta)_{p(\cdot)} := \sup_{0 \leq h \leq \delta} \left\|\sigma_h^{\alpha} f\right\|_{p,\pi}.$$

By inequality (3), we conclude that

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} \leq c \left\|f\right\|_{p,\pi},$$

where $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ possesses the property DL_1, $0 < h \leq 1$, and the constant $c > 0$ depends only on α, r, and p.

Remark 1. The modulus of smoothness $\Omega_{\alpha}(f,\delta)_{p(\cdot)}$, $\alpha \in \mathbb{R}^+$, has the following properties for $p \in \mathcal{P}$ possessing the property DL_1:

(i) $\Omega_{\alpha}(f,\delta)_{p(\cdot)}$ is a nonnegative nondecreasing function of $\delta \geq 0$;

(ii) $\Omega_{\alpha}(f_1 + f_2, \cdot)_{p(\cdot)} \leq \Omega_{\alpha}(f_1, \cdot)_{p(\cdot)} + \Omega_{\alpha}(f_2, \cdot)_{p(\cdot)}$;

(iii) $\lim\limits_{\delta \to 0} \Omega_{\alpha}(f,\delta)_{p(\cdot)} = 0$.

Let

$$E_n(f)_{p(\cdot)} := \inf_{T \in \mathcal{T}_n} \|f - T\|_{p,\pi}, \quad n = 0, 1, 2, \ldots,$$

be the error of approximation of a function $f \in L_{2\pi}^{p(\cdot)}$, where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

For a given $f \in L^1$, assuming that

$$\int_{\boldsymbol{T}} f(x)dx = 0, \tag{4}$$

we define the *αth fractional* $(\alpha \in \mathbb{R}^+)$ *integral* of f as follows [42, Vol. 2, p. 134]:

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k \,(ik)^{-\alpha}\, e^{ikx},$$

where

$$c_k := \int_{\boldsymbol{T}} f(x)e^{-ikx}dx \quad \text{for} \quad k \in \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$$

and

$$(ik)^{-\alpha} := |k|^{-\alpha}\, e^{(-1/2)\pi i\alpha\, \operatorname{sign} k}$$

as the principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define the *fractional derivative* of a function $f \in L^1$ satisfying (4) as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+[\alpha]-\alpha}(x, f),$$

provided that the right-hand side exists; here, $[x]$ denotes the integer part of a real number x.

Let $W_{p(\cdot)}^\alpha$, $p \in \mathcal{P}$, $\alpha > 0$, be the class of functions $f \in L_{2\pi}^{p(\cdot)}$ such that $f^{(\alpha)} \in L_{2\pi}^{p(\cdot)}$. The class $W_{p(\cdot)}^\alpha$ becomes a Banach space with the norm

$$\|f\|_{W_{p(\cdot)}^\alpha} := \|f\|_{p,\pi} + \left\|f^{(\alpha)}\right\|_{p,\pi}.$$

The main results of this work are the following:

Theorem 1. *Let $f \in W_{p(\cdot)}^\alpha$, $\alpha \in \mathbb{R}^+$, and let $p \in \mathcal{P}$ possess the property DL_γ with $\gamma \geq 1$. Then, for every natural n, there exists a constant $c > 0$ independent of n and such that*

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} E_n(f^{(\alpha)})_{p(\cdot)}.$$

Corollary 1. *Under the conditions of Theorem 1, the following relation is true:*

$$E_n(f)_{p(\cdot)} \le \frac{c}{(n+1)^{\alpha}} \left\| f^{(\alpha)} \right\|_{p,\pi},$$

where $c > 0$ *is a constant independent of* $n = 0, 1, 2, 3, \ldots$.

Theorem 2. *If* $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ *possesses the property* DL_γ *with* $\gamma \ge 1$, *and* $f \in L_{2\pi}^{p(\cdot)}$, *then there exists a constant* $c > 0$ *dependent only on* α *and* p *and such that the following relation holds for* $n = 0, 1, 2, 3, \ldots$:

$$E_n(f)_{p(\cdot)} \le c\, \Omega_\alpha \left(f, \frac{2\pi}{n+1} \right)_{p(\cdot)} .$$

The following inverse theorem of trigonometric approximation is true:

Theorem 3. *If* $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ *possesses the property* DL_γ *with* $\gamma \ge 1$, *and* $f \in L_{2\pi}^{p(\cdot)}$, *then the following relation holds for* $n = 0, 1, 2, 3, \ldots$:

$$\Omega_\alpha \left(f, \frac{\pi}{n+1} \right)_{p(\cdot)} \le \frac{c}{(n+1)^{\alpha}} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot)},$$

where the constant $c > 0$ *depends only on* α *and* p.

Corollary 2. *Let* $\alpha \in \mathbb{R}^+$, *let* $p \in \mathcal{P}$ *possess the property* DL_γ *with* $\gamma \ge 1$, *and let* $f \in L_{2\pi}^{p(\cdot)}$. *If*

$$E_n(f)_{p(\cdot)} = \mathcal{O}\left(n^{-\sigma}\right), \qquad \sigma > 0, \quad n = 1, 2, \ldots,$$

then

$$\Omega_\alpha (f, \delta)_{p(\cdot)} = \begin{cases} \mathcal{O}(\delta^{\sigma}), & \alpha > \sigma, \\ \mathcal{O}\left(\delta^{\sigma} \left|\log(1/\delta)\right|\right), & \alpha = \sigma, \\ \mathcal{O}(\delta^{\alpha}), & \alpha < \sigma. \end{cases}$$

Definition 1. *For* $0 < \sigma < \alpha$, *we set*

$$\operatorname{Lip} \sigma\, (\alpha, p(\cdot)) := \left\{ f \in L_{2\pi}^{p(\cdot)} : \Omega_\alpha (f, \delta)_{p(\cdot)} = \mathcal{O}\left(\delta^{\sigma}\right),\ \delta > 0 \right\}.$$

Corollary 3. *Let* $0 < \sigma < \alpha$, *let* $p \in \mathcal{P}$ *possess the property* DL_γ *with* $\gamma \ge 1$, *and let* $f \in L_{2\pi}^{p(\cdot)}$. *Then the following conditions are equivalent:*

(a) $f \in \operatorname{Lip} \sigma\, (\alpha, p(\cdot))$;

(b) $E_n(f)_{p(\cdot)} = \mathcal{O}\left(n^{-\sigma}\right)$, $n = 1, 2, \ldots$.

Theorem 4. *Let $p \in \mathcal{P}$ possess the property DL_γ with $\gamma \geq 1$ and let $f \in L_{2\pi}^{p(\cdot)}$. If $\beta \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1} E_\nu(f)_{p,\pi} < \infty,$$

then $f \in W_{p(\cdot)}^{\beta}$ and

$$E_n(f^{(\beta)})_{p(\cdot)} \leq c \left((n+1)^\beta E_n(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} E_\nu\,(f)_{p(\cdot)} \right),$$

where the constant $c > 0$ depends only on β and p.

Corollary 4. *Suppose that $p \in \mathcal{P}$ possesses the property DL_γ with $\gamma \geq 1$, $f \in L_{2\pi}^{p(\cdot)}$, $\beta \in (0, \infty)$, and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot)} < \infty$$

for some $\alpha > 0$. In this case, for $n = 0, 1, 2, \ldots,$ there exists a constant $c > 0$ dependent only on α, β, and p and such that

$$\Omega_\beta \left(f^{(\alpha)}, \frac{\pi}{n+1} \right)_{p(\cdot)} \leq \frac{c}{(n+1)^\beta} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+\beta-1}\, E_\nu(f)_{p(\cdot)} + c \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot)}.$$

The following theorem on simultaneous approximation is true:

Theorem 5. *Let $\beta \in [0, \infty)$, let $p \in \mathcal{P}$ possess the property DL_γ with $\gamma \geq 1$, and let $f \in L_{2\pi}^{p(\cdot)}$. Then there exist $T \in \mathcal{T}_n$ and a constant $c > 0$ dependent only on α and p and such that*

$$\left\| f^{(\beta)} - T^{(\beta)} \right\|_{p,\pi} \leq c E_n\big(f^{(\beta)}\big)_{p(\cdot)}.$$

Definition 2 (Hardy space of variable exponent $H^{p(\cdot)}$ on a unit disc $\mathbb{D}$ with boundary $\mathbb{T} := \partial\mathbb{D}$) [21]**.** *Let $p(z) \colon \mathbb{T} \to (1, \infty)$ be a measurable function. We say that a complex-valued analytic function Φ in $\mathbb{D}$ belongs to the Hardy space $H^{p(\cdot)}$ if*

$$\sup_{0<r<1} \int_0^{2\pi} \left| \Phi\big(re^{i\vartheta}\big) \right|^{p(\vartheta)} d\vartheta < +\infty,$$

where $p(\vartheta) := p\big(e^{i\vartheta}\big)$ and $\vartheta \in [0, 2\pi]$ (and, therefore, $p\,(\vartheta)$ is a 2π-periodic function). Let

$$\underline{p} := \inf_{z \in \mathbb{T}} p(z) \qquad \text{and} \qquad \overline{p} := \sup_{z \in \mathbb{T}} p(z).$$

If $\underline{p} > 0$, *then it is obvious that* $H^{\overline{p}} \subset H^{p(\cdot)} \subset H^{\underline{p}}$. *Therefore, if* $f \in H^{p(\cdot)}$ *and* $\underline{p} > 0$, *then nontangential boundary values* $f\left(e^{i\theta}\right)$ *exist a.e. on* $\mathbb{T}$ *and* $f\left(e^{i\theta}\right) \in L_{2\pi}^{p(\cdot)}(\mathbb{T})$. *Under the conditions* $1 < \underline{p}$ *and* $\overline{p} < \infty$, $H^{p(\cdot)}$ *becomes a Banach space with the norm*

$$\|f\|_{H^{p(\cdot)}} := \left\| f\left(e^{i\theta}\right)\right\|_{p,\pi,\mathbb{T}} = \inf\left\{\lambda > 0\colon \int_{\mathbb{T}} \left|\frac{f\left(e^{i\theta}\right)}{\lambda}\right|^{p(\theta)} d\theta \le 1\right\}.$$

Theorem 6. *If* $p \in \mathcal{P}$ *possesses the property* DL_γ *with* $\gamma \ge 1$, f *belongs to the Hardy space* $H^{p(\cdot)}$ *on* $\mathbb{D}$, *and* $r \in \mathbb{R}^+$, *then there exists a constant* $c > 0$ *independent of* n *and such that*

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} \le c\, \Omega_r\left(f\left(e^{i\theta}\right), \frac{1}{n+1}\right)_{p(\cdot)}, \quad n = 0, 1, 2, \ldots,$$

where $a_k(f)$, $k = 0, 1, 2, 3, \ldots,$ *are the Taylor coefficients of* f *at the origin.*

2. Some Auxiliary Results

We begin with the following lemma:

Lemma A [20]. *For* $r \in \mathbb{R}^+$, *let*

(i) $a_1 + a_2 + \ldots + a_n + \ldots$

and

(ii) $a_1 + 2^r a_2 + \ldots + n^r a_n + \ldots$

be two series in a Banach space $(B, \|\cdot\|)$. *Let*

$$R_n^{\langle r\rangle} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) a_k$$

and

$$R_n^{\langle r\rangle *} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) k^r a_k$$

for $n = 1, 2, \ldots$. *Then*

$$\left\| R_n^{\langle r\rangle *} \right\| \le c, \quad n = 1, 2, \ldots,$$

for some $c > 0$ *if and only if there exists* $R \in B$ *such that*

$$\left\| R_n^{\langle r\rangle} - R \right\| \le \frac{C}{n^r},$$

where c *and* C *are constants that depend only on one another.*

Lemma B [38]. *If* $p \in \mathcal{P}$ *possesses the property* DL_γ *with* $\gamma \geq 1$ *and* $f \in L_{2\pi}^{p(\cdot)}$, *then there are constants* $c, C > 0$ *such that*

$$\|\tilde{f}\|_{p,\pi} \leq c\, \|f\|_{p,\pi} \tag{5}$$

and

$$\|S_n(\cdot, f)\|_{p,\pi} \leq C\, \|f\|_{p,\pi} \tag{6}$$

for $n = 1, 2, \ldots$.

Remark 2. Under the conditions of Lemma B, the following conclusions can be made:

(i) it readily follows from (5) and (6) that there exists a constant $c > 0$ such that

$$\|f - S_n(\cdot, f)\|_{p,\pi} \leq c E_n(f)_{p(\cdot)} \asymp E_n(\tilde{f})_{p(\cdot)};$$

(ii) it follows from the generalized Hölder inequality [24] (Theorem 2.1) that

$$L_{2\pi}^{p(\cdot)} \subset L^1.$$

For a given $f \in L^1$, let

$$f(x) \backsim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{7}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

be the *Fourier series* and the *conjugate Fourier series* of f, respectively. Setting $A_k(x) := c_k e^{ikx}$ in (7), we define

$$S_n(f) := S_n(x, f) := \sum_{k=0}^{n} (A_k(x) + A_{-k}(x)) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 0, 1, 2, \ldots,$$

$$R_n^{\langle\alpha\rangle}(f, x) := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right) (A_k(x) + A_{-k}(x)),$$

and

$$\Theta_m^{\langle r\rangle} := \frac{1}{1 - \left(\frac{m+1}{2m+1}\right)^{r}} R_{2m}^{\langle r\rangle} - \frac{1}{\left(\frac{2m+1}{m+1}\right)^{r} - 1} R_m^{\langle r\rangle} \qquad \text{for} \quad m = 1, 2, 3, \ldots . \tag{8}$$

Under the conditions of Lemma B, using (6) and the Abel transformation, we get

$$\left\| R_n^{\langle\alpha\rangle}(f,x) \right\|_{p,\pi} \le c \, \|f\|_{p,\pi}, \quad n = 1,2,3,\ldots, \quad x \in \boldsymbol{T}, \quad f \in L_{2\pi}^{p(\cdot)}, \tag{9}$$

and, therefore, it follows from (8) and (9) that

$$\left\| \Theta_m^{\langle r\rangle}(f,x) \right\|_{p,\pi} \le c \, \|f\|_{p,\pi}, \quad m = 1,2,3,\ldots, \quad x \in \boldsymbol{T}, \quad f \in L_{2\pi}^{p(\cdot)}.$$

From the property (see (16) in [25])

$$\Theta_m^{\langle r\rangle}(f)(x) = \frac{1}{\sum_{k=m+1}^{2m}\left[(k+1)^r - k^r\right]} \sum_{k=m+1}^{2m} \left[(k+1)^r - k^r\right] S_k(x,f), \quad x \in \boldsymbol{T}, \quad f \in L^1,$$

it is known (see (18) in [25]) that

$$\Theta_m^{\langle r\rangle}(T_m) = T_m \tag{10}$$

for $T_m \in \mathcal{T}_m$, $m = 1,2,3,\ldots$.

Lemma 1. *Let $T_n \in \mathcal{T}_n$, let $p \in \mathcal{P}$ possess the property DL_γ with $\gamma \ge 1$, and let $r \in \mathbb{R}^+$. Then there exists a constant $c > 0$ independent of n and such that*

$$\left\| T_n^{(r)} \right\|_{p,\pi} \le c n^r \, \|T_n\|_{p,\pi}.$$

Proof. Without loss of generality, one can assume that $\|T_n\|_{p,\pi} = 1$. Since

$$T_n = \sum_{k=0}^{n} \left(A_k(x) + A_{-k}(x)\right),$$

we get

$$\frac{\tilde{T}_n}{n^r} = \sum_{k=1}^{n} \left[\left(A_k(x) - A_{-k}(x)\right)/n^r \right]$$

and

$$\frac{T_n^{(r)}}{n^r} = i^r \sum_{k=1}^{n} k^r \left[\left(A_k(x) - A_{-k}(x)\right)/n^r \right].$$

In this case, by virtue of (9) and (5), we have

$$\left\| R_n^{\langle r\rangle}\left(\frac{\tilde{T}_n}{n^r}\right) \right\|_{p,\pi} \le \frac{c}{n^r} \|\tilde{T}_n\|_{p,\pi} \le \frac{c}{n^r} \|T_n\|_{p,\pi} = \frac{c}{n^r},$$

whence, applying Lemma A (with $R=0$) to the series

$$\sum_{k=1}^{n}\left[(A_k(x)-A_{-k}(x))/n^r\right]+0+0+\ldots+0+\ldots,$$

$$\sum_{k=1}^{n}k^r\left[(A_k(x)-A_{-k}(x))/n^r\right]+0+0+\ldots+0+\ldots,$$

we obtain

$$\left\|\sum_{k=1}^{n}\left(1-\left(\frac{k}{n+1}\right)^r\right)k^r\left[(A_k(x)-A_{-k}(x))/n^r\right]\right\|_{p,\pi}\leq c,$$

namely,

$$\left\|R_n^{\langle r\rangle}\left(\frac{T_n^{(r)}}{n^r}\right)\right\|_{p,\pi}=\left\|i^r\sum_{k=1}^{n}\left(1-\left(\frac{k}{n+1}\right)^r\right)k^r\left[(A_k(x)-A_{-k}(x))/n^r\right]\right\|_{p,\pi}$$

$$=\left\|\sum_{k=1}^{n}\left(1-\left(\frac{k}{n+1}\right)^r\right)k^r\left[(A_k(x)-A_{-k}(x))/n^r\right]\right\|_{p,\pi}\leq c_*.$$

Since $R_n^{\langle r\rangle}(cf)=cR_n^{\langle r\rangle}(f)$ for every real c, it follows from relation (10) and the last inequality that

$$\|T_n^{(r)}\|_{p,\pi}=\left\|\Theta_n^{\langle r\rangle}\left(T_n^{(r)}\right)\right\|_{p,\pi}=n^r\left\|\frac{1}{n^r}\Theta_n^{\langle r\rangle}\left(T_n^{(r)}\right)\right\|_{p,\pi}$$

$$=n^r\left\|\Theta_n^{\langle r\rangle}\left(\frac{T_n^{(r)}}{n^r}\right)\right\|_{p,\pi}\leq c_*n^r=c_*n^r\|T_n\|_{p,\pi}.$$

The general case follows immediately from this.

Lemma 2. *If* $p\in\mathcal{P}$ *possesses the property* DL_γ *with* $\gamma\geq 1$, $f\in W_{p(\cdot)}^2$, *and* $r=1,2,3,\ldots,$ *then*

$$\Omega_r(f,\delta)_{p(\cdot)}\leq c\delta^2\Omega_{r-1}\left(f'',\delta\right)_{p(\cdot)},\quad \delta\geq 0,$$

with some constant $c>0$.

Proof. Setting

$$g(x):=\prod_{i=2}^{r}\left(I-\sigma_{h_i}\right)f(x),$$

we get

$$\left(I-\sigma_{h_1}\right)g(x)=\prod_{i=1}^{r}\left(I-\sigma_{h_i}\right)f(x)$$

and

$$\prod_{i=1}^{r}\left(I-\sigma_{h_i}\right)f(x)=\frac{1}{h_1}\int_{-h_1/2}^{h_1/2}\left(g(x)-g\left(x+t\right)\right)dt=-\frac{1}{2h_1}\int_{0}^{h_1/2}\int_{0}^{2t}\int_{-u/2}^{u/2}g''\left(x+s\right)dsdudt.$$

Therefore, it follows from (1) that

$$\left\|\prod_{i=1}^{r}\left(I-\sigma_{h_i}\right)f\left(x\right)\right\|_{p,\pi}$$

$$\leq\frac{c}{2h_1}\sup\left\{\int_{T}\left|\int_{0}^{h_1/2}\int_{0}^{2t}\int_{-u/2}^{u/2}g''\left(x+s\right)dsdudt\right|\left|g_0\left(x\right)\right|dx\colon\ g_0\in L_{2\pi}^{p'(\cdot)}\ \text{ and }\int_{T}\left|g_0(x)\right|^{p'(x)}dx\leq 1\right\}$$

$$\leq\frac{c}{2h_1}\int_{0}^{h_1/2}\int_{0}^{2t}u\left\|\frac{1}{u}\int_{-u/2}^{u/2}g''\left(x+s\right)ds\right\|_{p,\pi}dudt$$

$$\leq\frac{c}{2h_1}\int_{0}^{h_1/2}\int_{0}^{2t}u\left\|g''\right\|_{p,\pi}dudt=ch_1^2\left\|g''\right\|_{p,\pi}.$$

Since

$$g''(x)=\prod_{i=2}^{r}\left(I-\sigma_{h_i}\right)f''(x),$$

we obtain

$$\Omega_r\left(f,\delta\right)_{p(\cdot)}\leq\sup_{\substack{0<h_i\leq\delta\\ i=1,2,\ldots,r}}ch_1^2\left\|g''\right\|_{p,\pi}=c\delta^2\sup_{\substack{0<h_i\leq\delta\\ i=2,\ldots,r}}\left\|\prod_{i=2}^{r}\left(I-\sigma_{h_i}\right)f''(x)\right\|_{p,\pi}$$

$$=c\delta^2\sup_{\substack{0<h_j\leq\delta\\ j=2,\ldots,r-1}}\left\|\prod_{j=1}^{r-1}\left(I-\sigma_{h_j}\right)f''(x)\right\|_{p,\pi}=c\delta^2\Omega_{r-1}\left(f'',\delta\right)_{p(\cdot)}.$$

Lemma 2 is proved.

Corollary 5. *If* $r = 1, 2, 3, \ldots,$ $p \in \mathcal{P}$ *possesses the property* DL_γ *with* $\gamma \geq 1$, *and* $f \in W^{2r}_{p(\cdot)}$, *then*

$$\Omega_r\left(f, \delta\right)_{p(\cdot)} \leq c\delta^{2r} \left\| f^{(2r)} \right\|_{p,\pi}, \quad \delta \geq 0,$$

with some constant $c > 0$.

Lemma 3. *Let* $\alpha \in \mathbb{R}^+$, *let* $p \in \mathcal{P}$ *possess the property* DL_γ *with* $\gamma \geq 1$, *let* $n = 0, 1, 2, \ldots,$ *and let* $T_n \in \mathcal{T}_n$. *Then*

$$\Omega_\alpha\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} \left\| T_n^{(\alpha)} \right\|_{p,\pi},$$

where the constant $c > 0$ *depends only on* α *and* p.

Proof. First, we prove that if $0 < \alpha < \beta$, $\alpha, \beta \in \mathbb{R}^+$, then

$$\Omega_\beta\left(f, \cdot\right)_{p(\cdot)} \leq c\, \Omega_\alpha\left(f, \cdot\right)_{p(\cdot)}. \tag{11}$$

It is easily seen that if $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{Z}^+$, then

$$\Omega_\beta\left(f, \cdot\right)_{p(\cdot)} \leq c\left(\alpha, \beta, p\right) \Omega_\alpha\left(f, \cdot\right)_{p(\cdot)}. \tag{12}$$

We now assume that $0 < \alpha < \beta < 1$. In this case, setting $\Phi(x) := \sigma_h^\alpha f(x)$, we get

$$\begin{aligned}
\sigma_h^{\beta-\alpha}\Phi(x) &= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} \Phi\left(x + u_1 + \ldots u_j\right) du_1 \ldots du_j \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} \Bigg[\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{h^k} \\
&\qquad \times \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} f\left(x + u_1 + \ldots u_j + u_{j+1} + \ldots u_{j+k}\right) du_1 \ldots du_j du_{j+1} \ldots du_{j+k} \Bigg] \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-\alpha}{j} \binom{\alpha}{k} \left[\frac{1}{h^{j+k}} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} f(x + u_1 + \ldots u_{j+k}) du_1 \ldots du_{j+k} \right] \\
&= \sum_{\upsilon=0}^{\infty} (-1)^\upsilon \binom{\beta}{\upsilon} \frac{1}{h^\upsilon} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} f\left(x + u_1 + \ldots u_\upsilon\right) du_1 \ldots du_\upsilon = \sigma_h^\beta f(x) \quad \text{a.e.}
\end{aligned}$$

Then

$$\left\|\sigma_h^{\beta} f(x)\right\|_{p,\pi} = \left\|\sigma_h^{\beta-\alpha}\Phi(x)\right\|_{p,\pi} \le c\left\|\sigma_h^{\alpha} f(x)\right\|_{p,\pi}$$

and

$$\Omega_{\beta}(f,\cdot)_{p(\cdot)} \le c\,\Omega_{\alpha}(f,\cdot)_{p(\cdot)}. \tag{13}$$

Note that if $r_1, r_2 \in \mathbb{Z}^+$ and $\alpha_1, \beta_1 \in (0,1)$, then, taking $\alpha := r_1 + \alpha_1$ and $\beta := r_2 + \beta_1$ for the remaining cases $r_1 = r_2,\ \alpha_1 < \beta_1$, or $r_1 < r_2,\ \alpha_1 = \beta_1$, or $r_1 < r_2,\ \alpha_1 < \beta_1$ and using (12) and (13), one can easily verify that the required inequality (11) is true.

Using relation (11), Corollary 5, and Lemma 1, we get

$$\Omega_{\alpha}\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \le c\,\Omega_{[\alpha]}\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \le c\left(\frac{\pi}{n+1}\right)^{2[\alpha]}\left\|T_n^{(2[\alpha])}\right\|_{p,\pi}$$

$$\le \frac{c}{(n+1)^{2[\alpha]}}(n+1)^{[\alpha]-(\alpha-[\alpha])}\left\|T_n^{(\alpha)}\right\|_{p,\pi} = \frac{c}{(n+1)^{\alpha}}\left\|T_n^{(\alpha)}\right\|_{p,\pi},$$

which is the required result.

Definition 3. *For $p \in \mathcal{P}$, $f \in L_{2\pi}^{p(\cdot)}$, $\delta > 0$, and $r = 1,2,3,\dots$, the Peetre K-functional is defined as follows:*

$$K\left(\delta, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{r}\right) := \inf_{g \in W_{p(\cdot)}^{r}}\left\{\|f-g\|_{p,\pi} + \delta\left\|g^{(r)}\right\|_{p,\pi}\right\}. \tag{14}$$

Theorem 7. *If $p \in \mathcal{P}$ possesses the property DL_{γ} with $\gamma \ge 1$ and $f \in L_{2\pi}^{p(\cdot)}$, then the K-functional $K\left(\delta^{2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right)$ in (14) and the modulus $\Omega_r(f,\delta)_{p(\cdot)}$, $r = 1,2,3,\dots$, are equivalent.*

Proof. If $h \in W_{p(\cdot)}^{2r}$, then, by virtue of Corollary 5 and (14), we have

$$\Omega_r(f,\delta)_{p(\cdot)} \le c\|f-h\|_{p,\pi} + c\delta^{2r}\left\|h^{(2r)}\right\|_{p,\pi} \le cK\left(\delta^{2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right).$$

We estimate the reverse of the last inequality. The operator L_{δ} defined by

$$(L_{\delta} f)(x) := 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\int_{-u/2}^{u/2} f(x+s)\,ds\,du\,dt, \quad x \in \boldsymbol{T},$$

is bounded in $L_{2\pi}^{p(\cdot)}$ because

$$\|L_{\delta} f\|_{p,\pi} \le 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t} u\,\|\sigma_u f\|_{p,\pi}\,du\,dt \le c\,\|f\|_{p,\pi}.$$

We prove that

$$\frac{d^2}{dx^2} L_\delta f = \frac{c}{\delta^2} \left(I - \sigma_\delta\right) f,$$

where c is a real constant. Since

$$\left(L_\delta f\right)(x) = 3\delta^{-3} \int_0^{\delta/2} \int_0^{2t} \int_{-u/2}^{u/2} f\left(x+s\right) ds\, du\, dt$$

$$= 3\delta^{-3} \int_0^{\delta/2} \int_0^{2t} \left[\int_0^{x+u/2} f\left(s\right) ds - \int_0^{x-u/2} f\left(s\right) ds \right] du\, dt,$$

using the Lebesgue differentiation theorem we get

$$\frac{d}{dx} \left(L_\delta f\right)(x) = 3\delta^{-3} \int_0^{\delta/2} \int_0^{2t} \left[\frac{d}{dx} \int_0^{x+u/2} f\left(s\right) ds - \frac{d}{dx} \int_0^{x-u/2} f\left(s\right) ds \right] du\, dt$$

$$= 3\delta^{-3} \int_0^{\delta/2} \int_0^{2t} \left[f\left(x+u/2\right) - f\left(x-u/2\right) \right] du\, dt$$

$$= 6\delta^{-3} \int_0^{\delta/2} \left[\int_x^{x+t} f(u) du + \int_x^{x-t} f(u) du \right] dt \quad \text{a.e.}$$

Using the Lebesgue differentiation theorem once again, we obtain

$$\frac{d^2}{dx^2} \left(L_\delta f\right)(x) = 6\delta^{-3} \int_0^{\delta/2} \left[\frac{d}{dx} \int_x^{x+t} f(u) du + \frac{d}{dx} \int_0^{x-t} f(u) du \right] dt$$

$$= 6\delta^{-3} \int_0^{\delta/2} \left[f\left(x+t\right) - f\left(x\right) + f\left(x-t\right) - f(x) \right] dt$$

$$= \frac{6}{\delta^3} \left[\int_0^{\delta/2} f\left(x+t\right) dt + \int_0^{\delta/2} f\left(x-t\right) dt - \delta f(x) \right]$$

$$= \frac{6}{\delta^2}\left[\frac{1}{\delta}\int_0^{\delta/2} f(x+t)\,dt + \frac{1}{\delta}\int_{-\delta/2}^{0} f(x+t)\,dt - f(x)\right]$$

$$= \frac{6}{\delta^2}\left[\frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f(x+t)\,dt - f(x)\right]$$

$$= \frac{-6}{\delta^2}\left[f(x) - \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f(x+t)\,dt\right] = \frac{-6}{\delta^2}(I-\sigma_\delta) f(x) \quad \text{a.e.}$$

The last equality implies by induction on r that

$$\frac{d^{2r}}{dx^{2r}} L_\delta^r f = \frac{c}{\delta^{2r}}(I-\sigma_\delta)^r f, \quad r = 1, 2, 3, \dots \quad \text{a.e.}$$

Indeed, for $r = 2$, we have

$$\frac{d^4}{dx^4} L_\delta^2 f = \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2} L_\delta^2 f\right) = \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2} L_\delta\left(L_\delta f =: u\right)\right)$$

$$= \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2} L_\delta u\right) = \frac{d^2}{dx^2}\left(\frac{-6}{\delta^2}(I-\sigma_\delta) u\right)$$

$$= \frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}(I-\sigma_\delta) u\right) = \frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}(I-\sigma_\delta) L_\delta f\right) \quad \text{a.e.}$$

Since

$$\frac{d^2}{dx^2}\sigma_\delta\left(L_\delta f\right) = \sigma_\delta\left(\frac{d^2}{dx^2} L_\delta f\right),$$

we get

$$\frac{d^2}{dx^2}(I-\sigma_\delta) L_\delta f = \frac{d^2}{dx^2} L_\delta f - \frac{d^2}{dx^2}\sigma_\delta\left(L_\delta f\right)$$

$$= \frac{d^2}{dx^2} L_\delta f - \sigma_\delta\left(\frac{d^2}{dx^2} L_\delta f\right) = (I-\sigma_\delta)\left[\frac{d^2}{dx^2} L_\delta f\right] \quad \text{a.e.,}$$

and, therefore,

$$\frac{d^4}{dx^4} L_\delta^2 f = \frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}(I-\sigma_\delta) L_\delta f\right) = \frac{-6}{\delta^2}(I-\sigma_\delta)\left[\frac{d^2}{dx^2} L_\delta f\right]$$

$$= \frac{-6}{\delta^2}(I-\sigma_\delta)\left[\frac{-6}{\delta^2}(I-\sigma_\delta) f\right] = \frac{c}{\delta^4}(I-\sigma_\delta)^2 f \quad \text{a.e.}$$

Now let

$$\frac{d^{2(r-1)}}{dx^{2(r-1)}} L_\delta^{(r-1)} f = \frac{c}{\delta^{2(r-1)}}(I-\sigma_\delta)^{(r-1)} f \quad \text{a.e.}$$

Then

$$\frac{d^{2r}}{dx^{2r}} L_\delta^r f = \frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}} L_\delta^{(r-1)} (L_\delta f := u)\right] = \frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}} L_\delta^{(r-1)} u\right]$$

$$= \frac{d^2}{dx^2}\left[\frac{c}{\delta^{2(r-1)}}(I-\sigma_\delta)^{(r-1)} u\right] = \frac{d^2}{dx^2}\left[\frac{c}{\delta^{2(r-1)}}(I-\sigma_\delta)^{(r-1)} L_\delta f\right]$$

$$= \frac{c}{\delta^{2(r-1)}}(I-\sigma_\delta)^{(r-1)}\left[\frac{d^2}{dx^2} L_\delta f\right] = \frac{c}{\delta^{2r}}(I-\sigma_\delta)^r f \quad \text{a.e.}$$

Setting $A_\delta^r := I - \left(I - L_\delta^r\right)^r$, we prove that

$$\left\|\frac{d^{2r}}{dx^{2r}} A_\delta^r f\right\|_{p,\pi} \leq c \left\|\frac{d^{2r}}{dx^{2r}} L_\delta^r f\right\|_{p,\pi} \qquad \text{and} \qquad A_\delta^r f \in W_{p(\cdot)}^{2r}.$$

For $r = 1$, we have

$$A_\delta^1 f := I - \left(I - L_\delta^1 f\right)^1 = L_\delta^1 f \qquad \text{and} \qquad \left\|\frac{d^2}{dx^2} A_\delta^1 f\right\|_{p,\pi} = \left\|\frac{d^2}{dx^2} L_\delta^1 f\right\|_{p,\pi}.$$

Since

$$\frac{d^2}{dx^2} L_\delta f = \frac{c}{\delta^2}(I-\sigma_\delta) f,$$

we get $A_\delta^1 f \in W_{p(\cdot)}^2$. For $r = 2, 3, \ldots,$ using

$$A_\delta^r := I - \left(I - L_\delta^r\right)^r = \sum_{j=0}^{r-1} (-1)^{r-j+1} \binom{r}{j} L_\delta^{r(r-j)},$$

we obtain

$$\left\|\frac{d^{2r}}{dx^{2r}} A_\delta^r f\right\|_{p,\pi} \leq \sum_{j=0}^{r-1} \binom{r}{j} \left\|\frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f\right\|_{p,\pi}.$$

We estimate

$$\left\|\frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f\right\|_{p,\pi}$$

as follows:

$$\begin{aligned}\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r(r-j)}f\right\|_{p,\pi} &= \left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r}\left(L_\delta^{(r-j)}f =: u\right)\right\|_{p,\pi}\\ &= \left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r}u\right\|_{p,\pi} = \left\|\frac{c}{\delta^{2r}}\left(I-\sigma_\delta\right)^r u\right\|_{p,\pi}\\ &= \left\|\frac{c}{\delta^{2r}}\left(I-\sigma_\delta\right)^r \left[L_\delta^{(r-j)}f\right]\right\|_{p,\pi} = \frac{c}{\delta^{2r}}\left\|\left(I-\sigma_\delta\right)^r \left[L_\delta^{(r-j)}f\right]\right\|_{p,\pi}\\ &\le \frac{c}{\delta^{2r}}\left\|\sum_{i=0}^{r}(-1)^i\binom{r}{i}\sigma_\delta^i\left[L_\delta^{(r-j)}f\right]\right\|_{p,\pi}.\end{aligned}$$

Since $\sigma_\delta\left(L_\delta f\right) = L_\delta\left(\sigma_\delta f\right)$, we have $\sigma_\delta^i\left[L_\delta^{(r-j)}f\right] = L_\delta^{(r-j)}\left(\sigma_\delta^i f\right)$ and, hence,

$$\begin{aligned}\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r(r-j)}f\right\|_{p,\pi} &\le \frac{c}{\delta^{2r}}\left\|\sum_{i=0}^{r}(-1)^i\binom{r}{i}\sigma_\delta^i\left[L_\delta^{(r-j)}f\right]\right\|_{p,\pi}\\ &\le \frac{c}{\delta^{2r}}\left\|\sum_{i=0}^{r}(-1)^i\binom{r}{i}L_\delta^{(r-j)}\left(\sigma_\delta^i f\right)\right\|_{p,\pi}\\ &= \frac{c}{\delta^{2r}}\left\|L_\delta^{(r-j)}\left[\sum_{i=0}^{r}(-1)^i\binom{r}{i}\sigma_\delta^i f\right]\right\|_{p,\pi} \le \frac{C}{\delta^{2r}}\left\|\sum_{i=0}^{r}(-1)^i\binom{r}{i}\sigma_\delta^i f\right\|_{p,\pi}\\ &= \frac{C}{\delta^{2r}}\left\|\left(I-\sigma_\delta\right)^r f\right\|_{p,\pi} = \left\|\frac{C}{\delta^{2r}}\left(I-\sigma_\delta\right)^r f\right\|_{p,\pi} = c_1\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r}f\right\|_{p,\pi}.\end{aligned}$$

The last inequality yields

$$\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^{r}f\right\|_{p,\pi} \le c\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r}f\right\|_{p,\pi} \qquad \text{and} \qquad A_\delta^r f \in W_{p(\cdot)}^{2r}.$$

Therefore,

$$\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^{r}f\right\|_{p,\pi} \le c\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r}f\right\|_{p,\pi} = \frac{c}{\delta^{2r}}\left\|\left(I-\sigma_\delta\right)^r\right\|_{p,\pi} \le \frac{c}{\delta^{2r}}\Omega_r(f,\delta)_{p(\cdot)}.$$

Since

$$I - L_\delta^r = \left(I - L_\delta\right)\sum_{j=0}^{r-1}L_\delta^j,$$

we get

$$\left\|\left(I-L_{\delta}^{r}\right) g\right\|_{p,\pi} \leq c\left\|\left(I-L_{\delta}\right) g\right\|_{p,\pi}$$

$$\leq 3c\delta^{-3}\int\limits_{0}^{\delta/2}\int\limits_{0}^{2t} u\left\|\left(I-\sigma_{u}\right) g\right\|_{p,\pi} du dt \leq c \sup_{0<u\leq\delta}\left\|\left(I-\sigma_{u}\right) g\right\|_{p,\pi}.$$

Taking into account that

$$\left\|f-A_{\delta}^{r} f\right\|_{p,\pi}=\left\|\left(I-L_{\delta}^{r}\right)^{r} f\right\|_{p,\pi},$$

by a recursive procedure we obtain

$$\left\|f-A_{\delta}^{r} f\right\|_{p,\pi} \leq c \sup_{0<t_{1}\leq\delta}\left\|\left(I-\sigma_{t_{1}}\right)\left(I-L_{\delta}^{r}\right)^{r-1} f\right\|_{p,\pi}$$

$$\leq c \sup_{0<t_{1}\leq\delta} \sup_{0<t_{2}\leq\delta}\left\|\left(I-\sigma_{t_{1}}\right)\left(I-\sigma_{t_{2}}\right)\left(I-L_{\delta}^{r}\right)^{r-2} f\right\|_{p,\pi}$$

$$\leq \ldots \leq c \sup_{\substack{0<t_{i}\leq\delta \\ i=1,2,\ldots,r}}\left\|\prod_{i=1}^{r}\left(I-\sigma_{t_{i}}\right) f(x)\right\|_{p,\pi}=c\,\Omega_{r}(f,\delta)_{p(\cdot)}.$$

Theorem 7 is proved.

3. Proof of the Main Results

Proof of Theorem 1. We set $A_k(x,f):=a_k\cos kx+b_k\sin kx$. Since the set of trigonometric polynomials is dense [22] in $L_{2\pi}^{p(\cdot)}$, for a given $f\in L_{2\pi}^{p(\cdot)}$ we have $E_n(f)_{p(\cdot)}\to 0$ as $n\to\infty$. From the first inequality in Remark 2, we have

$$f(x)=\sum_{k=0}^{\infty} A_k(x,f)$$

in the norm $\|\cdot\|_{p,\pi}$. For $k=1,2,3,\ldots,$ we can find

$$A_k(x,f)=a_k\cos k\left(x+\frac{\alpha\pi}{2k}-\frac{\alpha\pi}{2k}\right)+b_k\sin k\left(x+\frac{\alpha\pi}{2k}-\frac{\alpha\pi}{2k}\right)$$

$$=A_k\left(x+\frac{\alpha\pi}{2k},f\right)\cos\frac{\alpha\pi}{2}+A_k\left(x+\frac{\alpha\pi}{2k},\tilde{f}\right)\sin\frac{\alpha\pi}{2}$$

and

$$A_k\left(x,f^{(\alpha)}\right)=k^{\alpha}A_k\left(x+\frac{\alpha\pi}{2k},f\right).$$

Therefore,

$$\sum_{k=0}^{\infty} A_k(x,f) = A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x+\frac{\alpha\pi}{2k}, f\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x+\frac{\alpha\pi}{2k}, \tilde{f}\right)$$

$$= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right)$$

and, hence,

$$f(x) - S_n(x,f) = \cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha}} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha}} A_k\left(x, \tilde{f}^{(\alpha)}\right).$$

Since

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right)$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) - \left(S_{k-1}\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)\right]$$

$$= \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) - (n+1)^{-\alpha}\left(S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)$$

and

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right)$$

$$= \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right) - (n+1)^{-\alpha}\left(S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right),$$

we obtain

$$\|f(\cdot) - S_n(\cdot, f)\|_{p,\pi} \le \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left\|S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi}$$

$$+ (n+1)^{-\alpha}\left\|S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi}$$

$$+ \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) \left\| S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot) \right\|_{p,\pi}$$

$$+ (n+1)^{-\alpha} \left\| S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot) \right\|_{p,\pi}$$

$$\leq c \left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(f^{(\alpha)}\right)_{p(\cdot)} \right]$$

$$+ c \left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) E_k\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)} \right].$$

Consequently, it follows from the equivalence in Remark 2(i) that

$$\| f(x) - S_n(x, f) \|_{p,\pi}$$

$$\leq c \left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) + (n+1)^{-\alpha} \right] \left\{ E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)} \right\}$$

$$\leq c E_n\left(f^{(\alpha)}\right)_{p(\cdot)} \left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) + (n+1)^{-\alpha} \right] \leq \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}.$$

Theorem 1 is proved.

Proof of Theorem 2. We put $r - 1 < \alpha < r,\ r \in \mathbb{Z}^{+}$. For $g \in W^{2r}_{p(\cdot)}$, by virtue of Corollary 1, relation (14), and Theorem 7, we have

$$E_n(f)_{p(\cdot)} \leq E_n(f - g)_{p(\cdot)} + E_n(g)_{p(\cdot)} \leq c \left[\| f - g \|_{p,\pi} + (n+1)^{-2r} \left\| g^{(2r)} \right\|_{p,\pi} \right]$$

$$\leq c K\left((n+1)^{-2r}, f; L^{p(\cdot)}_{2\pi}, W^{2r}_{p(\cdot)}\right) \leq c\, \Omega_r \left(f, \frac{1}{n+1}\right)_{p(\cdot)},$$

as required for $r \in \mathbb{Z}^{+}$. Therefore, by the last inequality, we have

$$E_n(f)_{p(\cdot)} \leq c\, \Omega_r \left(f, 1/(n+1)\right)_{p(\cdot)} \leq c\, \Omega_r \left(f, 2\pi/(n+1)\right)_{p(\cdot)}, \quad n = 0, 1, 2, 3, \ldots,$$

and, by (11), we get

$$E_n(f)_{p(\cdot)} \leq c\, \Omega_r \left(f, 2\pi/(n+1)\right)_{p(\cdot)} \leq c\, \Omega_{\alpha} \left(f, 2\pi/(n+1)\right)_{p(\cdot)},$$

whence the required assertion follows.

Proof of Theorem 3. Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial for $f \in L_{2\pi}^{p(\cdot)}$ and let $m \in \mathbb{Z}^+$. Then, by Remark 1(ii), we have

$$\Omega_\alpha\left(f, \pi/n+1\right)_{p(\cdot)} \le \Omega_\alpha\left(f - T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} + \Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)}$$

$$\le cE_{2^m}(f)_{p(\cdot)} + \Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)}.$$

Since

$$T_{2^m}^{(\alpha)}(x) = T_1^{(\alpha)}(x) + \sum_{\nu=0}^{m-1}\left\{T_{2^{\nu+1}}^{(\alpha)}(x) - T_{2^\nu}^{(\alpha)}(x)\right\},$$

by virtue of Lemma 3 we get

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \le \frac{c}{(n+1)^\alpha}\left\{\left\|T_1^{(\alpha)}\right\|_{p,\pi} + \sum_{\nu=0}^{m-1}\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi}\right\}.$$

Lemma 1 gives

$$\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi} \le c2^{\nu\alpha}\left\|T_{2^{\nu+1}} - T_{2^\nu}\right\|_{p,\pi} \le c2^{\nu\alpha+1}E_{2^\nu}(f)_{p(\cdot)}$$

and

$$\left\|T_1^{(\alpha)}\right\|_{p,\pi} = \left\|T_1^{(\alpha)} - T_0^{(\alpha)}\right\|_{p,\pi} \le cE_0(f)_{p(\cdot)}.$$

Hence,

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \le \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + \sum_{\nu=0}^{m-1}2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)}\right\}.$$

Using

$$2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)} \le c^*\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{\alpha-1}E_\mu(f)_{p(\cdot)}, \quad \nu = 1, 2, 3, \ldots,$$

we obtain

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)}$$

$$\le \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + 2^\alpha E_1(f)_{p(\cdot)} + c\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{\alpha-1}E_\mu(f)_{p(\cdot)}\right\}$$

$$\le \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + \sum_{\mu=1}^{2^m}\mu^{\alpha-1}E_\mu(f)_{p(\cdot)}\right\} \le \frac{c}{(n+1)^\alpha}\sum_{\nu=0}^{2^m-1}(\nu+1)^{\alpha-1}E_\nu(f)_{p(\cdot)}.$$

If we choose $2^m \le n+1 \le 2^{m+1}$, then

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \le \frac{c}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot)},$$

$$E_{2^m}(f)_{p(\cdot)} \le E_{2^{m-1}}(f)_{p(\cdot)} \le \frac{c}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot)}.$$

The last two inequalities complete the proof.

Proof of Theorem 4. For the polynomial T_n of the best approximation of f, according to Lemma 1, we have

$$\left\| T_{2^{i+1}}^{(\beta)} - T_{2^i}^{(\beta)} \right\|_{p,\pi} \le C(\beta) 2^{(i+1)\beta} \left\| T_{2^{i+1}} - T_{2^i} \right\|_{p,\pi} \le 2C(\beta) 2^{(i+1)\beta} E_{2^i}(f)_{p(\cdot)}.$$

Hence,

$$\sum_{i=1}^{\infty} \left\| T_{2^{i+1}} - T_{2^i} \right\|_{W^\beta_{p(\cdot)}} = \sum_{i=1}^{\infty} \left\| T_{2^{i+1}}^{(\beta)} - T_{2^i}^{(\beta)} \right\|_{p,\pi} + \sum_{i=1}^{\infty} \left\| T_{2^{i+1}} - T_{2^i} \right\|_{p,\pi}$$

$$\le c \sum_{m=2}^{\infty} m^{\beta-1} E_m(f)_{p(\cdot)} < \infty.$$

Therefore,

$$\left\| T_{2^{i+1}} - T_{2^i} \right\|_{W^\beta_{p(\cdot)}} \to 0 \quad \text{as} \quad i \to \infty.$$

This means that $\{T_{2^i}\}$ is a Cauchy sequence in $L^{p(\cdot)}_{2\pi}$. Since $T_{2^i} \to f$ in $L^{p(\cdot)}_{2\pi}$ and $W^\beta_{p(\cdot)}$ is a Banach space, we conclude that $f \in W^\beta_{p(\cdot)}$.

On the other hand, since

$$\left\| f^{(\beta)} - S_n(f^{(\beta)}) \right\|_{p,\pi} \le \left\| S_{2^{m+2}}(f^{(\beta)}) - S_n(f^{(\beta)}) \right\|_{p,\pi} + \sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}(f^{(\beta)}) - S_{2^k}(f^{(\beta)}) \right\|_{p,\pi},$$

for $2^m < n < 2^{m+1}$ we have

$$\left\| S_{2^{m+2}}(f^{(\beta)}) - S_n(f^{(\beta)}) \right\|_{p,\pi} \le c 2^{(m+2)\beta} E_n(f)_{p(\cdot)} \le c\,(n+1)^\beta E_n(f)_{p(\cdot)}.$$

On the other hand,

$$\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}(f^{(\beta)})-S_{2^k}(f^{(\beta)})\right\|_{p,\pi}\le c\sum_{k=m+2}^{\infty}2^{(k+1)\beta}E_{2^k}(f)_{p(\cdot)}$$

$$\le c\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^k}\mu^{\beta-1}E_{\mu}(f)_{p(\cdot)}$$

$$= c\sum_{\nu=2^{m+1}+1}^{\infty}\nu^{\beta-1}E_{\nu}(f)_{p(\cdot)}\le c\sum_{\nu=n+1}^{\infty}\nu^{\beta-1}E_{\nu}(f)_{p(\cdot)}.$$

Theorem 4 is proved.

Proof of Theorem 5. We set

$$W_n(f):=W_n(x,f):=\frac{1}{n+1}\sum_{\nu=n}^{2n}S_\nu(x,f),\quad n=0,1,2,\dots .$$

Since

$$W_n(\cdot,f^{(\alpha)})=W_n^{(\alpha)}(\cdot,f),$$

we have

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi}$$

$$\le\left\|f^{(\alpha)}(\cdot)-W_n(\cdot,f^{(\alpha)})\right\|_{p,\pi}+\left\|T_n^{(\alpha)}(\cdot,W_n(f))-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi}+\left\|W_n^{(\alpha)}(\cdot,f)-T_n^{(\alpha)}(\cdot,W_n(f))\right\|_{p,\pi}$$

$$:=I_1+I_2+I_3.$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n for f in $L_{2\pi}^{p(\cdot)}$. In this case, the boundedness of the operator S_n in $L_{2\pi}^{p(\cdot)}$ implies the boundedness of the operator W_n in $L_{2\pi}^{p(\cdot)}$, and we obtain

$$I_1\le\left\|f^{(\alpha)}(\cdot)-T_n^*(\cdot,f^{(\alpha)})\right\|_{p,\pi}+\left\|T_n^*(\cdot,f^{(\alpha)})-W_n(\cdot,f^{(\alpha)})\right\|_{p,\pi}$$

$$\le cE_n(f^{(\alpha)})_{p(\cdot)}+\left\|W_n(\cdot,T_n^*(f^{(\alpha)})-f^{(\alpha)})\right\|_{p,\pi}\le cE_n(f^{(\alpha)})_{p(\cdot)}.$$

Using Lemma 1, we get

$$I_2\le cn^{\alpha}\left\|T_n(\cdot,W_n(f))-T_n(\cdot,f)\right\|_{p,\pi}$$

and

$$I_3 \le c\,(2n)^{\alpha}\,\|W_n(\cdot,f)-T_n(\cdot,W_n(f))\|_{p,\pi} \le c\,(2n)^{\alpha}\,E_n\,(W_n(f))_{p(\cdot)}\,.$$

We now have

$$\begin{aligned}
&\|T_n(\cdot,W_n(f))-T_n(\cdot,f)\|_{p,\pi}\\
&\quad\le \|T_n(\cdot,W_n(f))-W_n(\cdot,f)\|_{p,\pi}+\|W_n(\cdot,f)-f(\cdot)\|_{p,\pi}+\|f(\cdot)-T_n(\cdot,f)\|_{p,\pi}\\
&\quad\le cE_n\,(W_n(f))_{p(\cdot)}+cE_n(f)_{p(\cdot)}+cE_n(f)_{p(\cdot)}.
\end{aligned}$$

Since

$$E_n\,(W_n(f))_{p(\cdot)} \le cE_n(f)_{p(\cdot)},$$

we get

$$\begin{aligned}
\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi} &\le cE_n(f^{(\alpha)})_{p(\cdot)}+cn^{\alpha}E_n\,(W_n(f))_{p(\cdot)}+cn^{\alpha}E_n(f)_{p(\cdot)}+c\,(2n)^{\alpha}\,E_n\,(W_n(f))_{p(\cdot)}\\
&\le cE_n(f^{(\alpha)})_{p(\cdot)}+cn^{\alpha}E_n(f)_{p(\cdot)}.
\end{aligned}$$

Since, according to Theorem 1,

$$E_n(f)_{p(\cdot)} \le \frac{c}{(n+1)^{\alpha}}E_n(f^{(\alpha)})_{p(\cdot)},$$

we obtain

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi} \le cE_n(f^{(\alpha)})_{p(\cdot)}.$$

Theorem 5 is proved.

Proof of Theorem 6. Let $f\in H^{p(\cdot)}(\mathbb{D})$. First of all, if $p(x)$ defined on $\boldsymbol{T}$ possesses the Dini–Lipschitz property DL_{γ} for $\gamma\ge 1$ on $\boldsymbol{T}$, then $p\left(e^{ix}\right)$, $x\in\boldsymbol{T}$, defined on $\mathbb{T}$ possesses the Dini–Lipschitz property DL_{γ} for $\gamma\ge 1$ on $\mathbb{T}$. Since $H^{p(\cdot)}\subset H^{1}(\mathbb{D})$ for $1<\underline{p}$, let

$$\sum_{k=-\infty}^{\infty}\beta_k e^{ik\theta}$$

be the Fourier series of the function $f\left(e^{i\theta}\right)$ and let

$$S_n(f,\theta):=\sum_{k=-n}^{n}\beta_k e^{ik\theta}$$

be its nth partial sum. Since $f\left(e^{i\theta}\right) \in H^1(\mathbb{D})$, we have [11, p. 38]

$$\beta_k = \begin{cases} 0, & \text{for } k < 0, \\ a_k(f), & \text{for } k \geq 0. \end{cases}$$

Therefore,

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} = \| f - S_n(f, \cdot) \|_{p,\pi} . \tag{15}$$

If t_n^* is the best approximating trigonometric polynomial for $f(e^{i\theta})$ in $L_{2\pi}^{p(\cdot)}$, then, using relations (6) and (15) and Theorem 2, we get

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} \leq \left\| f\left(e^{i\theta}\right) - t_n^*(\theta) \right\|_{p,\pi} + \left\| S_n\left(f - t_n^*, \theta\right) \right\|_{p,\pi}$$

$$\leq c E_n \left(f\left(e^{i\theta}\right) \right)_{p(\cdot)} \leq c\, \Omega_r \left(f\left(e^{i\theta}\right), \frac{1}{n+1} \right)_{p(\cdot)} .$$

Theorem 6 is proved.

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