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ON SOME EXTREMAL PROBLEMS OF DIFFERENT METRICS FOR DIFFERENTIABLE FUNCTIONS ON THE AXIS

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For an arbitrary fixed segment $[\alpha, \beta] \subset \mathbf{R}$ and given $r \in \mathbf{N}$, A_r, A_0, and $p > 0$, we solve the extremal problem

$$\int_\alpha^\beta \left| x^{(k)}(t) \right|^q dt \to \sup, \quad q \geq p, \quad k = 0, \quad q \geq 1, \quad 1 \leq k \leq r-1,$$

on the set of all functions $x \in L^r_\infty$ such that $\left\| x^{(r)} \right\|_\infty \leq A_r$ and $L(x)_p \leq A_0$, where

$$L(x)_p := \left\{ \left(\int_a^b |x(t)|^p dt \right)^{1/p} : a, b \in \mathbf{R}, |x(t)| > 0, \quad t \in (a, b) \right\}.$$

In the case where $p = \infty$ and $k \geq 1$, this problem was solved earlier by Bojanov and Naidenov.

1. Introduction

Let G denote either the real axis $\mathbf{R}$ or a finite segment $[\alpha, \beta] \subset \mathbf{R}$. We consider the spaces $L_p(G)$, $0 < p \leq \infty$, of all measurable functions $x: G \to \mathbf{R}$ such that $\| x \|_{L_p(G)} < \infty$, where

$$\| x \|_{L_p(G)} := \begin{cases} \left(\int_G |x(t)|^p dt \right)^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname{vrai\,sup}_{t \in G} |x(t)| & \text{if } p = \infty. \end{cases}$$

For $r \in \mathbf{N}$ and $p, s \in (0, \infty]$, we denote by $L^r_{p,s}$ the space of all functions $x: \mathbf{R} \to \mathbf{R}$ for which $x^{(r-1)}$ is locally absolutely continuous, $x \in L_p(\mathbf{R})$, and $x^{(r)} \in L_s(\mathbf{R})$. We write $\| x \|_p$ instead of $\| x \|_{L_p(\mathbf{R})}$ and L^r_∞ instead of $L^r_{\infty,\infty}$.

In the present paper, we study some modifications of the known extremal problem

$$\left\| x^{(k)} \right\|_q \to \sup, \quad 0 \leq k \leq r-1, \quad q \geq 1, \tag{1}$$

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on the set of all functions $x \in L^r_{p,s}$ such that

$$\left\| x^{(r)} \right\|_s \leq A_r, \qquad \| x \|_p \leq A_0. \tag{2}$$

It is known (see, e.g., [1, p. 47]) that problem (1) is equivalent to the determination of the exact constant C in the Kolmogorov–Nagy-type inequality

$$\left\| x^{(k)} \right\|_q \leq C \| x \|_p^{\alpha} \left\| x^{(r)} \right\|_s^{1-\alpha}, \qquad 0 \leq k \leq r-1, \tag{3}$$

for functions $x \in L^r_{p,s}$, where

$$\alpha = \frac{r - k + 1/q - 1/s}{r + 1/p - 1/s}.$$

Only in some cases is the exact constant in inequality (3) known for all $r \in \mathbf{N}$. For a detailed description of the cases in which the exact constant in inequality (3) is known, see [1–3].

For an arbitrary segment $[\alpha, \beta] \subset \mathbf{R}$, Bojanov and Naidenov [4] solved the problem

$$\int_\alpha^\beta \Phi\left(\left| x^{(k)}(t) \right| \right) dt \to \sup, \qquad 1 \leq k \leq r-1,$$

on the set of all functions $x \in L^r_\infty$ that satisfy (2) with $p = s = \infty$; here, the function Φ is continuously differentiable on $[0, \infty)$, positive on $(0, \infty)$, and such that $\Phi(t)/t$ does not decrease and $\Phi(0) = 0$.

Consider the class W of functions Φ continuous, nonnegative, and convex on $[0, \infty)$ and such that $\Phi(0) = 0$. For $p > 0$, we set

$$L(x)_p := \sup \left\{ \| x \|_{L_p[a,b]} : a, b \in \mathbf{R}, | x(t) | > 0, t \in (a, b) \right\}.$$

Functionals of this type were studied in [5]. Note that $L(x)_\infty = \| x \|_\infty$ and $L(x')_1 \leq 2 \| x \|_\infty$.

In the present paper, we solve the modifications of the Bojanov–Naidenov problem

$$\int_\alpha^\beta \Phi\left(| x(t) |^p \right) dt \to \sup, \qquad \Phi \in W, \quad p > 0,$$

and

$$\int_\alpha^\beta \Phi\left(\left| x^{(k)}(t) \right| \right) dt \to \sup, \qquad \Phi \in W, \quad 1 \leq k \leq r-1,$$

on the class of all functions $x \in L^r_\infty$ that satisfy the conditions

$$\left\| x^{(r)} \right\|_\infty \le A_r, \qquad L(x)_p \le A_0$$

instead of conditions (2) with $s = p = \infty$.

2. Auxiliary Statements

Let $\varphi_r(t)$, $r \in \mathbf{N}$, denote the rth 2π-periodic integral of the function $\varphi_0(t) = \operatorname{sgn} \sin t$ with mean value zero on a period. For $\lambda > 0$, we set $\varphi_{\lambda,r}(t) := \lambda^{-r} \varphi_r(\lambda t)$. Denote a rearrangement of the function $|x|$, $x \in L_1[a, b]$, by $r(x, t)$ (see, e.g., [6], Sec. 1.3). We set $r(x, t) = 0$ for $t \ge b - a$.

Note that if a function $x \in L^r_\infty$ satisfies the condition $L(x)_p < \infty$ for a certain $p > 0$ and $|x(t)| > 0$, $t \in (a, b)$, where $a = -\infty$ or $b = +\infty$, then $x(t) \to 0$ as $t \to -\infty$ or $t \to +\infty$. In this case, we assume that $x(-\infty) = 0$ and $x(+\infty) = 0$.

Lemma 1. *Let* $r \in \mathbf{N}$, *let* $A_r, p > 0$, *and let an interval* (a, b), $-\infty \le a < b \le \infty$, *and a function* $x \in L^r_\infty$ *be such that*

$$\left\| x^{(r)} \right\|_\infty \le A_r, \qquad L(x)_p \le \infty,$$

$$x(a) = x(b) = 0, \qquad |x(t)| > 0, \qquad t \in (a, b).$$

Also assume that $\lambda > 0$ *satisfies the condition*

$$L(x)_p \le A_r L(\varphi_{\lambda,r})_p. \tag{4}$$

Then, for any function $\Phi \in W$ *and any measurable set* $E \subset (a, b)$, $\mu E \le \pi/\lambda$, *the following inequalities are true:*

$$\int_a^b \Phi\left(|x(t)|^p\right) dt \le \int_0^{\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt \tag{5}$$

and

$$\int_E \Phi\left(|x(t)|^p\right) dt \le \int_{m-\Theta}^{m+\Theta} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt, \qquad \Theta = \frac{\mu E}{2}, \tag{6}$$

where m *is a point of local maximum of the spline* $\varphi_{\lambda,r}$.

Furthermore, if $-\infty < a < b < \infty$, *then*

$$\frac{1}{b-a} \int_a^b \Phi\left(|x(t)|^p\right) dt \le \frac{\lambda}{\pi} \int_0^{\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt. \tag{7}$$

Proof. We fix an arbitrary function $x \in L_\infty^r$ and an interval (a, b) that satisfy the conditions of Lemma 1. First, we prove the inequality

$$\|x\|_\infty \le A_r \left\|\varphi_{\lambda,r}\right\|_\infty. \tag{8}$$

Assume that (8) is not true. Then there exists $\omega < \lambda$ such that

$$\|x\|_\infty = A_r \left\|\varphi_{\omega,r}\right\|_\infty. \tag{9}$$

Assume that $t_0 \in \mathbf{R}$ satisfies the condition

$$\left\|\varphi_{\omega,r}\right\|_\infty = \varphi_{\omega,r}(t_0) \tag{10}$$

and c is the maximum zero of the spline $\varphi_{\omega,r}$ such that $c < t_0$. We fix an arbitrary $\varepsilon > 0$. There exists a point $t_\varepsilon \in (c, t_0)$ for which $\varphi_{\omega,r}(t_\varepsilon) = \left\|\varphi_{\omega,r}\right\|_\infty - \varepsilon$. We set $\delta := t_0 - t_\varepsilon$. It is clear that $\delta \to 0$ as $\varepsilon \to 0$. For sufficiently small $\varepsilon > 0$, we define a function $\psi_\varepsilon(t)$ on $[c, c + \pi/\omega]$ as follows:

$$\psi_\varepsilon(t) := \begin{cases} \varphi_{\omega,r}(t-\delta) & \text{if } t \in [c+\delta, t_0], \\ \varphi_{\omega,r}(t+\delta) & \text{if } t \in [t_0, c+\pi/\omega-\delta], \\ 0 & \text{if } t \in [c, c+\delta] \cup [c+\pi/\omega-\delta, c+\pi/\omega]. \end{cases}$$

It is obvious that $\psi_\varepsilon(t_0) = \left\|\varphi_{\omega,r}\right\|_\infty - \varepsilon$ and $\psi_\varepsilon(t) \to \varphi_{\omega,r}(t)$, $t \in \left[c, c+\pi/\omega\right]$, as $\varepsilon \to 0$. Since $L(x)_p < \infty$, it follows from (9) and (10) that there exists a shift $x_\varepsilon(t) := x(t + \tau_\varepsilon)$ such that $x_\varepsilon'(t_0) = 0$ and

$$|x_\varepsilon(t_0)| \ge A_r\left(\left\|\varphi_{\omega,r}\right\|_\infty - \varepsilon\right) = A_r \psi_\varepsilon(t_0). \tag{11}$$

Note that, by virtue of (9), the function x satisfies the conditions of the Kolmogorov comparison theorem [7]. According to this theorem [7], one has

$$|x_\varepsilon(t)| \ge A_r \psi_\varepsilon(t), \qquad t \in [c+\delta, c+\pi/\omega-\delta].$$

Therefore,

$$L(x)_p = L(x_\varepsilon)_p \ge A_r \|\psi_\varepsilon\|_{L_p[c+\delta, c+\pi/\omega-\delta]}.$$

Passing to the limit as ε tends to zero, we get

$$L(x)_p \ge A_r L(\varphi_{\omega,r})_p > A_r L(\varphi_{\lambda,r})_p,$$

which contradicts condition (4). Thus, inequality (8) is proved.

We now prove inequality (5). Let $\bar{x}$ denote the restriction of the function x to (a, b) and let $\bar{\varphi}$ denote the restriction of the spline $A_r \varphi_{\lambda,r}$ to $[c, c + \pi/\lambda]$, where c is a zero of the spline $\varphi_{\lambda,r}$. By virtue of the Hardy–Littlewood theorem (see, e.g., [6], Proposition 1.3.11), to prove (5) it suffices to show that

$$\int_0^\xi r\left(|\bar{x}|^p, t\right) dt \;\le\; \int_0^\xi r\left(\left|\overline{\varphi_{\lambda,r}}\right|^p, t\right) dt, \qquad \xi > 0. \tag{12}$$

By virtue of relation (8) and the condition $x(a) = x(b) = 0$ of Lemma 1, for any $z \in \left(0, \|\bar{x}\|_{L_\infty(a,b)}\right)$ there exist points $t_i \in (a, b)$, $i = 1, \ldots, m$, $m \ge 2$, and two points $y_j \in (c, c + \pi/\lambda)$ such that

$$z = |\bar{x}(t_i)| = \left|\bar{\varphi}(y_j)\right|. \tag{13}$$

According to the Kolmogorov comparison theorem, we have

$$|\bar{x}'(t_i)| \le \left|\bar{\varphi}'(y_j)\right|. \tag{14}$$

Therefore, if points θ_1 and θ_2 are chosen so that

$$z = r(\bar{x}, \theta_1) = r(\bar{\varphi}, \theta_2),$$

then, according to the theorem on the derivative of a rearrangement (see, e.g., [6], Proposition 1.3.2), we get

$$|r'(\bar{x}, \theta_1)| = \left[\sum_{i=1}^m |\bar{x}'(t_i)|^{-1}\right]^{-1} \le \left[\sum_{j=1}^2 \left|\bar{\varphi}'(y_j)\right|^{-1}\right]^{-1} = |r'(\bar{\varphi}, \theta_2)|.$$

This implies that the difference $\Delta(t) := r(\bar{x}, t) - r(\bar{\varphi}, t)$ changes its sign (from "–" to "+") at most once. The same is true for the difference $\Delta_p(t) := r^p(\bar{x}, t) - r^p(\bar{\varphi}, t)$.

Consider the integral

$$I(\xi) := \int_0^\xi \Delta_p(t)\, dt.$$

It is clear that $I(0) = 0$. We set $M := \max\{b - a, \pi/\lambda\}$. Then, by virtue of (4), for any $\xi \ge M$, we have

$$I(\xi) = L(x)_p^p - A_r^p\, L(\varphi_{\lambda,r})_p^p \le 0.$$

Furthermore, the derivative $I'(t) = \Delta_p(t)$ changes its sign (from "–" to "+") at most once. Hence, $I(\xi) \le 0$ for all $\xi \ge 0$. Thus, inequalities (12) and (5) are proved.

We now prove (6). Note that $|\overline{\varphi}|$ is a comparison function for $|\overline{x}|$, i.e., relation (14) follows from (13). The proof of (5) was based on exactly this fact and inequality (4). Therefore, using (5) instead of (4), we can prove by analogy that

$$\int_0^{\xi} r\big(\Phi(|\overline{x}|^p, t)\big)\,dt \;\le\; \int_0^{\xi} r\big(\Phi(|\overline{\varphi}|^p, t)\big)\,dt, \qquad \xi > 0.$$

This yields

$$\int_E \Phi\big(|x(t)|^p\big)\,dt \;\le\; \int_0^{\mu E} r\big(\Phi(|\overline{x}|^p, t)\big)\,dt \;\le\; \int_0^{\mu E} r\big(\Phi(|\overline{\varphi}|^p, t)\big)\,dt \;=\; \int_{m-\Theta}^{m+\Theta} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt,$$

which proves (6).

It remains to prove (7). Let $-\infty < a < b < +\infty$. We choose $d \in (a, b)$ so that

$$\int_a^d \Phi\big(|x(t)|^p\big)\,dt \;=\; \int_d^b \Phi\big(|x(t)|^p\big)\,dt \;:=\; I.$$

By virtue of (5), there exists $y \in [0, \pi/(2\lambda)]$ for which

$$I \;=\; \int_c^{c+y} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt,$$

where c is a zero of the spline $\varphi_{\lambda,r}$. By virtue of the Kolmogorov comparison theorem, we obtain $d - a \ge y$ and $b - d \ge y$. Therefore,

$$\begin{aligned}
\int_a^b \Phi\big(|x(t)|^p\big)\,dt \;&=\; \int_a^d \Phi\big(|x(t)|^p\big)\,dt \;+\; \int_d^b \Phi\big(|x(t)|^p\big)\,dt \\
&\le\; \frac{d-a}{y}\int_c^{c+y} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt \;+\; \frac{b-d}{y}\int_c^{c+y} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt \\
&=\; (b-a)\,\frac{1}{y}\int_c^{c+y} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt.
\end{aligned}$$

It is easy to see that the function

$$\frac{1}{y}\int_c^{c+y} \Phi\Big(\big|A_r\,\varphi_{\lambda,r}(t)\big|^p\Big)\,dt$$

does not decrease on $[0, \pi/(2\lambda)]$. Therefore,

$$\int_a^b \Phi\left(|x(t)|^p\right)dt \;\le\; (b-a)\frac{2\lambda}{\pi}\int_c^{c+\pi/(2\lambda)} \Phi\left(\left|A_r\,\varphi_{\lambda,r}(t)\right|^p\right)dt \;=\; (b-a)\frac{\lambda}{\pi}\int_c^{c+\pi/\lambda} \Phi\left(\left|A_r\,\varphi_{\lambda,r}(t)\right|^p\right)dt,$$

which is equivalent to (7).

Lemma 1 proved.

Setting $\Phi(t) = t^{q/p}$, $q \ge p$, we obtain the following corollary:

Corollary 1. *Under the conditions of Lemma 1, one has*

$$L(x)_q \;\le\; A_r\,L(\varphi_{\lambda,r})_q, \quad q \ge p.$$

In particular,

$$\|x\|_\infty \;\le\; A_r\left\|\varphi_{\lambda,r}\right\|_\infty.$$

Lemma 2. *Let* $r \in \mathbf{N}$ *and* $A_r,\ p > 0$. *Suppose that a function* $x \in L^r_\infty$ *has zeros and satisfies the condition*

$$\left\|x^{(r)}\right\|_\infty \;\le\; A_r, \qquad L(x)_p < \infty,$$

and $\lambda > 0$ *is chosen so that*

$$L(x)_p \;\le\; A_r\,L(\varphi_{\lambda,r})_p.$$

If t_0 *is a zero of the function* x *and* c *is a zero of the spline* $\varphi_{\lambda,r}$, *then, for an arbitrary function* $\Phi \in W$ *and any* $\xi \in (0, \pi/\lambda]$, *one has*

$$\int_{t_0}^{t_0+\xi} \Phi\left(|x(t)|^p\right)dt \;\le\; \int_c^{c+\xi} \Phi\left(\left|A_r\,\varphi_{\lambda,r}(t)\right|^p\right)dt \tag{15}$$

and

$$\int_{t_0-\xi}^{t_0} \Phi\left(|x(t)|^p\right)dt \;\le\; \int_{c-\xi}^{c} \Phi\left(\left|A_r\,\varphi_{\lambda,r}(t)\right|^p\right)dt.$$

Proof. Passing to the shift $x(\cdot + \tau)$ if necessary, we can assume that $t_0 = c$.

Let us prove inequality (15) (the second inequality of Lemma 2 is proved by analogy). We set $\overline{\varphi}(t) := A_r\,\varphi_{\lambda,r}(t)$. In the proof of Lemma 1, we have established that the spline $\overline{\varphi}$ is a comparison function for the function x, i.e., the inequality $|x(\xi)| = |\overline{\varphi}(\eta)|$ implies that $|x'(\xi)| \le |\overline{\varphi}'(\eta)|$. Therefore, $|x(t)| \le |\overline{\varphi}(t)|$, $t \in (c, c + \pi/(2\lambda))$. If the last inequality holds for all $t \in (c, c+\xi)$, then inequality (15) is obvious. For this reason, we can assume that the difference $\Delta(t) := |x(t)| - |\overline{\varphi}(t)|$ changes its sign on $(c, c+\xi)$. Moreover, it has at most one change of sign (from "–" to "+") on $(c, c+\pi/\lambda)$ because $\overline{\varphi}$ is a comparison function for x. The same is true for the difference

$$\Delta_\Phi(t) := \Phi\left(|x(t)|^p\right) - \Phi\left(|\overline{\varphi}(t)|^p\right).$$

Let a point $d \in (c, c+\pi/\lambda)$ be such that $\Delta(t) \le 0$, $t \in (c, d)$, and $\Delta(t) \ge 0$, $t \in (d, c+\pi/\lambda)$. Then $\Delta_\Phi(t) \le 0$, $t \in (c, d)$, and $\Delta_\Phi(t) \ge 0$, $t \in (d, c+\pi/\lambda)$.

Consider the following two cases:

(i) $|x(t)| > 0$, $t \in (c, c+\xi)$;

(ii) $x(t)$ has a zero on $(c, c+\xi)$.

We set

$$I_\Phi(t) := \int_c^{c+t} \Delta_\Phi(u)\, du .$$

Let us prove the inequality $I_\Phi(t) \le 0, t \in (0, \pi/\lambda)$, which is equivalent to (15).

First, assume that $|x(t)| > 0$, $t \in (c, c+\xi)$. By assumption, we have $d < c+\xi$. Hence, $|x(t)| \ge |\overline{\varphi}(t)| > 0$, $t \in (d, c+\pi/\lambda)$, and, therefore, $|x(t)| > 0$, $t \in (c, c+\pi/\lambda)$. Then, according to inequality (5), $I_\Phi(\pi/\lambda) \le 0$. Furthermore, $I_\Phi(0) = 0$, and the derivative $I'_\Phi(t) = \Delta_\Phi(c+t)$ changes its sign (from "–" to "+") at most once on $(0, \pi/\lambda)$. Thus, $I_\Phi(t) \le 0$, $t \in (0, \pi/\lambda)$.

Now assume that $x(t)$ has a zero on $(c, c+\xi)$. We set $c_1 := \sup\{t \in (c, c+\pi/\lambda) : x(t) = 0\}$. It is clear that $x(c_1) = 0$ and $|x(t)| \le |\overline{\varphi}(t)|$, $t \in (c, c_1)$. Therefore,

$$\int_c^\gamma \Phi\left(|x(t)|^p\right) dt \le \int_c^\gamma \Phi\left(|\overline{\varphi}(t)|^p\right) dt, \qquad \gamma \in [c, c_1]. \tag{16}$$

If $c+\xi \le c_1$, then relation (16) follows from (15). Now let $c_1 < c+\xi$. Then $|x(t)| > 0$, $t \in (c_1, c+\pi/\lambda)$. In this case, inequality (15) is already proved. Assuming that $t_0 := c_1$ and using (15) with $c+\xi-c_1$ instead of ξ, we obtain

$$\int_{c_1}^{c+\xi} \Phi\left(|x(t)|^p\right) dt \le \int_c^{2c+\xi-c_1} \Phi\left(|\overline{\varphi}(t)|^p\right) dt \le \int_{c_1}^{c+\xi} \Phi\left(|\overline{\varphi}(t)|^p\right) dt. \tag{17}$$

The last inequality follows from the obvious relation

$$\inf_{a\in(c,\pi/\lambda-\delta)} \int_a^{a+\delta} \Phi\left(|\bar{\varphi}(t)|^p\right)dt = \int_c^{c+\delta} \Phi\left(|\bar{\varphi}(t)|^p\right)dt, \qquad \delta \le \frac{\pi}{\lambda}.$$

Adding (17) and (16) with $\gamma = c_1$ together, we get (15).

Lemma 2 is proved.

Lemma 3. *Suppose that* $r \in \mathbf{N}$, $A_r, p > 0$, *and a function* $x \in L_\infty^r$ *are such that*

$$\left\| x^{(r)} \right\|_\infty \le A_r, \qquad L(x)_p < \infty,$$

and $\lambda > 0$ *satisfies the condition*

$$L(x)_p \le A_r L(\varphi_{\lambda,r})_p.$$

Then, for any function $\Phi \in W$ *and an arbitrary segment* $[a, b] \subset \mathbf{R}$, $b - a \le \pi/\lambda$, *the following inequality is true:*

$$\int_a^b \Phi\left(|x(t)|^p\right)dt \le \int_{m-\Theta}^{m+\Theta} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt, \qquad \Theta = \frac{b-a}{2}, \tag{18}$$

where m *is a point of a local maximum of the spline* $\varphi_{\lambda,r}$. *In particular,*

$$\int_a^b \Phi\left(|x(t)|^p\right)dt \le \int_0^{\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt.$$

Proof. If $|x(t)| > 0$ for $t \in (a, b)$, then inequality (6) yields (18). Assume that $x(t)$ has a zero $t_0 \in (a, b)$. Then, according to Lemma 2,

$$\int_a^{t_0} \Phi\left(|x(t)|^p\right)dt \le \int_{c+\pi/\lambda-(t_0-a)}^{c+\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt \tag{19}$$

and

$$\int_{t_0}^b \Phi\left(|x(t)|^p\right)dt \le \int_c^{c+b-t_0} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt, \tag{20}$$

where c is a zero of the spline $\varphi_{\lambda,r}$. Adding (19) and (20) together, we obtain (18) because

$$\sup_{\mu E=\delta} \int_E \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt = \int_{m-\delta/2}^{m+\delta/2} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right)dt, \qquad \delta \le \frac{\pi}{\lambda}.$$

Lemma 3 is proved.

3. Main Results

We fix an arbitrary segment $[\alpha, \beta] \subset \mathbf{R}$, $r \in \mathbf{N}$, and A_r, $A_0, p > 0$. Recall the structure of the extremal function $\varphi_{[\alpha,\beta],r}$ in the Bojanov–Naidenov problem [4]. To this end, we first choose $\lambda > 0$ that satisfies the equality

$$A_0 = A_r L(\varphi_{\lambda,r})_p \tag{21}$$

and then represent the length of the segment $[\alpha, \beta]$ in the form

$$\beta - \alpha = n\frac{\pi}{\lambda} + 2\Theta, \qquad \Theta \in (0, \pi/(2\lambda)), \tag{22}$$

where $n \in \mathbf{N}$ or $n = 0$. We now set

$$\varphi_{[\alpha,\beta],r}(t) := A_r \varphi_{\lambda,r}(t + \tau), \tag{23}$$

where τ is chosen so that

$$\left|\varphi_{[\alpha,\beta],r}(\alpha + \Theta)\right| = \left|\varphi_{[\alpha,\beta],r}(\beta - \Theta)\right| = A_r \left\|\varphi_{\lambda,r}\right\|_\infty.$$

It is clear that $\varphi_{[\alpha,\beta],r} \in L^r_\infty$ and

$$\left\|\varphi^{(r)}_{[\alpha,\beta],r}\right\|_\infty = A_r, \qquad L\left(\varphi_{[\alpha,\beta],r}\right)_p = A_0.$$

Theorem 1. *Let* $r \in \mathbf{N}$, A_0, $A_r, p > 0$, $\Phi \in W$, *and* $[\alpha, \beta] \subset \mathbf{R}$. *Then*

$$\sup\left\{\int_\alpha^\beta \Phi\left(|x(t)|^p\right) dt : x \in L^r_\infty, \left\|x^{(r)}\right\|_\infty \le A_r, L(x)_p \le A_0\right\} = \int_\alpha^\beta \Phi\left(\left|\varphi_{[\alpha,\beta],r}(t)\right|^p\right) dt.$$

Proof. We fix an arbitrary function $x \in L^r_\infty$ such that

$$\left\|x^{(r)}\right\|_\infty \le A_r, \qquad L(x)_p \le A_0.$$

According to (21), we have

$$L(x)_p \le A_r L(\varphi_{\lambda,r})_p.$$

We set $a_k := \alpha + k\pi/\lambda$, $k = 0, 1, \ldots, n$. By virtue of Lemma 3, we get

$$\int_{a_k}^{a_k+1} \Phi\left(|x(t)|^p\right) dt \le \int_0^{\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt, \qquad k = 0, 1, \ldots, n-1,$$

and

$$\int_{a_n}^{\beta} \Phi\left(|x(t)|^p\right) dt \le \int_{m-\Theta}^{m+\Theta} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt,$$

where m is a point of a local maximum of the spline $\varphi_{\lambda,r}$ and Θ is defined by (22). Thus,

$$\int_{\alpha}^{\beta} \Phi\left(|x(t)|^p\right) dt \le n \int_0^{\pi/\lambda} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt + \int_{m-\Theta}^{m+\Theta} \Phi\left(\left|A_r \varphi_{\lambda,r}(t)\right|^p\right) dt = \int_{\alpha}^{\beta} \Phi\left(\left|\varphi_{[\alpha,\beta],r}(t)\right|^p\right) dt.$$

The equality here is realized for $x = \varphi_{[\alpha,\beta],r}$.

Theorem 1 is proved.

Let $q \ge p$. Setting $\Phi(t) = t^{q/p}$, we obtain the following corollary:

Corollary 2. *Under the conditions of Theorem 1, the following relation holds for any* $q \ge p > 0$:

$$\sup\left\{\int_{\alpha}^{\beta} |x(t)|^q\, dt : x \in L_\infty^r,\ \left\|x^{(r)}\right\|_\infty \le A_r,\ L(x)_p \le A_0\right\} = \int_{\alpha}^{\beta} \left|\varphi_{[\alpha,\beta],r}(t)\right|^q dt.$$

For $[\alpha, \beta] \subset \mathbf{R}$ and $k, r \in \mathbf{N}$, $k < r$, we consider the function

$$\varphi_{[\alpha,\beta],r,k}(t) := \varphi_{[\alpha,\beta],r}(t + \tau_k), \qquad \tau_k := \frac{\pi}{4\lambda}\left(1 + (-1)^{k+1}\right),$$

where $\varphi_{[\alpha,\beta],r}$ is defined by (23). It is clear that

$$\varphi^{(k)}_{[\alpha,\beta],r,k}(t) = \varphi_{[\alpha,\beta],r-k}(t).$$

Furthermore, $\varphi_{[\alpha,\beta],r,k} \in L_\infty^r$ and

$$\left\|\varphi^{(r)}_{[\alpha,\beta],r,k}\right\|_\infty = A_r, \qquad L\left(\varphi_{[\alpha,\beta],r,k}\right)_p = A_0.$$

Theorem 2. *Let* $k, r \in \mathbf{N}$, $k < r$, $A_0, A_r, p > 0$, $\Phi \in W$, *and* $[\alpha, \beta] \subset \mathbf{R}$. *Then*

$$\sup\left\{\int_{\alpha}^{\beta} \Phi\left(\left|x^{(k)}(t)\right|\right) dt : x \in L_\infty^r,\ \left\|x^{(r)}\right\|_\infty \le A_r,\ L(x)_p \le A_0\right\} = \int_{\alpha}^{\beta} \Phi\left(\left|\varphi^{(k)}_{[\alpha,\beta],r,k}(t)\right|\right) dt.$$

Proof. We fix an arbitrary function $x \in L_\infty^r$ such that

$$\left\| x^{(r)} \right\|_\infty \le A_r, \qquad L(x)_p \le A_0.$$

According to (21), we have

$$L(x)_p \le A_r L(\varphi_{\lambda,r})_p.$$

According to Corollary 1, we get

$$\| x \|_\infty \le A_r \left\| \varphi_{\lambda,r} \right\|_\infty.$$

By virtue of the Kolmogorov theorem, we obtain

$$\left\| x^{(i)} \right\|_\infty \le A_r \left\| \varphi_{\lambda,r-i} \right\|_\infty, \qquad i = 1, \ldots, r-1.$$

Therefore,

$$L\left(x^{(k)}\right)_1 \le 2 \left\| x^{(k-1)} \right\|_\infty \le 2 A_r \left\| \varphi_{\lambda,r+1-k} \right\|_\infty = A_r L(\varphi_{\lambda,r-k})_1.$$

Applying Theorem 1 with $p = 1$ to the function $x^{(k)} \in L_\infty^{r-k}$, we get

$$\int_\alpha^\beta \Phi\left(\left| x^{(k)}(t) \right|^p\right) dt \le \int_\alpha^\beta \Phi\left(\left| \varphi_{[\alpha,\beta],\, r-k}(t) \right|^p\right) dt = \int_\alpha^\beta \Phi\left(\left| \varphi_{[\alpha,\beta],\, r,\, k}^{(k)}(t) \right|^p\right) dt.$$

The equality here is realized for the function $x = \varphi_{[\alpha,\beta],r,k}$.

Remark 1. For $p = \infty$, Theorem 2 was proved by Bojanov and Naidenov in [4].

Corollary 3. *Under the conditions of Theorem 2, the following relation holds for any* $q \ge 1$ *and* $p > 0$:

$$\sup \left\{ \int_\alpha^\beta \left| x^{(k)}(t) \right|^q dt : x \in L_\infty^r, \left\| x^{(r)} \right\|_\infty \le A_r,\ L(x)_p \le A_0 \right\} = \int_\alpha^\beta \left| \varphi_{[\alpha,\beta],\, r,\, k}^{(k)}(t) \right|^q dt.$$

The theorem below specifies Theorems 1 and 2 for functions that have zeros and for periodic functions.

Theorem 3. *Let* $r \in \mathbf{N}$, $p > 0$, *and* $\Phi \in W$. *Then the following inequality holds for any segment* $[\alpha, \beta] \subset \mathbf{R}$ *and any function* $x \in L_\infty^r$ *such that* $L(x)_p < \infty$ *and* $x(\alpha) = x(\beta) = 0$:

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \Phi\left(| x(t) |^p\right) dt \le \frac{1}{\pi} \int_0^\pi \Phi\left(\left| \left(\frac{L(x)_p}{L(\varphi_r)_p}\right)^{\frac{r}{r+1/p}} \left\| x^{(r)} \right\|_\infty^{\frac{1/p}{r+1/p}} \varphi_r(t) \right|^p\right) dt. \tag{24}$$

In particular, for any $q \ge p$, *one has*

$$\left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}|x(t)|^q\,dt\right)^{1/q} \le \left(\frac{1}{\pi}\int_0^{\pi}|\varphi_r(t)|^q\,dt\right)^{1/q}\left(\frac{L(x)_p}{L(\varphi_r)_p}\right)^{\frac{r}{r+1/p}}\|x^{(r)}\|_\infty^{\frac{1/p}{r+1/p}}. \tag{25}$$

Furthermore, if $q \ge 1$ *and* $k = 1, \ldots, r-1$, *then the following inequality holds for any segment* $[a, b] \subset \mathbf{R}$ *and any function* $x \in L^r_\infty$ *that satisfies the condition* $x^{(k)}(a) = x^{(k)}(b) = 0$:

$$\left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left|x^{(k)}(t)\right|^q dt\right)^{1/q} \le \left(\frac{1}{\pi}\int_0^{\pi}\left|\varphi_{r-k}(t)\right|^q dt\right)^{1/q}\left(\frac{\|x\|_\infty}{\|\varphi_r\|_\infty}\right)^{\frac{r-k}{r}}\|x^{(r)}\|_\infty^{k/r}. \tag{26}$$

Proof. We fix an arbitrary segment $[\alpha, \beta] \subset \mathbf{R}$ and a function $x \in L^r_\infty$ such that $L(x)_p < \infty$ and $x(\alpha) = x(\beta) = 0$. We set $A_r := \|x^{(r)}\|_\infty$ and choose $\lambda > 0$ that satisfies the condition

$$L(x)_p = A_r L(\varphi_{\lambda,r})_p = A_r \lambda^{-r-1/p} L(\varphi_r)_p,$$

i.e.,

$$\lambda^{-1} = \left(\frac{L(x)_p}{A_r L(\varphi_r)_p}\right)^{\frac{1}{r+1/p}}. \tag{27}$$

Consider the set of all segments $[a_j, b_j] \subset [\alpha, \beta]$ such that

$$x(a_j) = x(b_j) = 0, \quad |x(t)| > 0, \quad t \in (a_j, b_j).$$

It is clear that

$$\int_{\alpha}^{\beta}\Phi\left(|x(t)|^p\right)dt = \sum_j \int_{a_j}^{b_j}\Phi\left(|x(t)|^p\right)dt$$

and

$$\sum_j (b_j - a_j) \le \beta - \alpha.$$

Note that, on each interval (a_j, b_j), the function x satisfies all conditions of Lemma 1. Estimating the integrals

$$\int_{a_j}^{b_j}\Phi\left(|x(t)|^p\right)dt$$

with the use of inequality (7) and taking into account the definition $\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t)$, we obtain

$$\int_\alpha^\beta \Phi\left(|x(t)|^p\right)dt \le \sum_j (b_j - a_j)\frac{\lambda}{\pi}\int_0^{\pi/\lambda}\Phi\left(\left|A_r\varphi_{\lambda,r}(t)\right|^p\right)dt$$

$$\le (\beta-\alpha)\frac{\lambda}{\pi}\int_0^{\pi/\lambda}\Phi\left(\left|A_r\varphi_{\lambda,r}(t)\right|^p\right)dt = (\beta-\alpha)\frac{1}{\pi}\int_0^{\pi}\Phi\left(\left|A_r\lambda^{-r}\varphi_r(s)\right|^p\right)ds.$$

Hence, taking into account that $A_r := \left\|x^{(r)}\right\|_\infty$ and λ is defined by (27), we obtain (24). Setting $\Phi(t) = t^{q/p}$ in (24), we get (25).

It remains to prove (26). We fix an arbitrary $k = 1, \ldots, r-1$, a segment $[a, b] \subset \mathbf{R}$, and a function $x \in L^r_\infty$ that satisfies the condition $x^{(k)}(a) = x^{(k)}(b) = 0$. Applying inequality (25) with $p = 1$ to the function $x^{(k)} \in L^{r-k}_\infty$, for $q \ge 1$ we obtain

$$\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\left|x^{(k)}(t)\right|^q dt\right)^{1/q} \le \left(\frac{1}{\pi}\int_0^\pi\left|\varphi_{r-k}(t)\right|^q dt\right)^{1/q}\left(\frac{L\left(x^{(k)}\right)_1}{L(\varphi_{r-k})_1}\right)^{\frac{r-k}{r-k+1}}\left\|x^{(r)}\right\|_\infty^{\frac{1}{r-k+1}}. \tag{28}$$

Taking into account the obvious relations

$$L\left(x^{(k)}\right)_1 \le 2\left\|x^{(k-1)}\right\|_\infty, \qquad L(\varphi_{r-k})_1 = 2\left\|\varphi_{r-k+1}\right\|_\infty$$

and estimating $\left\|x^{(k-1)}\right\|_\infty$ (for $k > 1$) with the use of the Kolmogorov inequality

$$\left\|x^{(k-1)}\right\|_\infty \le \left\|\varphi_{r-k+1}\right\|_\infty\left(\frac{\|x\|_\infty}{\|\varphi_r\|_\infty}\right)^{\frac{r-k+1}{r}}\left\|x^{(r)}\right\|_\infty^{\frac{k-1}{r}},$$

we deduce (26) from (28).

Theorem 3 is proved.

Remark 2. For a 2π-periodic functions $x \in L^r_\infty$, inequality (26) with $b - a = 2\pi$ transforms into the well-known Ligun inequality [8]

$$\left\|x^{(k)}\right\|_{L_q[0,2\pi]} \le \left\|\varphi_{r-k}\right\|_{L_q[0,2\pi]}\left(\frac{\|x\|_\infty}{\|\varphi_r\|_\infty}\right)^{\frac{r-k}{r}}\left\|x^{(r)}\right\|_\infty^{k/r}.$$

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