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CONDITIONS FOR THE STABILITY OF AN IMPULSIVE LINEAR EQUATION WITH PURE DELAY

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We establish necessary and sufficient conditions for the stability of one class of impulsive linear differential equations with delay.

Stability of solutions of impulsive differential equations, including periodic systems, was studied in numerous works (see, e.g., [1–3]). Systems of impulsive differential equations with delay were investigated in [4–6]. These investigations were based on the direct Lyapunov method combined with Razumikhin's concept. Of special importance is the problem of the construction of an analog of the Floquet theory for this class of differential equations. In the present paper, for a scalar equation with pure delay whose value coincides with the period of pulse action, under certain assumptions, we construct an analog of the monodromy operator in a functional space and establish necessary and sufficient conditions for the stability of a linear equation. The method is based on the comparison principle for discrete mappings [7]. The investigation of stability is reduced to the determination of real roots of a certain transcendental equation.

Consider the problem of the stability of the differential equation

$$\begin{aligned} \dot{x} &= bx(t-\theta), \quad t \neq k\theta, \\ x(t^+) &= cx(t), \qquad t = k\theta \qquad \forall\, k \in \mathbb{N}_0, \end{aligned} \tag{1}$$

where $bc \geq 0$ and $\theta > 0$, in the space of functions

$$X = C\,[0,\theta) \,\cap\, C^1\left(\bigcup_{k=1}^{\infty} \big(k\theta,\ (k+1)\,\theta\big)\right).$$

We have assumed here that $bc \geq 0$ to guarantee the solvability on the real axis for the transcendental equation introduced below.

Note that, for $b = 0$, this equation is trivial; this special case is considered in what follows. We now assume that $b \neq 0$.

Since $bc \geq 0$, the following two cases are possible:

(i) $b > 0$ and $c \geq 0$;

(ii) $b < 0$ and $c \leq 0$.

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In what follows, we restrict ourselves to case (i), addressing case (ii) only in remarks.

Denote $\Omega = (0, \theta)$. For Eq. (1), we formulate the initial conditions

$$x(t) = f(t), \qquad t \in \bar{\Omega}, \tag{2}$$

where $f(t)$ is a continuous function.

We take a sequence $\{\varphi_n\}_{n \in \mathbb{N}_0}$ of functions $\varphi_n \colon \bar{\Omega} \to \mathbb{R}$ and consider the problem

$$\frac{d\varphi_n(t)}{dt} = b\varphi_{n-1}(t), \quad t \in \bar{\Omega}, \quad n \in \mathbb{N}, \tag{3}$$

$$\varphi_n(-\theta) = c\varphi_{n-1}(0),$$

$$\varphi_0(t) = f(t), \quad t \in \bar{\Omega}, \tag{4}$$

where the functions φ_n, $n \in \mathbb{N}$, are continuously differentiable in the domain of definition (φ_0 is continuous).

Definition 1. *System (3) is called stable if, for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if*

$$\|\varphi_0(t)\|_{C(\bar{\Omega})} < \delta,$$

then

$$\|\varphi_n(t)\|_{C(\bar{\Omega})} < \varepsilon$$

uniformly in n.

Definition 2. *System (3) is called asymptotically stable if it is stable and* $\|\varphi_n(t)\|_{C(\bar{\Omega})} \to 0$ *as* $n \to \infty$.

It is easy to see that the solutions of problems (1), (2) and (3), (4) are related to one another as follows:

$$\varphi_n(t) = x(n\theta + t), \qquad t \in (0, \theta], \tag{5}$$

Therefore, conditions for stability and asymptotic stability of system (1) are equivalent to conditions for stability and asymptotic stability of system (3).

Definition 3. *Let* $\{\varphi_n\}$ *be a solution of (3). Then the operator* $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ *defined by the equality* $T\varphi_n = \varphi_{n+1}$ *for an arbitrary* $n \in \mathbb{N}$ *is called the monodromy operator for (3).*

It is obvious that this operator admits the representation

$$T\varphi_n(t) = c\varphi_n(0) + b\int_{-\theta}^{t} \varphi_n(s)\,ds. \tag{6}$$

It can be shown that the operator T is linear. Using the Banach–Steinhaus theorem and the definitions of monodromy operator and stabilities, one can easily verify that the stability of (3) is equivalent to the boundedness of the sequence $\left\| T^n \right\|_{C(\bar{\Omega})}$ (here, the norm of an operator is the ordinary operator norm generated by the norm $\|\cdot\|_{C(\bar{\Omega})}$), and the asymptotic stability is equivalent to the relation

$$\lim_{n \to \infty} \left\| T^n \right\|_{C(\bar{\Omega})} = 0.$$

The introduced monodromy operator has the form

$$T\varphi_n(t) = c\varphi_n(\theta) + b\int_0^t \varphi_n(s)\,ds. \tag{7}$$

Consider the problem of finding the general form of the expression $T^n 1$.

We investigate $T^n 1$ by taking first several values of n:

$$T^0 1 = 1 \quad \text{on} \quad \bar{\Omega},$$

$$T1 = c \cdot 1 + b\int_0^t 1\,ds = c + bt \quad \text{on} \quad \bar{\Omega},$$

$$T^2 1 = T\big(T(1)\big) = c(c + b\theta) + b\int_0^t (c + bt)\,ds = c^2 + cb\theta + b\left(ct + \frac{1}{2}bt^2\right) \quad \text{on} \quad \bar{\Omega}.$$

This implies that the general form of $T^n 1$ is $T^n 1 = P_n(t)$, where $P_n(t)$ is a polynomial of degree n.

It can be shown that

$$Tt^k = c\theta^k + \frac{1}{k+1}bt^{k+1}.$$

Let b_n denote the free term of the polynomial $P_n(t)$ for an arbitrary $n \in \mathbb{N}_0$. For the polynomial $P_n(t)$, we obtain the representation

$$P_n(t) = b_0 \frac{b^n}{n!}t^n + b_1 \frac{b^{n-1}}{(n-1)!}t^{n-1} + b_2 \frac{b^{n-2}}{(n-2)!}t^{n-2} + \ldots + b_{n-1}bt + b_n. \tag{8}$$

In this case, we have

$$P_{n+1}(t) = TP_n(t) = b_0 \frac{b^{n+1}}{(n+1)!}t^{n+1} + b_1 \frac{b^n}{n!}t^n + b_2 \frac{b^{n-1}}{(n-1)!}t^{n-1} + \ldots b_n t + cP_n(\theta),$$

whence

$$b_{n+1} = cP_n(\theta) = c\left(b_n + \beta b_{n-1} + \frac{\beta^2}{2!}b_{n-2} + \ldots + \frac{\beta^n}{n!}b_0\right), \tag{9}$$

where $\beta = b\theta$.

Equality (9) is a recurrence relation. Using this relation, we can determine b_{n+1} for known $b_1, b_2, \ldots,$ b_n. Parallel with this relation, we also consider the relation

$$\tilde{b}_{n+1} = c\sum_{k=0}^{\infty}\frac{1}{k!}\tilde{b}_{n-k}\beta^k \tag{10}$$

for a certain sequence $\{\tilde{b}_n\}$. Since $b_0 = 1$, we set $\tilde{b}_0 = 1$.

We seek a solution of (10) in the form $\tilde{b}_n = q^n$ because, as $n \to +\infty$, any solution of system (10) can always be majorized by the solution proposed above multiplied by a certain constant. We are interested here in a sequence with maximum growth. Then relation (10) takes the form

$$q^{n+1} = c\sum_{k=0}^{\infty}q^{n-k}\frac{\beta^k}{k!},$$

or, on performing transformations and finding the sum of the series,

$$q = ce^{\beta/q}. \tag{11}$$

It can be shown that, for $b > 0$ and $c > 0$ (the case $c = 0$ corresponds to an equation with trivially stable zero solution), the transcendental equation (11) has a unique real root, which is positive. This follows from the fact that the function on the right-hand side of the equation takes only positive values and decreases monotonically on the right half axis from $+\infty$ to 1. By analogy, we establish that, for $b < 0$ and $c < 0$, this equation has a unique real solution, which is negative. Thus, we assume that q satisfies (11). Then relation (10) holds for $\tilde{b}_n = q^n$, $n \in \mathbb{Z}$.

For an arbitrary $n \in \mathbb{N}_0$, we denote

$$\theta_n = \frac{b_n}{\tilde{b}_n}. \tag{12}$$

One can verify that θ_n satisfies the relation

$$\theta_{n+1} = e^{-\frac{\beta}{q}}\left(\theta_n + \frac{\beta}{q}\theta_{n-1} + \frac{1}{2!}\left(\frac{\beta}{q}\right)^2\theta_{n-2} + \ldots + \frac{1}{n!}\left(\frac{\beta}{q}\right)^n\theta_0\right). \tag{13}$$

Denoting

$$\beta_1 = \frac{\beta}{q},$$

we obtain

$$\theta_{n+1} = e^{-\beta_1}\left(\theta_n + \beta_1\theta_{n-1} + \frac{\beta_1^2}{2!}\theta_{n-2} + \ldots + \frac{\beta_1^n}{n!}\theta_0\right).$$

Performing the substitution $A_k = e^{-\beta_1(\beta^k/k!)}$, we get

$$\theta_{n+1} = \sum_{k=0}^{n} A_k\theta_{n-k}. \tag{14}$$

Denote

$$S_n = \sum_{k=0}^{n} A_k$$

and $r_n = 1 - S_n$.

Lemma 1. *Let a sequence* θ_n *be defined by (12). Then there exists* θ^* *such that* $\theta^* \le \theta_n \le 1$ *uniformly in* n.

Proof. Thus, $\theta_{n+1} = \nu_n(1 - r_n)$, where ν_n is a "weighted mean":

$$\nu_n = \frac{\sum_{k=0}^{n} A_k\theta_{n-k}}{\sum_{k=0}^{n} A_k}.$$

First, we show that the second inequality in the statement of the lemma is true. We prove it by the method of mathematical induction.

For $n = 0$, we have $\theta_0 \le 1$. Assuming that the required inequality holds for an arbitrary $k \le n$ $(\theta_k \le 1)$, we prove it for $n + 1$. Indeed,

$$\theta_{n+1} = \nu_n(1 - r_n) \le \nu_n \le \max_k\{\theta_k\} \le 1,$$

which was to be proved.

Now consider the first inequality. Consider one more inequality defined by the recurrence relation

$$\tilde{\theta}_{n+1} = \tilde{\theta}_l(1 - r_n),$$

where l is such that $\tilde{\theta}_l < \tilde{\theta}_k$ for any $k \le n$, $k \ne l$, and $\tilde{\theta}_0 = 1$. Using the method of mathematical induction, one can easily prove that $l = n$, i.e., $\{\tilde{\theta}_n\}$ decreases monotonically because $1 - \tilde{r}_n < 1$.

Using the method of mathematical induction, we show that $\theta_n \ge \tilde{\theta}_n$ for an arbitrary n.

For $n = 0$, we have $\theta_0 \ge \tilde{\theta}_0$. Assuming that the required inequality holds for $l \le n$ (i.e., $\theta_l \ge \tilde{\theta}_l$), we prove that $\theta_{n+1} \ge \tilde{\theta}_{n+1}$. Indeed,

$$\theta_{n+1} = \nu_n(1 - r_n) \ge \min_i \{\theta_i\}(1 - r_n) \ge \min_i \{\tilde{\theta}_i\}(1 - r_n) = \tilde{\theta}_n(1 - r_n) = \tilde{\theta}_{n+1},$$

which was to be proved.

Let us show that $\tilde{\theta}_n$ is bounded from below by a positive number. Indeed,

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n(1 - r_n) = \tilde{\theta}_{n-1}(1 - r_n)(1 - r_{n-1}) = \ldots = \tilde{\theta}_0 \sum_{k=0}^{n} (1 - r_k).$$

Thus, the problem of the boundedness of $\tilde{\theta}_n$ is equivalent to the problem of the convergence (to a nonzero value) of the product $\prod_{n=0}^{\infty}(1 - r_n)$. This product converges if and only if there exists the sum of the series $\sum_{n=0}^{\infty} r_n$ [8]. However,

$$r_n = \sum_{k=n+1}^{\infty} \beta_1^k \frac{1}{k!}.$$

We take the minimal $l \ge n$ for which $l > \beta_1$ and obtain

$$\sum_{k=n+1}^{\infty} \beta_1^k \frac{1}{k!} = \sum_{k=n+1}^{l} \beta_1^k \frac{1}{k!} + \sum_{k=l+1}^{\infty} \beta_1^k \frac{1}{k!}$$

$$= \sum_{k=n+1}^{l} \beta_1^k \frac{1}{k!} + \frac{\beta_1^l}{l!} \sum_{k=l+1}^{\infty} \frac{\beta_1^{k-l}}{(l+1)(l+2)\ldots k}$$

$$< \sum_{k=n+1}^{l} \beta_1^k \frac{1}{k!} + \frac{\beta_1^l}{l!} \sum_{k=l+1}^{\infty} \frac{\beta_1^{k-l}}{(l+1)^{k-l}} = \sum_{k=n+1}^{l} \frac{\beta_1^k}{k!} + \frac{\beta_1^l}{l!} \frac{\beta_1}{l+1} \frac{1}{1 - \frac{\beta_1}{l+1}}$$

$$= \sum_{k=n+1}^{l} \frac{\beta_1^k}{k!} + \frac{\beta_1^{l+1}}{l!} \frac{1}{l + 1 - \beta_1}.$$

We now eliminate r_n for which $l > n$. This does not affect the convergence of the series. Now let $n = l$. Then

$$r_n < \frac{\beta_1^{n+1}}{n!}\frac{1}{n+1-\beta_1} < \beta_1\frac{\beta_1^n}{n!} = \beta_1\frac{\beta_1^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\delta_n}},$$

where $\delta_n \in \left(0, \frac{1}{12n}\right)$ [9]. Further, we get

$$\beta_1\frac{\beta_1^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\delta_n}} = \beta_1\frac{1}{\sqrt{2\pi n}\left(\frac{n}{\beta_1 e}\right)^n e^{\delta_n}} < \beta_1\frac{1}{\left(\frac{n}{\beta_1 e}\right)^n} < \frac{\beta_1}{2^n}$$

if we take n such that

$$\frac{n}{\beta_1 e} > 2\,.$$

However,

$$\sum_{n=1}^{\infty}\frac{\beta_1}{2^n} = \beta_1\,.$$

Therefore, the series $\sum_{n=0}^{\infty} r_n$ is convergent, and the product $\prod_{n=0}^{\infty}(1-r_n)$ is also convergent. Consequently, there exists θ^* such that $\theta_n > \theta^*$ for all $n \in \mathbb{N}$.

Lemma 1 is proved.

Lemma 1 yields the following obvious corollary:

Corollary 1. *Let b_n be the sequence of free terms of $P_n(t) = T^n(1)$, where T is the monodromy operator for (4) for $b > 0$ and $c \geq 0$, and let q be a solution of the equation $q = ce^{\beta/q}$. Then the following assertions are true:*

(i) if $q < 1$, then $b_n \to 0$ as $n \to \infty$;

(ii) if $q = 1$, then there exist n^*, c_1, and c_2, $0 < c_1 < c_2$, such that $c_1 < b_n < c_2$ for all $n > n^*$;

(iii) if $q > 1$, then $b_n \to \infty$ as $n \to \infty$.

Lemma 2. *Suppose that $P_n(t) = T^n(1)$, where T is the monodromy operator for (3), $b > 0$, $c \geq 0$, and $b_n > 0$ are the free terms of $P_n(t)$. Then the following assertions are true:*

(i) if $b_n \to \infty$ *as* $n \to \infty$, *then* $\| P_n \|_{C(\bar{\Omega})} \to \infty$ *as* $n \to \infty$;

(ii) if there exists $n^* \in \mathbb{N}$ *such that* $0 < c_1 < b_n < c_2$ *for all* $n > n^*$, *then there exists* n^{**} *such that* $\gamma_1 < \| P_n \|_{C(\bar{\Omega})} < \gamma_2$ *for all* $n > n^{**}$;

(iii) if $b_n \to 0$ *as* $n \to \infty$, *then* $\| P_n \|_{C(\bar{\Omega})} \to 0$ *as* $n \to \infty$.

Proof. 1. Assume that $b_n \to \infty$. However,

$$\| P_n \|_{C(\bar{\Omega})} = \left\| \sum_{k=0}^{n} b_k \frac{b^{n-k}}{(n-k)!} t^{n-k} \right\|_{C(\bar{\Omega})} \geq |P_n(0)| = b_n .$$

Therefore, $\| P_n \|_{C(\bar{\Omega})} \to \infty$ as $n \to \infty$.

2. Assume that there exists $n^* \in \mathbb{N}$ such that $0 < c_1 < b_n < c_2$ for all $n > n^*$. Then

$$\| P_n \|_{C(\bar{\Omega}_1)} = \left\| \sum_{k=0}^{n} b_k \frac{b^{n-k}}{(n-k)!} t^{n-k} \right\|_{C(\bar{\Omega}_1)} \geq |P_n(0)| = b_n > c_1 .$$

On the other hand,

$$\| P_n \|_{C(\bar{\Omega}_1)} = \left\| \sum_{k=0}^{n} b_k \frac{b^{n-k}}{(n-k)!} t^{n-k} \right\|_{C(\bar{\Omega}_1)} = \left| \sum_{k=0}^{n} b_k \frac{b^{n-k}}{(n-k)!} \theta^{n-k} \right|$$

$$= \left| \sum_{k=0}^{n} b_{n-k} \frac{(b\theta)^k}{k!} \right| = \frac{1}{c} b_{n+1} < \frac{1}{c} c_2 .$$

Thus,

$$c_1 < \| P_n \|_{C(\bar{\Omega}_1)} < \frac{c_2}{c} \qquad \text{for} \quad n > n^* .$$

3. Assume that $b_n \to 0$ as $n \to \infty$. Then

$$\| P_n \|_{C(\bar{\Omega}_1)} = \frac{b_{n+1}}{c} \to 0 \qquad \text{as} \quad n \to \infty.$$

Lemma 2 is proved.

Lemma 3. *Suppose that* T *is the monodromy operator for (3),* $b > 0$, *and* $c \geq 0$. *Then*

$$\left\| T^n f \right\|_{C(\bar{\Omega})} \leq \| f \|_{C(\bar{\Omega})} \left\| T^n (1) \right\|_{C(\bar{\Omega})} \quad \textit{for any} \quad n.$$

Proof. It suffices to show that

$$\left| T^n f(t) \right| \leq \| f \|_{C(\bar{\Omega})} \left| T^n (1)(t) \right|.$$

We prove this by the method of mathematical induction. For $n = 0$, this relation is true because

$$\left| T^0 f(t) \right| = | f(t) | \leq \| f \|_{C(\bar{\Omega})} = \| f \|_{C(\bar{\Omega})} \left| T^0 (1) \right|.$$

Assume that

$$\left| T^n f(t) \right| \leq \| f \|_{C(\bar{\Omega})} \left| T^n (1)(t) \right|.$$

We get

$$\left| T^{n+1} f(t) \right| = \left| T T^n f(t) \right| = \left| b T^n f(\theta) + c \int_0^t T^n f(s)\, ds \right|$$

$$\leq b \left| T^n f(\theta) \right| + c \left| \int_0^t T^n f(s)\, ds \right|$$

$$\leq b \left| T^n (1)(\theta) \right| \| f \|_{C(\bar{\Omega})} + c \left| \int_0^t T^n (1)(s)\, ds \right| \| f \|_{C(\bar{\Omega})}$$

$$= \| f \|_{C(\bar{\Omega})} \left| b T^n (1)(\theta) + c \int_0^t T^n (1)(s)\, ds \right| = \| f \|_{C(\bar{\Omega})} \left| T^{n+1} (1)(t) \right|.$$

The lemma is proved.

Thus, we have established that if q is a solution of the equation $q = c e^{\beta\theta/q}$, then the behavior of the sequence b_n is determined by the location of q with respect to 1 (Corollary 1), the behavior of $\left\| T^n 1 \right\|_{C(\bar{\Omega})}$ is determined by the behavior of the sequence b_n (Lemma 2), and the behavior of $\left\| T^n f \right\|_{C(\bar{\Omega})}$ is determined by the behavior of $\left\| T^n 1 \right\|_{C(\bar{\Omega})}$. Therefore, we can formulate the following statement:

Corollary 2. *Let q be a solution of the equation $q = ce^{\beta\theta/q}$ and let $b > 0$ and $c \geq 0$ in system (3). Then the following assertions are true:*

(i) if $q < 1$, then Eq. (1) is asymptotically stable;

(ii) if $q = 1$, then Eq. (1) is stable;

(iii) if $q > 1$, then Eq. (1) is unstable.

Note that if $b = 0$, then the stability of (3) is determined by the location of the modulus of the parameter c with respect to 1 because a solution of this system admits the analytic representation

$$\varphi_n(t) = c^n f(\theta), \quad t \in \bar{\Omega}, \quad n \in \mathbb{N}.$$

In the case where $b < 0$ and $c \leq 0$, a solution of problem (3), (4) can be represented in the form $\varphi_n = (-1)^n \tilde{\varphi}_n$, where $\tilde{\varphi}_n$ is a solution of problem (3), (4) in which the coefficients b and c are replaced by their moduli. Therefore, the problems of stability of φ_n and $\tilde{\varphi}_n$ are equivalent.

In this connection, taking into account that a solution of system (1) is associated with the solution of system (3) given by (5), we can formulate the following theorem:

Theorem 1. *Let q be a solution of the equation $q = ce^{\beta\theta/q}$ and let system (1) be such that $bc \geq 0$. Then the following assertions are true:*

(i) if $|q| < 1$, then system (1) is asymptotically stable;

(ii) if $|q| = 1$, then system (1) is stable;

(iii) if $|q| > 1$, then system (1) is unstable.

Thus, the problem of the stability of solutions of the considered equation reduces to the determination of the location of the modulus of a solution of the transcendental equation (1) with respect to 1.

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