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MONOGENIC FUNCTIONS IN A BIHARMONIC ALGEBRA

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We present a constructive description of monogenic functions that take values in a commutative biharmonic algebra by using analytic functions of complex variables. We establish an isomorphism between algebras of monogenic functions defined in different biharmonic planes. It is proved that every biharmonic function in a bounded simply connected domain is the first component of a certain monogenic function defined in the corresponding domain of a biharmonic plane.

An associative algebra of the second rank with identity commutative over the field of complex numbers $\mathbb{C}$ is called *biharmonic* if it contains a basis $\{e_1, e_2\}$ that satisfies the conditions

$$\left(e_1^2 + e_2^2\right)^2 = 0, \qquad e_1^2 + e_2^2 \neq 0, \tag{1}$$

which is also called *biharmonic*.

It was shown in [1] that there exists a unique biharmonic algebra $\mathbb{B}$ whose basis (note that it is not biharmonic) consists of the identity of the algebra 1 and an element ρ for which $\rho^2 = 0$. In the same work, all biharmonic bases $\{e_1, e_2\}$ were described and it was shown that they form the two-parameter family

$$e_1 = \alpha_1 + \alpha_2\rho, \qquad e_2 = \pm i\left(\alpha_1 + \left(\alpha_2 - \frac{1}{2\alpha_1}\right)\rho\right), \tag{2}$$

where i is the imaginary unit and the complex numbers $\alpha_1 \neq 0$ and α_2 can be chosen arbitrarily. Here and in what follows, the presence of the symbol $\pm$ in a relation means that either the upper signs or the lower signs should be chosen simultaneously.

The *biharmonic plane* μ_{e_1, e_2} is understood as the linear span $\mu_{e_1, e_2} := \{\zeta = xe_1 + ye_2 : x, y \in \mathbb{R}\}$ of the elements e_1 and e_2 over the field of real numbers $\mathbb{R}$.

We associate a domain D of the Cartesian plane xOy with the domain $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}$ congruent to it in the plane μ_{e_1, e_2}.

Since $\alpha_1 \neq 0$ in (2), any nonzero element of a biharmonic plane is invertible. Therefore, the derivatives of functions defined in domains of a biharmonic plane are defined in the same way as in the complex plane.

A function $\Phi : D_\zeta \to \mathbb{B}$ is called *monogenic* in a domain D_ζ if, at every point $\zeta \in D_\zeta$, there exists the finite limit

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$$\lim_{h \to 0,\, h \in \mu_{e_1, e_2}} \big(\Phi(\zeta + h) - \Phi(\zeta)\big) h^{-1} = \Phi'(\zeta),$$

which is called the derivative of the function Φ at the point ζ.

If a function

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) i e_1 + U_3(x, y) e_2 + U_4(x, y) i e_2, \qquad \zeta = x e_1 + y e_2, \tag{3}$$

where $U_k : D \to \mathbb{R}$, $k = \overline{1, 4}$, has continuous derivatives up to the fourth order inclusive in the domain D_ζ, then, by virtue of the equality

$$\Delta^2 \Phi := \frac{\partial^4 \Phi(\zeta)}{\partial x^4} + 2 \frac{\partial^4 \Phi(\zeta)}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi(\zeta)}{\partial y^4} = \Phi^{(4)}(\zeta) \left(e_1^2 + e_2^2\right)^2 \tag{4}$$

and condition (1), each component $U_k(x, y)$, $k = \overline{1, 4}$, of this function is a *biharmonic function,* i.e., it satisfies the biharmonic equation

$$\Delta^2 U(x, y) = 0. \tag{5}$$

in the domain D.

Similarly to monogenic functions in the complex plane, the monogenic functions defined in a domain D_ζ of an arbitrary biharmonic plane μ_{e_1, e_2} and taking values in the biharmonic algebra $\mathbb{B}$ form an algebra, which is denoted by $\mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$.

In [2], monogenic functions defined in domains of a biharmonic plane whose biharmonic basis is formed by the elements

$$e_1 = 1, \qquad e_2 = i - \frac{i}{2} \rho \tag{6}$$

were considered and necessary and sufficient conditions for their monogeneity (Cauchy–Riemann conditions) were established; we write these conditions here in a folded form:

$$\frac{\partial \Phi(\zeta)}{\partial y} = \frac{\partial \Phi(\zeta)}{\partial x} e_2 \qquad \forall \zeta = x e_1 + y e_2 \in D_\zeta. \tag{7}$$

It can be established by analogy that function (3) is monogenic in a domain D_ζ of an arbitrary biharmonic plane μ_{e_1, e_2} if and only if its components $U_k(x, y)$, $k = \overline{1, 4}$, are differentiable in the domain D and the following equality is true:

$$\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2 \qquad \forall \zeta = x e_1 + y e_2 \in D_\zeta. \tag{8}$$

In the present paper, we give a constructive description of all monogenic functions in a plane μ_{e_1, e_2} by using monogenic functions of complex variables and establish an isomorphism between algebras of monogenic functions defined in different biharmonic planes. We also show that every biharmonic function in a bounded simply connected domain is the first component of a certain monogenic function (3) and determine the latter in an explicit form.

1. Constructive Description of Monogenic Functions in a Plane μ_{e_1, e_2}

The unique maximum ideal $\mathcal{J} \equiv \{c\rho : c \in \mathbb{C}\}$ of the algebra $\mathbb{B}$ is associated with a linear continuous functional $f : \mathbb{B} \to \mathbb{C}$ with kernel $\mathcal{J}$ such that $f(1) = 1$. Let G denote the domain in $\mathbb{C}$ onto which the functional f maps the domain D_ζ. Consider the linear operator $\mathcal{A}$ that associates every function $\Phi : D_\zeta \to \mathbb{B}$ with a function $F_\Phi : G \to \mathbb{C}$ according to the relation $F_\Phi(z) := f(\Phi(\zeta))$, where $\zeta = xe_1 + ye_2$ and $z := f(\zeta) = \alpha_1(x \pm iy)$.

In this case, it is obvious that if a function Φ is monogenic in the domain D_ζ, then $F_\Phi(z) = (\mathcal{A}\Phi)(z)$ is a monogenic function of the complex variable z in the domain G, i.e., it is holomorphic if $z = \alpha_1(x + iy)$ and antiholomorphic if $z = \alpha_1(x - iy)$.

By analogy with Theorem 2.4 in [3], one can prove the following statement:

Theorem 1. *Every function* $\Phi : D_\zeta \to \mathbb{B}$ *monogenic in the domain* D_ζ *can be represented in the form*

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_\gamma (t - \zeta)^{-1} (\mathcal{A}\Phi)(t)\, dt + \Phi_0(\zeta) \qquad \forall\, \zeta \in D_\zeta, \tag{9}$$

where γ *is an arbitrary closed rectifiable Jordan curve in the domain* G *that encloses the point* $f(\zeta)$, *and* $\Phi_0 : D_\zeta \to \mathcal{J}$ *is a function monogenic in the domain* D_ζ *and taking values in the ideal* $\mathcal{J}$.

Note that the complex number $z = f(\zeta)$ is the spectrum of an element ζ of the algebra $\mathbb{B}$, and the integral in equality (9) is the principal extension of the monogenic function $F(z) = (\mathcal{A}\Phi)(z)$ of the complex variable z to the domain D_ζ.

It follows from Theorem 1 that the algebra $\mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$ can be decomposed into the direct sum of the algebra of principal extensions of monogenic functions of complex variables to D_ζ and the algebra of functions monogenic in D_ζ and taking values in the ideal $\mathcal{J}$.

The theorem below describes all monogenic functions defined in a domain D_ζ of an arbitrary biharmonic plane μ_{e_1, e_2} and taking values in the ideal $\mathcal{J}$ in terms of monogenic functions of complex variables.

Theorem 2. *Every function* $\Phi_0 : D_\zeta \to \mathcal{J}$ *monogenic in the domain* D_ζ *and taking values in the ideal* $\mathcal{J}$ *can be represented in the form*

$$\Phi_0(\zeta) = F_0(z)\rho \qquad \forall\, \zeta \in D_\zeta, \tag{10}$$

where $F_0 : G \to \mathbb{C}$ *is a monogenic function of the complex variable* $z = f(\zeta)$.

Proof. Since Φ_0 takes value in the ideal $\mathcal{J}$, the following equality is true:

$$\Phi_0(\zeta) = \varphi_0(x, y)\rho \qquad \forall\, \zeta = xe_1 + ye_2 \in D_\zeta. \tag{11}$$

where $\varphi_0 : D \to \mathbb{C}$.

Function (11) satisfies the monogeneity condition (8) for $\Phi = \Phi_0$:

$$\frac{\partial \varphi_0(x, y)}{\partial y}\rho e_1 = \frac{\partial \varphi_0(x, y)}{\partial x}\rho e_2 \qquad \forall\,(x, y) \in D. \tag{12}$$

Using relation (2) and the expression

$$e_1^{-1} = \frac{1}{\alpha_1}\left(1 - \frac{\alpha_2}{\alpha_1}\rho\right)$$

for the element inverse to e_1, we obtain

$$\rho e_1^{-1} e_2 = \pm i\rho.$$

Taking this equality into account, we reduce condition (12) to the form

$$\frac{\partial \varphi_0(x, y)}{\partial y}\rho = \pm i\,\frac{\partial \varphi_0(x, y)}{\partial x}\rho \qquad \forall\,(x, y) \in D.$$

Taking into account the uniqueness of a decomposition of elements of the algebra $\mathbb{B}$ in the basis $\{1, \rho\}$, we get

$$\frac{\partial \varphi_0(x, y)}{\partial y} = \pm i\,\frac{\partial \varphi_0(x, y)}{\partial x} \qquad \forall\,(x, y) \in D.$$

Therefore, $F_0(z) := \varphi_0(x, y)$ is a monogenic function of the complex variable $z = f(xe_1 + ye_2)$ in the domain G.

The theorem is proved.

Theorem 3. *Every function* $\Phi : D_\zeta \to \mathbb{B}$ *monogenic in the domain* D_ζ *has derivatives of all orders in* D_ζ.

Proof. The function Φ is defined by equality (9) in which the integral has derivatives of all orders in the domain D_ζ and the monogenic function Φ_0 can be represented in the form (10) and, hence, is infinitely differentiable with respect to the variables x and y in the domain D. Therefore, the derivative Φ_0' satisfies conditions of the form (8) in D_ζ, i.e., it is a monogenic function. By analogy, we establish that derivatives of all orders of the function Φ_0 are monogenic functions in the domain D_ζ.

The theorem is proved.

It follows from Theorem 3, equality (4), and condition (1) that the components $U_k(x, y)$, $k = \overline{1, 4}$, of every function (3) monogenic in the domain D_ζ satisfy the biharmonic equation (5) in the domain D.

By virtue of equalities (9) and (10), all monogenic functions $\Phi: D_\zeta \to \mathbb{B}$ can be represented by using two arbitrary complex-valued monogenic functions $F(z)$ and $F_0(z)$ of a complex variable $z \in G$ as follows:

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_\gamma F(t)(t-\zeta)^{-1}\,dt + F_0\left(f(\zeta)\right)\rho \quad \forall \zeta \in D_\zeta. \tag{13}$$

In [2], the principal extensions of holomorphic functions of complex variables to a biharmonic plane μ_{e_1,e_2} based on vectors (6) were constructed in an explicit form.

To obtain the principal extension of a monogenic function $F(z)$ of a complex variable $z = \alpha_1(x \pm iy) \in G$ to the domain D_ζ of an arbitrary biharmonic plane μ_{e_1, e_2}, we use the resolvent decomposition

$$(t-\zeta)^{-1} = \frac{1}{t-z} - \frac{1}{2\alpha_1}\frac{2\alpha_2 z \pm iy}{(t-z)^2}\rho \quad \forall \zeta = xe_1 + ye_2 \in D_\zeta \quad \forall t \in \gamma$$

in the basis $\{1, \rho\}$. As a result, we get

$$\frac{1}{2\pi i}\int_\gamma F(t)(t-\zeta)^{-1}\,dt = F(z) - \frac{F'(z)}{\alpha_1}\left(\alpha_2 z \pm \frac{iy}{2}\right)\rho \quad \forall \zeta = xe_1 + ye_2 \in D_\zeta. \tag{14}$$

In particular, if $\alpha_1 = 1$, $\alpha_2 = 0$, and the basis elements e_1 and e_2 of the biharmonic plane μ_{e_1, e_2} are defined by (6), then the right-hand side of equality (14) is simplified and equality (13) takes the form

$$\Phi(\zeta) = F(z) - \left(\frac{iy}{2}F'(z) - F_0(z)\right)\rho \quad \forall \zeta = x + ye_2 \in D_\zeta, \tag{15}$$

where $z \equiv f(\zeta) = x + iy \in G$.

Note that, in [14], equality (15) was obtained (in a different form) for monogenic functions under additional assumptions on the geometry of the domain D_ζ.

2. On an Isomorphism of Algebras of Monogenic Functions Defined in Different Biharmonic Planes

First, we consider several auxiliary statements.

Lemma 1. *Suppose that biharmonic bases* $\{e_1, e_2\}$ *and* $\{\tilde{e}_1, \tilde{e}_2\}$ *are connected by the relations*

$$\tilde{e}_1 = e_1 + r_1\rho, \quad \tilde{e}_2 = \pm(e_2 + r_2\rho), \quad r_1, r_2 \in \mathbb{C}. \tag{16}$$

If a function $\Phi: D_\zeta \to \mathbb{B}$ *is monogenic in the domain* D_ζ *of the biharmonic plane* μ_{e_1, e_2}, *then the function*

$$\tilde{\Phi}(\tilde{\zeta}) = \Phi(\zeta) + \Phi'(\zeta)(xr_1 + yr_2)\rho \tag{17}$$

is monogenic in the domain $\tilde{D}_{\tilde{\zeta}} := \left\{\tilde{\zeta} = x\tilde{e}_1 \pm y\tilde{e}_2 : \zeta = xe_1 + ye_2 \in D_\zeta\right\}$ *of the biharmonic plane* $\mu_{\tilde{e}_1, \tilde{e}_2}$.

Proof. First, we prove that the monogeneity of a function $\Phi(\zeta)$ in the domain D_ζ yields the monogeneity of function (17) in the domain $\tilde{D}_{\tilde{\zeta}}$. To this end, we show that the function $\tilde{\Phi}$ satisfies necessary and sufficient monogeneity conditions of the form (8), i.e., the conditions

$$\frac{\partial\tilde{\Phi}(\tilde{\zeta})}{\partial y}\tilde{e}_1 = \pm\frac{\partial\tilde{\Phi}(\tilde{\zeta})}{\partial x}\tilde{e}_2 \qquad \forall\,\tilde{\zeta} = x\tilde{e}_1 \pm y\tilde{e}_2 \in D_{\tilde{\zeta}}. \tag{18}$$

By virtue of the monogeneity of the functions Φ and Φ', the following equalities hold for all $\zeta \in D_\zeta$:

$$\frac{\partial\Phi(\zeta)}{\partial y} = \Phi'(\zeta)\,e_2, \qquad \frac{\partial\Phi'(\zeta)}{\partial y} = \Phi''(\zeta)\,e_2. \tag{19}$$

Using relations (16), (17), and (19), we obtain

$$\frac{\partial\tilde{\Phi}(\tilde{\zeta})}{\partial y}\tilde{e}_1 = \left(\frac{\partial\Phi(\zeta)}{\partial y} + \frac{\partial\Phi'(\zeta)}{\partial y}(xr_1 + yr_2)\rho + r_2\Phi'(\zeta)\rho\right)(e_1 + r_1\rho)$$

$$= \left(\tilde{\Phi}'(\zeta)\,e_2 + \Phi''(\zeta)(xr_1 + yr_2)\,e_2\rho + r_2\Phi'(\zeta)\rho\right)(e_1 + r_1\rho)$$

$$= \Phi'(\zeta)\left(e_1e_2 + (r_2e_1 + r_1e_2)\rho\right) + \Phi''(\zeta)(xr_1 + yr_2)\,e_1e_2.$$

By analogy, using the equalities

$$\frac{\partial\Phi(\zeta)}{\partial x} = \Phi'(\zeta)\,e_1, \qquad \frac{\partial\Phi'(\zeta)}{\partial x} = \Phi''(\zeta)\,e_1,$$

which are valid for all $\zeta \in D_\zeta$, and the monogeneity of the functions Φ and Φ', we get

$$\pm\frac{\partial\tilde{\Phi}(\tilde{\zeta})}{\partial x}\tilde{e}_2 = \Phi'(\zeta)\left(e_1e_2 + (r_2e_1 + r_1e_2)\rho\right) + \Phi''(\zeta)(xr_1 + yr_2)\,e_1e_2.$$

Thus, the function $\tilde{\Phi}$ satisfies conditions (18), i.e., it is monogenic in the domain $\tilde{D}_{\tilde{\zeta}}$.

The lemma is proved.

Lemma 2. *Suppose that biharmonic bases* $\{e_1, e_2\}$ *and* $\{\tilde{e}_1, \tilde{e}_2\}$ *are connected by relations (16) and a function* $\tilde{\Phi}: \tilde{D}_{\tilde{\zeta}} \to \mathbb{B}$ *is monogenic in the domain* $\tilde{D}_{\tilde{\zeta}}$ *of the biharmonic plane* $\mu_{\tilde{e}_1, \tilde{e}_2}$. *Then there exists a unique function* $\Phi(\zeta)$ *monogenic in the domain* $D_\zeta := \left\{\zeta = xe_1 + ye_2 : \tilde{\zeta} = x\tilde{e}_1 \pm y\tilde{e}_2 \in \tilde{D}_{\tilde{\zeta}}\right\}$ *of the biharmonic plane* μ_{e_1, e_2} *that satisfies equality (17).*

Proof. Consider the function

$$\Phi(\zeta) = \tilde{\Phi}(\tilde{\zeta}) - \tilde{\Phi}'(\tilde{\zeta})(xr_1 + yr_2)\tilde{e}_1^2\rho \quad \forall\, \zeta \in D_\zeta. \tag{20}$$

The monogeneity of this function in the domain D_ζ can be proved by analogy with the monogeneity of function (17) in the domain $\tilde{D}_{\tilde{\zeta}}$ (see the proof of Lemma 1).

Let us prove that function (20) satisfies equality (17). For this purpose, we multiply both sides of equality (20) by ρ and then differentiate them with respect to x. As a result, we get

$$\frac{\partial \Phi(\zeta)}{\partial x}\rho = \frac{\partial \tilde{\Phi}(\tilde{\zeta})}{\partial x}\rho \quad \forall\, \zeta \in D_\zeta. \tag{21}$$

Taking into account the relations

$$\frac{\partial \tilde{\Phi}(\tilde{\zeta})}{\partial x} = \tilde{\Phi}'(\tilde{\zeta})\,\tilde{e}_1, \qquad \frac{\partial \Phi(\zeta)}{\partial x} = \Phi'(\zeta)\,e_1, \qquad \tilde{e}_1\rho = e_1\rho,$$

and equality (21), we obtain

$$\tilde{\Phi}'(\tilde{\zeta})\,\tilde{e}_1^2\,\rho = \frac{\partial \tilde{\Phi}(\tilde{\zeta})}{\partial x}\tilde{e}_1\rho = \frac{\partial \tilde{\Phi}(\tilde{\zeta})}{\partial x}e_1\rho = \frac{\partial \Phi(\zeta)}{\partial x}e_1\rho = \Phi'(\zeta)\,e_1.$$

By virtue of these relations, function (20) satisfies equality (17).

Finally, we prove the uniqueness of a monogenic function Φ that satisfies equality (17). To this end, it suffices to show that the function $\tilde{\Phi} \equiv 0$ in $D_{\tilde{\zeta}}$ is associated only with the function $\Phi \equiv 0$ in D_ζ. Indeed, for $\tilde{\Phi} \equiv 0$, relation (17) takes the form

$$\Phi(\zeta) + \Phi'(\zeta)(xr_1 + yr_2)\rho \equiv 0. \tag{22}$$

Multiplying identity (22) by ρ term by term, we get $\Phi(\zeta)\rho \equiv 0$, which, in turn, yields

$$\Phi'(\zeta)(xr_1 + yr_2)\rho \equiv 0. \tag{23}$$

Finally, comparing identities (22) and (23), we conclude that $\Phi \equiv 0$.

The lemma is proved.

Theorem 4. *Let a biharmonic basis $\{e_1, e_2\}$ be formed by elements (6) and let $\{\tilde{e}_1, \tilde{e}_2\}$ be an arbitrary biharmonic basis whose elements are represented by equalities of the form (2). Also assume that D_ζ is a domain of the biharmonic plane μ_{e_1, e_2} and $\tilde{D}_{\tilde{\zeta}} := \{\tilde{\zeta} = x\tilde{e}_1 \pm y\tilde{e}_2 : \zeta = xe_1 + ye_2 \in D_\zeta\}$ is the corresponding domain of the biharmonic plane $\mu_{\tilde{e}_1, \tilde{e}_2}$. Then the algebras $\mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$ and $\mathcal{M}(\mu_{\tilde{e}_1, \tilde{e}_2}, \tilde{D}_{\tilde{\zeta}})$ are isomorphic, and the correspondence between the functions $\Phi \in \mathcal{M}(\mu_{e_1,e_2}, D_\zeta)$ and $\tilde{\Phi} \in \mathcal{M}(\mu_{\tilde{e}_1,\tilde{e}_2}, \tilde{D}_{\tilde{\zeta}})$ is established by equality (17), where $r_1 := \alpha_2/\alpha_1$, $r_2 := i(\alpha_1^2 + 2\alpha_1\alpha_2 - 1)/(2\alpha_1^2)$, and α_1 and α_2 are the same complex numbers as in equalities (2) for the elements of the basis $\{\tilde{e}_1, \tilde{e}_2\}$.*

Proof. We consider a biharmonic basis $\left\{\tilde e_1^{(1)}, \tilde e_2^{(1)}\right\}$ such that

$$\tilde e_1^{(1)} = \tilde e_1/\alpha_1, \qquad \tilde e_2^{(1)} = \tilde e_2/\alpha_1$$

and define the domain $\tilde D^{(1)}_{\tilde\zeta^{(1)}} := \left\{\tilde\zeta^{(1)} = x\tilde e_1^{(1)} \pm y\tilde e_2^{(1)} : \zeta = xe_1 + ye_2 \in D_\zeta\right\}$ in the plane $\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}$.

We associate every function $\Phi \in \mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$ with a function $\tilde\Phi^{(1)} \in \mathcal{M}\left(\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}, \tilde D^{(1)}_{\tilde\zeta^{(1)}}\right)$ by a relation of the form (17). Since the elements of the basis $\left\{\tilde e_1^{(1)}, \tilde e_2^{(1)}\right\}$ are associated with the elements e_1 and e_2 by relations of the form (16), by virtue of Lemmas 1 and 2 the indicated correspondence between the algebras $\mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$ and $\mathcal{M}\left(\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}, \tilde D^{(1)}_{\tilde\zeta^{(1)}}\right)$ is bijective. It follows from the equalities

$$\tilde\Phi_1^{(1)}\left(\tilde\zeta^{(1)}\right)\tilde\Phi_2^{(1)}\left(\tilde\zeta^{(1)}\right) = \left(\Phi_1(\zeta) + \Phi_1'(\zeta)(xr_1 + yr_2)\rho\right)\left(\Phi_2(\zeta) + \Phi_2'(\zeta)(xr_1 + yr_2)\rho\right)$$

$$= \Phi_1(\zeta)\Phi_2(\zeta) + \left(\Phi_1(\zeta)\Phi_2(\zeta)\right)'(xr_1 + yr_2)\rho$$

that the product of the functions $\tilde\Phi_1^{(1)}, \tilde\Phi_2^{(1)} \in \mathcal{M}\left(\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}, \tilde D^{(1)}_{\tilde\zeta^{(1)}}\right)$ corresponds to the product of the functions $\Phi_1, \Phi_2 \in \mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$, i.e., the algebras $\mathcal{M}(\mu_{e_1, e_2}, D_\zeta)$ and $\mathcal{M}\left(\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}, \tilde D^{(1)}_{\tilde\zeta^{(1)}}\right)$ are isomorphic.

Finally, we establish an isomorphism between the algebras $\mathcal{M}\left(\mu_{\tilde e_1^{(1)}, \tilde e_2^{(1)}}, \tilde D^{(1)}_{\tilde\zeta^{(1)}}\right)$ and $\mathcal{M}(\mu_{\tilde e_1, \tilde e_2}, \tilde D_{\tilde\zeta})$ by the equality

$$\tilde\Phi(\tilde\zeta) := \tilde\Phi^{(1)}\left(\tilde\zeta^{(1)}\right), \qquad \tilde\zeta = \alpha_1\tilde\zeta^{(1)}.$$

The monogeneity of the function $\tilde\Phi$ in the domain $\tilde D_{\tilde\zeta}$ obviously follows from the monogeneity conditions (8) for the function $\tilde\Phi^{(1)}$ and the inequality $\alpha_1 \neq 0$.

The theorem is proved.

By virtue of Theorem 4, it suffices to study monogenic functions in the biharmonic plane μ_{e_1, e_2} constructed on the basis of vectors (6).

3. Representation of Biharmonic Functions in the Form of Components of Monogenic Functions

In what follows, the basis elements e_1 and e_2 of a biharmonic plane μ_{e_1, e_2} are defined by (6).

Let us show that every function $U_1(x, y)$ biharmonic in a bounded simply connected domain D of the Cartesian plane xOy is the first component of a certain function (3) monogenic in the corresponding domain D_ζ of the biharmonic plane μ_{e_1, e_2}.

Consider several auxiliary statements.

Lemma 3. *Any monogenic function (3) for which* $U_1 \equiv 0$ *has the form*

$$\Phi(\zeta) = i\left(-ax^2 + kx - ay^2 - by + n\right) + e_2\left(2ay^2 + 2by + c\right)$$

$$+ ie_2(-2axy - bx + ky + m) \quad \forall \zeta = xe_1 + ye_2, \tag{24}$$

where a, b, c, k, m, *and* n *are arbitrary real constants.*

Proof. The monogeneity condition (7) has the following componentwise form (see [2]):

$$\frac{\partial U_1(x, y)}{\partial y} = \frac{\partial U_3(x, y)}{\partial x}, \tag{25}$$

$$\frac{\partial U_2(x, y)}{\partial y} = \frac{\partial U_4(x, y)}{\partial x}, \tag{26}$$

$$\frac{\partial U_3(x, y)}{\partial y} = \frac{\partial U_1(x, y)}{\partial x} - 2\frac{\partial U_4(x, y)}{\partial x}, \tag{27}$$

$$\frac{\partial U_4(x, y)}{\partial y} = \frac{\partial U_2(x, y)}{\partial x} + 2\frac{\partial U_3(x, y)}{\partial x}. \tag{28}$$

Substituting the function $U_1 \equiv 0$ in equality (25) and integrating the latter with respect to the variable x, we get

$$U_3(x, y) = u_3(y) \quad \forall (x, y) \in D. \tag{29}$$

Here and in the remaining part of the proof, u_k, $k = 2, 3, 4$, are certain infinitely differentiable functions $u_k : D^y \to \mathbb{R}$, where D^y is the projection of a domain D to the axis Oy.

Integrating equality (27) with respect to the variable x and taking into account the identity $U_1 \equiv 0$ and equality (29), we obtain

$$U_4(x, y) = -\frac{x}{2}u_3'(y) + u_4(y) \quad \forall (x, y) \in D. \tag{30}$$

Substituting relations (29) and (30) into equality (28) and integrating it with respect to x, we get

$$U_2(x, y) = -\frac{1}{4}x^2 u_3''(y) + xu_4'(y) + u_2(y) \quad \forall (x, y) \in D. \tag{31}$$

Taking (30) and (31) into account, we reduce equality (26) to the form

$$-\frac{x^2}{4}u_3'''(y) + xu_4''(y) + u_2'(y) + \frac{1}{2}u_3'(y) = 0 \quad \forall (x, y) \in D. \tag{32}$$

Further, differentiating equality (32) twice with respect to the variable x, we obtain

$$u_3'''(y) = u_4''(y) = 0 \quad \forall y \in D^y, \tag{33}$$

$$u_2'(y) + \frac{1}{2}u_3'(y) = 0 \quad \forall y \in D^y. \tag{34}$$

Integrating equalities (33) the corresponding number of times with respect to the variable y, we determine the functions u_3 and u_4:

$$u_3(y) = 2ay^2 + 2by + c, \qquad u_4(y) = ky + m \quad \forall y \in D^y, \tag{35}$$

where a, b, c, k, and m are arbitrary real constants.

Substituting functions (35) into equalities (29) and (30), we get

$$U_3(x, y) = 2ay^2 + 2by + c \quad \forall (x, y) \in D, \tag{36}$$

$$U_4(x, y) = -2axy - bx + ky + m \quad \forall (x, y) \in D. \tag{37}$$

By analogy, substituting the function u_3 into equality (34) and integrating the latter with respect to y, we obtain

$$u_2(y) = -ay^2 - by + n \quad \forall y \in D^y, \tag{38}$$

where n is an arbitrary real constant. Substituting functions (35) and (38) in (31), we get

$$U_2(x, y) = -ax^2 + kx - ay^2 - by + n \quad \forall (x, y) \in D. \tag{39}$$

Finally, substituting the components $U_1 \equiv 0$, (36), (37), and (39) in decomposition (3) of the monogenic function Φ, we obtain equality (24).

The lemma is proved.

Lemma 4. *Let D be a bounded simply connected domain of the Cartesian plane xOy. If F is a holomorphic function in the domain $G := \{z = x + iy : (x, y) \in D\}$ of the complex plane, then the functions*

$$\Phi_1(\zeta) = u(x, y) + iv(x, y) - e_2 v(x, y) + ie_2 u(x, y),$$

$$\Phi_2(\zeta) = yu(x, y) + iyv(x, y) + e_2\big(\mathcal{U}(x, y) - yv(x, y)\big) + ie_2\big(\mathcal{V}(x, y) + yu(x, y)\big),$$

$$\Phi_3(\zeta) = xu(x, y) + ixv(x, y) + e_2\big(\mathcal{V}(x, y) - xv(x, y)\big)$$

$$+ ie_2\big(xu(x, y) - \mathcal{U}(x, y)\big) \quad \forall\, \zeta = xe_1 + ye_2 \in D_\zeta$$

are monogenic in the domain D_ζ *of the biharmonic plane* μ_{e_1, e_2}; *here,*

$$u(x, y) := \operatorname{Re} F(z), \qquad v(x, y) := \operatorname{Im} F(z),$$

$$\mathcal{U}(x, y) := \operatorname{Re} \mathcal{F}(z), \qquad \mathcal{V}(x, y) := \operatorname{Im} \mathcal{F}(z) \quad \forall\, z = x + iy,$$

and $\mathcal{F}$ *is the antiderivative of the function* F *in the domain* G.

The lemma is proved by the direct verification of the monogeneity conditions (7) for the functions Φ_1, Φ_2, and Φ_3.

It is known that every biharmonic function $U_1(x, y)$ in a domain D is represented by the Goursat formula (see, e.g., [5, p. 108])

$$U_1(x, y) = \operatorname{Re}\big(\varphi(z) + \bar{z}\psi(z)\big), \qquad z = x + iy, \tag{40}$$

where φ and ψ are holomorphic functions in the domain G defined in Lemma 4 and $\bar{z} := x - iy$.

Theorem 5. *Every function* $U_1(x, y)$ *biharmonic in a bounded simply connected domain* D *of the Cartesian plane* xOy *is the first component in decomposition (3) of the following monogenic function in the domain* D_ζ *of the biharmonic plane* μ_{e_1, e_2}:

$$\Phi(\zeta) = \varphi(z) + \bar{z}\psi(z) + ie_2\big(\varphi(z) + \bar{z}\psi(z) - 2\,\mathcal{F}(z)\big), \tag{41}$$

$$\zeta = xe_1 + ye_2, \qquad z = x + iy,$$

where φ *and* ψ *are the functions from equality (40), which are holomorphic in the domain* $G := \{z = x + iy : (x, y) \in D\}$, *and* $\mathcal{F}$ *is the antiderivative of the function* ψ *in the domain* G. *All functions monogenic in the domain* D_ζ *whose first component in decomposition (3) is the function* U_1 *can be represented in the form of the sum of functions (24) and (41).*

Proof. Denoting $u_1(x, y) := \operatorname{Re}\varphi(z)$, $u_2(x, y) := \operatorname{Re}\psi(z)$, and $v_2(x, y) := \operatorname{Im}\psi(z)$, we rewrite equality (40) in the form

$$U_1(x, y) = u_1(x, y) + xu_2(x, y) + yv_2(x, y). \tag{42}$$

Using relation (42) and Lemma 4, we establish that function (41) is monogenic in the domain D_ζ and that its first component in decomposition (3) is the function U_1. Finally, by using Lemma 3, we obtain a description of all functions monogenic in the domain D_ζ whose first component in decomposition (3) is the function U_1 in the form of the sum of functions (24) and (41).

The theorem is proved.

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