

# TAME COMODULE TYPE, ROITER BOCSSES, AND A GEOMETRY CONTEXT FOR COALGEBRAS

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We study the class of coalgebras  $C$  of  $f_c$ -tame comodule type introduced by the author. With any basic computable  $K$ -coalgebra  $C$  and a bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , we associate a bimodule matrix problem  $\text{Mat}_C^v(\mathbb{H})$ , an additive Roiter bocses  $\mathbf{B}_v^C$ , an affine algebraic  $K$ -variety  $\mathbf{Comod}_v^C$ , and an algebraic group action  $\mathbf{G}_v^C \times \mathbf{Comod}_v^C \rightarrow \mathbf{Comod}_v^C$ . We study the  $f_c$ -tame comodule type and the  $f_c$ -wild comodule type of  $C$  by means of  $\text{Mat}_C^v(\mathbb{H})$ , the category  $\text{rep}_K(\mathbf{B}_v^C)$  of  $K$ -linear representations of  $\mathbf{B}_v^C$ , and geometry of  $\mathbf{G}_v^C$ -orbits of  $\mathbf{Comod}_v^C$ . For computable coalgebras  $C$  over an algebraically closed field  $K$ , we give an alternative proof of the  $f_c$ -tame-wild dichotomy theorem. A characterization of  $f_c$ -tameness of  $C$  is given in terms of geometry of  $\mathbf{G}_v^C$ -orbits of  $\mathbf{Comod}_v^C$ . In particular, we show that  $C$  is  $f_c$ -tame of discrete comodule type if and only if the number of  $\mathbf{G}_v^C$ -orbits in  $\mathbf{Comod}_v^C$  is finite for every  $v = (v'|v'') \in K_0(C) \times K_0(C)$ .

## 1. Introduction

Throughout this paper, we use the terminology and notation introduced in [21, 22, 28]. We fix a field  $K$ . Given a  $K$ -coalgebra  $C$ , we denote by  $C\text{-Comod}$  and  $C\text{-comod}$  the categories of left  $C$ -comodules and left  $C$ -comodules of finite  $K$ -dimension. Recall that  $C$  is said to be *basic* if the left  $C$ -comodule  ${}_C C$  has a decomposition

$${}_C C = \bigoplus_{j \in I_C} E(j) \tag{1.1}$$

into a direct sum of pairwise nonisomorphic indecomposable injective left comodules  $E(j)$ . Throughout this paper, given  $j \in I_C$ , we denote by  $S(j)$  the unique simple subcomodule of  $E(j)$ . Hence,  $\text{soc } C = \bigoplus_{j \in I_C} S(j)$ . Following [26], the coalgebra  $C$  is called *Hom-computable* (or *computable*, in short) if  $\dim_K \text{Hom}_C(E(i), E(j))$  is finite for all  $i, j \in I_C$ . A left  $C$ -comodule  $M$  is said to be *computable* if  $\dim_K \text{Hom}_C(M, E(j))$  is finite for all  $j \in I_C$ .

Given a computable comodule  $M$ , we denote by  $\text{lgth } M = (\ell_j(M))_{j \in I_C} \in \mathbb{Z}^{I_C}$  the *composition length vector* of  $M$ , where  $\ell_j(M) < \infty$  is the number of simple composition factors of  $M$  isomorphic to the simple comodule  $S(j)$ . It is clear that  $\text{lgth } M \in \mathbb{Z}^{(I_C)}$  if  $M$  is of finite  $K$ -dimension. We recall from [21] that the map  $M \mapsto \text{lgth } M$  defines a group isomorphism  $\text{lgth}: K_0(C) \xrightarrow{\cong} \mathbb{Z}^{(I_C)}$ , where  $K_0(C) = K_0(C\text{-comod})$  is the Grothendieck group of the category  $C\text{-comod}$  and  $\mathbb{Z}^{(I_C)}$  is the direct sum of  $I_C$  copies of  $\mathbb{Z}$ .

We recall from [21] and [25] that an arbitrary  $K$ -coalgebra  $C$  is defined to be of  *$K$ -wild comodule type* (or  *$K$ -wild*, in short) if the category  $C\text{-comod}$  of finite-dimensional  $C$ -comodules is of  $K$ -wild representation type [18, 21, 23] in the sense that there exists an exact  $K$ -linear representation imbedding

$$T: \text{mod } \Gamma_3(K) \rightarrow C\text{-comod}, \text{ where } \Gamma_3(K) = \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}. \text{ A } K\text{-coalgebra } C \text{ is defined to be of } K\text{-tame}$$

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comodule type [25] (or  $K$ -tame, in short) if the category  $C\text{-comod}$  of finite-dimensional left  $C$ -comodules is of  $K$ -tame representation type ([18], Sec. 14.4, [22]), i.e., for every vector  $v \in K_0(C) \cong \mathbb{Z}^{(I_C)}$ , there exist  $C$ - $K[t]$ -bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  that are finitely generated free  $K[t]$ -modules such that all but finitely many indecomposable left  $C$ -comodules  $M$  with  $\text{lgth}M = v$  are of the form  $M \cong L^{(s)} \otimes K_\lambda^1$ , where  $s \leq r_v$  and

$$K_\lambda^1 = K[t]/(t - \lambda), \quad \lambda \in K. \tag{1.2}$$

Equivalently, there exist a nonzero polynomial  $h(t) \in K[t]$  and  $C$ - $K[t]_h$ -bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  that are finitely generated free  $K[t]_h$ -modules such that all but finitely many indecomposable left  $C$ -comodules  $M$  with  $\text{lgth}M = v$  are of the form  $M \cong L^{(s)} \otimes K_\lambda^1$ , where  $s \leq r_v$  and  $K[t]_h = K[t, h(t)^{-1}]$  is a rational  $K$ -algebra, see [7] or [18] (Sec. 14.4). In this case, we say that  $L^{(1)}, \dots, L^{(r_v)}$  form an *almost parametrizing family* for the family  $\text{ind}_v(C\text{-comod})$  of all indecomposable  $C$ -comodules  $M$  with  $\text{lgth}M = v$ .

Here, by a  $C$ - $K[t]_h$ -bicomodule  ${}_C L_{K[t]_h}$  we mean a  $K$ -vector space  $L$  equipped with a left  $C$ -comodule structure and a right  $K[t]_h$ -module structure satisfying the obvious associativity conditions. In [28], a  $K$ -tame-wild dichotomy theorem is proved for left (or right) semiperfect coalgebras and for acyclic hereditary coalgebras over an algebraically closed field  $K$  by reducing the problem to the  $fc$ -tame-wild dichotomy theorem [28] (Theorem 2.11) and, consequently, to the tame-wild dichotomy theorem for finite-dimensional  $K$ -algebras proved in [7] and [3].

The aim of the paper is to study the classes of coalgebras  $C$  of  $fc$ -tame comodule type and of  $fc$ -wild comodule type introduced in [28]. We recall that  $C$  is of  $fc$ -tame comodule type if, for every coordinate vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , the indecomposable finitely cogenerated  $C$ -comodules  $N$  such that  $\text{cdn}(N) = (v'|v'')$  form at most finitely many one-parameter families, see Sec. 2 for a precise definition.

We study mainly computable  $fc$ -tame and  $fc$ -wild basic coalgebras  $C$  by means of a bimodule matrix problem  $\text{Mat}_C^v(\mathbb{H})$ , the additive category  $\text{rep}_K(\mathbf{B}_v^C)$  of  $K$ -linear representations, an additive Roiter boc  $\mathbf{B}_v^C$ , an affine algebraic  $K$ -variety  $\text{Map}_v^C$ , an algebraic (parabolic) group action  $\mathbf{G}_v^C \times \text{Map}_v^C \rightarrow \text{Map}_v^C$ , and a Zariski open  $\mathbf{G}_v^C$ -invariant subset  $\text{Comod}_v^C \subseteq \text{Map}_v^C$  associated with  $C$  and with any bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ . It is shown in Sec. 4 that there is a bijection between the  $\mathbf{G}_v^C$ -orbits of  $\text{Comod}_v^C$  and the isomorphism classes of comodules in  $C\text{-Comod}_{fc}$ . On this way, we get in Theorem 4.1 a characterization of  $fc$ -tameness and  $fc$ -wildness of computable coalgebras by means of  $\text{Mat}_C^v(\mathbb{H})$ , the  $K$ -linear representations of the Roiter boc  $\mathbf{B}_v^C$ , and in terms of geometry of the  $\mathbf{G}_v^C$ -orbits of  $\text{Comod}_v^C$ .

We show in Sec. 4 that a computable coalgebra  $C$  is  $fc$ -tame of discrete comodule type if and only if the number of  $\mathbf{G}_v^C$ -orbits in  $\text{Comod}_v^C$  is finite for every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ . Moreover, we prove that a computable coalgebra  $C$  is  $fc$ -tame if and only if, for every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , there exists a constructible subset  $\mathcal{C}(v)$  of the constructible set  $\text{indComod}_v^C \subseteq \text{Comod}_v^C$  (defined by the indecomposable  $C$ -comodules) such that  $\mathbf{G}_v^C * \mathcal{C}(v) = \text{indComod}_v^C$  and  $\dim \mathcal{C}(v) \leq 1$ , see Theorem 4.1.

We also give an alternative proof of the following  $fc$ -tame-wild dichotomy theorem proved in [28]: *If  $C$  is a basic computable coalgebra over an algebraically closed field  $K$ , then  $C$  is either  $fc$ -tame or  $fc$ -wild, and these two types are mutually exclusive.*

We prove it in Sec. 3 by a reduction to the tame-wild dichotomy theorem of Drozd [7] for representations of additive Roiter bocses, by applying the bimodule problems technique introduced in [5] and developed in [3, 4, 9, 17, 19, 20].

Throughout this paper, we freely use the coalgebra representation theory notation and terminology introduced in [2, 16, 21, 22, 28]. The reader is referred to [1, 8, 10, 18] for representation theory terminology and notation, and to [3, 4, 7, 9, 13] for a background on the representation theory of bocses.

In particular, given a ring  $R$  with an identity element, we denote by  $\text{Mod}(R)$  the category of all unitary right  $R$ -modules, and by  $\text{mod}(R) \supseteq \text{fin}(R)$  the full subcategories of  $\text{Mod}(R)$  formed by the finitely generated

$R$ -modules and the finite-dimensional  $R$ -modules, respectively. Given a  $K$ -coalgebra  $C$  and a left  $C$ -comodule  $M$ , we denote by  $\text{soc } M$  the *socle* of  $M$ , i.e., the sum of all simple  $C$ -subcomodules of  $M$ .

A comodule  $N$  in  $C\text{-Comod}$  is said to be *socle-finite* if  $N$  is a subcomodule of a finite direct sum of indecomposable injective comodules, or equivalently,  $\dim_K \text{soc } N$  is finite. We say that  $N$  is *finitely copresented* if  $N$  admits a socle-finite injective copresentation, i.e., an exact sequence  $0 \rightarrow N \rightarrow E_0 \xrightarrow{\psi} E_1$  in  $C\text{-Comod}$ , where each of the comodules  $E_0$  and  $E_1$  is a finite direct sum of indecomposable injective comodules. If  $E_0, E_1 \in \text{add}(E)$ , for some socle-finite injective  $C$ -comodule  $E$ , the comodule  $N$  is called *finitely  $E$ -copresented*. We denote by  $C\text{-Comod}_{fc} \supseteq C\text{-Comod}_{fc}^E$  the full subcategories of  $C\text{-Comod}$  whose objects are the finitely copresented comodules and finitely  $E$ -copresented comodules, respectively. Here by  $\text{add}(E)$  we mean the full additive subcategory of  $C\text{-Comod}$  whose objects are finite direct sum of indecomposable injective comodules isomorphic to direct summands of  $E$ .

## 2. Preliminaries on $fc$ -Comodule Types for Coalgebras

Throughout we assume that  $K$  is an algebraically closed field and  $C$  is a basic  $K$ -coalgebra with a fixed decomposition (1.1). Following [28], given a finitely copresented  $C$ -comodule  $N$  in  $C\text{-Comod}_{fc}$ , with a minimal injective copresentation  $0 \rightarrow N \rightarrow E_0^N \xrightarrow{g} E_1^N$ , we define the *coordinate vector* of  $N$  to be the bipartite vector

$$\mathbf{cdn}(N) = (\mathbf{cdn}_0^N \mid \mathbf{cdn}_1^N) \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}, \tag{2.1}$$

where  $\mathbf{cdn}_0^N = \mathbf{lgth}(\text{soc } E_0^N)$  and  $\mathbf{cdn}_1^N = \mathbf{lgth}(\text{soc } E_1^N)$ . We call a bipartite vector  $v = (v' \mid v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  *proper* if  $v' \neq 0$  and  $v''$  has nonnegative coordinates. Note that an indecomposable comodule  $N$  in  $C\text{-Comod}_{fc}$  is injective if and only if the vector  $\mathbf{cdn}(N)$  is proper and has the form  $v = (e_j \mid v'')$ , where  $v'' = 0$  and  $e_j$  is the  $j$ th standard basis vector of  $\mathbb{Z}^{(I_C)}$ , for some  $j \in I_C$ .

The *support* of a bipartite vector  $v = (v' \mid v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  is the finite subset  $\text{supp}(v) = \{j \in I_C; v'_j \neq 0 \text{ or } v''_j \neq 0\}$  of  $I_C$ .

We recall from [28] that  $K$ -coalgebra  $C$  is defined to be of  *$fc$ -wild comodule type* (or  *$fc$ -wild*, in short) if the category  $C\text{-Comod}_{fc}$  of finitely copresented  $C$ -comodules is of  $K$ -wild representation type [18, 23, 25] in the sense that there exists an exact  $K$ -linear representation imbedding  $T: \text{mod}\Gamma_3(K) \rightarrow C\text{-Comod}_{fc}$ , where

$$\Gamma_3(K) = \begin{bmatrix} K & K^3 \\ 0 & K \end{bmatrix}.$$

A  $C\text{-}K[t]_h$ -bicomodule  ${}_C L_{K[t]_h}$  is defined to be *finitely copresented* if there is a  $C\text{-}K[t]_h$ -bicomodule exact sequence  $0 \rightarrow {}_C L_{K[t]_h} \rightarrow E' \otimes K[t]_h \xrightarrow{\psi} E'' \otimes K[t]_h$ , such that  $E', E''$  are socle-finite injective  $C$ -comodules. If  $E', E''$  are finitely  $E$ -copresented, we call  ${}_C L_{K[t]_h}$  *finitely  $E$ -copresented*.

A  $K$ -coalgebra  $C$  is defined to be of  *$fc$ -tame comodule type* (or  *$fc$ -tame*, in short) if the category  $C\text{-Comod}_{fc}$  is of  $fc$ -tame representation type [18] (Sec. 14.4), i.e., for every bipartite vector  $v = (v' \mid v'') \in K_0(C) \times K_0(C) \cong \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ , there exist  $C\text{-}K[t]_h$ -bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  that are finitely copresented and such that all but finitely many indecomposable left  $C$ -comodules  $N$  in  $C\text{-Comod}_{fc}$ , with  $\mathbf{cdn}(N) = v$ , are of the form  $N \cong L^{(s)} \otimes K_\lambda^1$ , where  $s \leq r_v$ ,

$$K_\lambda^1 = K[t]/(t - \lambda),$$

and  $\lambda \in K$ . In this case, we say that  $L^{(1)}, \dots, L^{(r_v)}$  is a *finitely copresented almost parametrizing family* for the family  $\text{ind}_v(C\text{-Comod}_{f_c})$  of all indecomposable  $C$ -comodules  $N$  with  $\text{cdn}(N) = v$ . Obviously, one can restrict the definition to proper bipartite vectors  $v = (v'|v'')$ .

We recall from [28] that the *growth function*  $\widehat{\mu}_C^1: K_0(C) \times K_0(C) \longrightarrow \mathbb{N}$  of  $C$  associates with any bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$  the minimal number  $\widehat{\mu}_C^1(v) = r_v \geq 1$  of nonzero finitely copresented  $C\text{-}K[t]_h$ -bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  forming an almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{f_c})$ . We set  $\widehat{\mu}_C^1(v) = r_v = 0$  if there is no such a family of bicomodules, i.e., there is only a finite number of comodules  $N$  in  $\text{ind}_v(C\text{-Comod}_{f_c})$ , up to isomorphism.

An  $f_c$ -tame coalgebra  $C$  is defined to be of  *$f_c$ -discrete comodule type* if  $\widehat{\mu}_C^1 = 0$ , i.e., the number of the isomorphism classes of the indecomposable  $C$ -comodules  $N$  in  $C\text{-Comod}_{f_c}$  with  $\text{cdn}(N) = v$  is finite, for every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ .

By the main result in [28], the definition is left-right symmetric for any computable coalgebra  $C$ . Note also that the  $K$ -tameness and  $K$ -wildness of a coalgebra are defined by means of finite-dimensional comodules, but the  $f_c$ -tame comodule type and  $f_c$ -wild comodule type are defined by means of the category  $C\text{-Comod}_{f_c}$  of finitely copresented comodules that usually contains a lot of infinite-dimensional comodules.

In the proof of our main results, we need the following construction, which associates with any  $v = (v'|v'') \in K_0(C) \times K_0(C)$  and any finitely copresented  $C\text{-}K[t]_h$ -bicomodule  ${}_C L_{K[t]_h}$  a new one  ${}_C \widetilde{L}_{K[t]_h}$ , called  *$f_c$ -localizing  $v$ -corrected  $C\text{-}K[t]_h$ -bicomodule*:

**Construction 2.1.** Let  $C$  be a basic  $K$ -coalgebra with a decomposition (1.1), and let  $v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  be a proper bipartite vector.

Let  $U_v = \text{supp}(v) \subseteq I_C$  be the support of  $v = (v'|v'')$ . We call the socle-finite injective  $C$ -comodules

$$\mathbf{E}(v') = \bigoplus_{i \in I_C} E(i)^{v'_i} \quad \text{and} \quad \mathbf{E}(v'') = \bigoplus_{j \in I_C} E(j)^{v''_j} \tag{2.2}$$

the *standard injective  $C$ -comodules* with  $\text{cdn}\mathbf{E}(v') = (v'|0)$  and  $\text{cdn}\mathbf{E}(v'') = (v''|0)$ .

We fix a rational  $K$ -algebra  $S = K[t]_h$  and note that

$$E_v = E_{U_v} = \bigoplus_{a \in U_v} E(a) \tag{2.3}$$

is a socle-finite injective direct summand of  ${}_C C$ .

Assume that  ${}_C L_S$  is a finitely copresented  $C\text{-}S$ -bicomodule with a fixed injective  $C\text{-}S$ -bicomodule copresentation

$$0 \longrightarrow {}_C L_S \longrightarrow E_0 \otimes S \xrightarrow{\psi} E_1 \otimes S, \tag{2.4}$$

where  $E_0, E_1$  are socle-finite injective comodules such that  $\mathbf{E}(v') \subseteq E_0$  and  $\mathbf{E}(v'') \subseteq E_1$ .

We construct in three steps a finitely  $E_v$ -copresented  $C\text{-}S$ -bicomodule  ${}_C \widetilde{L}_S$ , called a *localizing  $f_c$ -correction* of  ${}_C L_S$  as follows:

*Step 1.* Fix a decomposition  $E_0 = E'_0 \oplus E''_0$ , where  $E'_0$  is the injective envelope of the semisimple subcomodule  $S(v)$  generated by the simple subcomodules of  $E_0$  that are isomorphic to  $S(j)$ , with  $j \in U_v$ . Obviously, every simple subcomodule  $S$  of  $E''_0$  has the form  $S \cong S(a)$ , where  $a \notin U_v$ .

*Step 2.* Define a  $C\text{-}S$ -subbicomodule  ${}_C L'_S$  of  ${}_C L_S$  to be the kernel of the composite  $C\text{-}S$ -bicomodule homomorphism  $E'_0 \otimes S \xrightarrow{u'_0 \otimes S} E_0 \otimes S \xrightarrow{\psi} E_1 \otimes S$ , where  $u'_0: E'_0 \hookrightarrow E_0$  is the canonical imbedding.

*Step 3.* Let  $e_v: C \rightarrow K$  be the idempotent of the algebra  $C^* = \text{Hom}_K(C, K)$  defined by the direct summand  $E_v$  of  ${}_C C$ . An  $f_c$ -localizing correction of  ${}_C L_S$  is the  $C$ - $S$ -bicomodule

$${}_C \tilde{L}_S = e_v C \square_{e_v C e_v} [\text{res}_{E_v}({}_C L'_S)], \tag{2.5}$$

where  $\text{res}_{E_v}: C\text{-Comod}_{f_c} \rightarrow e_v C e_v\text{-Comod}_{f_c}$  is the exact restriction functor and  $e_v C \square_{e_v C e_v}(-): e_v C e_v\text{-Comod}_{f_c} \rightarrow C\text{-Comod}_{f_c}$  is the left exact cotensor product functor defined in [11] and [25] ((2.9), see also [29]).

The following  $f_c$ -localizing correction lemma is of importance:

**Lemma 2.1.** *Let  $K$  be an algebraically closed field,  $C$  a basic  $K$ -coalgebra with the decomposition (1.1),  $v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  a proper bipartite vector,  $S = K[t]_h$ , and  ${}_C L_S$  a finitely cogenerated  $C$ - $S$ -bicomodule with a fixed injective  $C$ - $S$ -bicomodule copresentation (2.4) as in Construction 2.1.*

(a) *The  $C$ - $S$ -bicomodule  ${}_C \tilde{L}_S$  (2.5) has an injective  $C$ - $S$ -bicomodule copresentation*

$$0 \rightarrow {}_C \tilde{L}_S \rightarrow \tilde{E}_0 \otimes S \xrightarrow{\tilde{\psi}} \tilde{E}_1 \otimes S \tag{2.6}$$

and the comodules  $\tilde{E}_0 = E'_0, \tilde{E}_1$  lie in  $\text{add}(E_{U_v})$ .

(b) *If  $N$  is an indecomposable comodule in  $C\text{-Comod}_{f_c}$  such that  $\text{cdn}(N) = v$  and  $N \cong {}_C L_S \otimes K_\lambda^1$ , with  $\lambda \in K$ , then the restriction  $\hat{u}'_0: {}_C L'_S \hookrightarrow {}_C L_S$  of the splitting monomorphism  $u'_0 \otimes S: E'_0 \otimes S \hookrightarrow E_0 \otimes S$  to  ${}_C L'_S$  is an imbedding of  $C$ - $S$ -bicomodules and induces isomorphisms  ${}_C \tilde{L}_S \otimes K_\lambda^1 \cong {}_C L'_S \otimes K_\lambda^1 \cong N$  of  $C$ -comodules.*

**Proof.** (a) By the construction, there are a decomposition  $E_0 = E'_0 \oplus E''_0$  and exact sequence

$$0 \rightarrow {}_C L'_S \rightarrow E'_0 \otimes S \xrightarrow{\psi'} E_1 \otimes S$$

of  $C$ - $S$ -bicomodules, where  $\psi' = \psi \circ (u'_0 \otimes S)$  and  $u'_0 = (\text{id}_{E'_0}, 0): E'_0 \hookrightarrow E_0 = E'_0 \oplus E''_0$  is the canonical imbedding into the direct summand  $E'_0$  of  $E_0$ . We recall from [11] and [25] (Sec. 2) that the restriction functor  $\text{res}_{E_v}: C\text{-Comod}_{f_c} \rightarrow e_v C e_v\text{-Comod}_{f_c}$  is exact and the cotensor product functor  $e_v C \square_{e_v C e_v}(-): e_v C e_v\text{-Comod}_{f_c} \rightarrow C\text{-Comod}_{f_c}$  is left exact. Then we derive an exact sequence

$$0 \rightarrow {}_C \tilde{L}_S \rightarrow \tilde{E}_0 \otimes S \xrightarrow{\psi'} E_1^\vee \otimes S$$

of  $C$ - $S$ -bicomodules, where  $\tilde{E}_0 = e_v C \square_{e_v C e_v} \text{res}_{E_v}(E'_0)$  and

$$E_1^\vee = e_v C \square_{e_v C e_v} \text{res}_{E_v}(E_1).$$

Since  $E_0$  is a direct summand of  $E_{U_v}$ , by [11] and [25] (Proposition 2.7 and Theorem 2.10) there is an isomorphism  $\tilde{E}_0 \cong E_0$ , the socle of  $\text{res}_{E_v}(E_1)$  is a finite-dimensional subcomodule of the coalgebra  $e_v C e_v$  and the socle of  $E_1^\vee = e_v C \square_{e_v C e_v} \text{res}_{E_v}(E_1)$  is a finite direct sum of comodules  $S(a)$ , with  $a \in U_v$ . It follows that the injective envelope  $\tilde{E}_1 = E_C(E_1^\vee)$  of the  $C$ -comodule  $E_1^\vee$  lies in  $\text{add}(E_{U_v})$ . Hence we get the exact sequence (2.6) and (a) follows.

(b) The canonical imbedding  $u'_0 = (\text{id}_{E'_0}, 0): E'_0 \hookrightarrow E_0 = E'_0 \oplus E''_0$  into the direct summand  $E'_0$  of  $E_0$  induces the commutative diagram of  $C$ - $S$ -bicomodules

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_C L_S & \longrightarrow & (E'_0 \oplus E''_0) \otimes S & \xrightarrow{\psi} & E_1 \otimes S \\ & & \tilde{u}'_0 \uparrow & & u'_0 \otimes S \uparrow & & \text{id}_{E_1} \otimes S \uparrow \\ 0 & \longrightarrow & {}_C L'_S & \longrightarrow & E'_0 \otimes S & \xrightarrow{\psi'} & E_1 \otimes S \end{array}$$

with exact rows, where  $\tilde{u}'_0$  is the restriction of the monomorphism  $u'_0 \otimes S: E'_0 \otimes S \hookrightarrow E_0 \otimes S$  to  ${}_C L'_S$ . Obviously,  $\tilde{u}'_0$  is an imbedding of  $C$ - $S$ -bicomodules.

Let  $N$  be an indecomposable comodule in  $C\text{-Comod}_{f_c}$  such that  $\mathbf{cdn}(N) = v = (v'|v'')$  and  $N \cong {}_C L_S \otimes K_\lambda^1$ , with  $\lambda \in K$ . Then  $N$  has a minimal injective copresentation  $0 \longrightarrow N \longrightarrow \mathbf{E}(v') \xrightarrow{g} \mathbf{E}(v'')$ . Recall that  $\mathbf{cdn}\mathbf{E}(v') = (v'|0)$  and  $\mathbf{cdn}\mathbf{E}(v'') = (v''|0)$ . Then we get a commutative diagram of  $C$ -comodules in  $C\text{-Comod}_{f_c}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & \mathbf{E}(v') & \xrightarrow{g} & \mathbf{E}(v'') \\ & & \downarrow \cong & & f_0 \downarrow & & f_1 \downarrow \\ 0 & \longrightarrow & {}_C L \otimes_S K_\lambda^1 & \longrightarrow & (E'_0 \oplus E''_0) \otimes K_\lambda^1 & \xrightarrow{\psi \otimes \text{id}} & E_1 \otimes K_\lambda^1 \\ & & \tilde{u}'_0 \uparrow & & u'_0 \otimes \text{id} \uparrow & & \text{id} \uparrow \\ 0 & \longrightarrow & {}_C L' \otimes_S K_\lambda^1 & \longrightarrow & E'_0 \otimes K_\lambda^1 & \xrightarrow{\psi'} & E_1 \otimes K_\lambda^1 \end{array}$$

with exact rows. Since the upper row is a minimal injective copresentation of  $N$ ,  $f_0$  and  $f_1$  are monomorphisms, and  $f_0$  has a factorization  $\mathbf{E}(v') \xrightarrow{f'_0} E'_0 \otimes K_\lambda^1 \xrightarrow{u'_0 \otimes S} E'_0 \otimes K_\lambda^1$  through the subcomodule  $E'_0 \otimes K_\lambda^1$  of  $E_0 \otimes K_\lambda^1$ , because the socle of  $E'_0 \otimes K_\lambda^1$  contains no simple comodules  $S(a)$ , with  $a \in U_v$ . It follows that  $f'_0$  restricts to a monomorphism  $\hat{f}'_0: N \rightarrow {}_C L' \otimes_S K_\lambda^1$  such that the composite map  $N \xrightarrow{\hat{f}'_0} {}_C L' \otimes_S K_\lambda^1 \xrightarrow{\tilde{u}'_0} {}_C L \otimes_S K_\lambda^1$  is an isomorphism. Consequently,  $\hat{f}'_0: N \rightarrow {}_C L' \otimes_S K_\lambda^1$  is an isomorphism of  $C$ -comodules. Hence, in the notation of Construction 2.1, we get the isomorphisms

$$\begin{aligned} {}_C L \otimes_S K_\lambda^1 &= [e_v C \square_{e_v C e_v} \text{res}_{E_v}({}_C L')] \otimes_S K_\lambda^1 \\ &\cong e_v C \square_{e_v C e_v} [\text{res}_{E_v}({}_C L' \otimes_S K_\lambda^1)] \cong e_v C \square_{e_v C e_v} [\text{res}_{E_v}(N)] \cong N \end{aligned}$$

of  $C$ -comodules, because  $N$  is finitely  $E_{U_v}$ -copresented and [25] (Theorem 2.10 (d)) applies to  $N$ .

The lemma is proved.

### 3. $f_c$ -Tameness, $f_c$ -Wildness and Roiter Bocses for Coalgebras

We show in this section how the study of  $f_c$ -tame and  $f_c$ -wild coalgebras can be reduced to the study of bimodule matrix problems in the sense of Drozd [5], to representations of additive Roiter bocses [3–7], and to the study of propartite modules over a class of bipartite algebras [19, 20].

To formulate our main results on  $f_c$ -tame and  $f_c$ -wild computable coalgebras, we recall some notation, see [25] and [26]. Given a socle-finite injective direct summand

$$E = E_U = \bigoplus_{u \in U} E(u) \tag{3.1}$$

of  ${}_C C = \bigoplus_{j \in I_C} E(j)$ , with a finite subset  $U$  of  $I_C$ , we define the category  $C\text{-Comod}_{fc}^{E_U}$  to be *fc-tame* if for every bipartite vector  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$ , there is a finitely  $E$ -copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$ .

We start with the following *fc*-parametrization correction lemma:

**Lemma 3.1.** *Let  $K$  be an algebraically closed field,  $C$  a basic  $K$ -coalgebra with the decomposition (1.1), and  $E = E_U$  a socle-finite injective direct summand (3.1) of  ${}_C C$ .*

- (a) *If  $C$  is *fc-tame* then the category  $C\text{-Comod}_{fc}^{E_U}$  is *fc-tame*.*
- (b) *If  $v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  is a proper bipartite vector,  $S = K[t]_h$ , and  $L^{(1)}, \dots, L^{(r_v)}$  is a finitely copresented almost parametrizing family of  $C$ - $S$ -bicomodules for  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$  then the *fc*-localizing  $v$ -corrected  $C$ - $S$ -bicomodules  $\tilde{L}^{(1)}, \dots, \tilde{L}^{(r_v)}$  in the sense of Construction 2.1 form a finitely  $E_U$ -copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$ .*

**Proof.** It is sufficient to prove (b), because (a) is a direct consequence of (b). Assume that  $v = (v'|v'') \in K_0(C) \times K_0(C) = \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  is a proper bipartite vector and  $L^{(1)}, \dots, L^{(r_v)}$  is a finitely copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$ . Assume that  $r_v \geq 0$  is a minimal number of such nonzero bimodules. If  $r_v = 0$  then there is nothing to prove, because the number of the isomorphism classes of indecomposable comodules in  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$  is finite.

Assume that  $r_v \geq 1$ . Then, for each  $1 \leq j \leq r_v$ , there is an indecomposable comodule  $N$  such that  $\text{cdn}(N) = v$  and  $N \cong L^{(j)} \otimes_S K_{\lambda(j)}^1$ , for some  $\lambda(j) \in K$ . Then  $N$  has a minimal injective copresentation  $0 \rightarrow N \rightarrow \mathbf{E}(v') \xrightarrow{g} \mathbf{E}(v'')$ .

Since  ${}_C L_S^{(j)}$  is a finitely copresented  $C$ - $S$ -bicomodule, it has an injective  $C$ - $S$ -bicomodule copresentation

$$0 \rightarrow {}_C L_S^{(j)} \rightarrow E_0^{(j)} \otimes S \xrightarrow{\psi^{(j)}} E_1^{(j)} \otimes S,$$

where  $E_0^{(j)}, E_1^{(j)}$  are socle-finite injective  $C$ -comodules. Since  $N \cong L^{(j)} \otimes_S K_{\lambda(j)}^1$ , there are  $C$ -comodule monomorphisms  $\mathbf{E}(v') \subseteq E_0$  and  $\mathbf{E}(v'') \subseteq E_1$ , because the sequence

$$0 \rightarrow {}_C L^{(j)} \otimes_S K_{\lambda(j)}^1 \rightarrow E_0^{(j)} \otimes K_{\lambda(j)}^1 \xrightarrow{\hat{\psi}^{(j)}} E_1^{(j)} \otimes K_{\lambda(j)}^1$$

induced by the previous one is exact and is a socle-finite injective copresentation of  $N \cong L^{(j)} \otimes_S K_{\lambda(j)}^1$ . Then the Construction 2.1 applies to  ${}_C L_S^{(j)}$ , for  $j = 1, \dots, r_v$ .

By applying Lemma 2.1 to the finitely copresented  $C$ - $S$ -bicomodule  $L^{(j)}$  we get a finitely  $E_U$ -copresented  $C$ - $S$ -bicomodule  $\tilde{L}^{(j)}$  such that the *fc*-localizing  $v$ -corrected  $C$ - $S$ -bicomodules  $\tilde{L}^{(1)}, \dots, \tilde{L}^{(r_v)}$  form a finitely  $E_U$ -copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{fc}^{E_U})$ .

The lemma is proved.

Following [25, 26, 28] given a socle-finite injective direct summand  $E = E_U$  (3.1), we consider the  $K$ -algebra

$$R_E = \text{End}_C E = \bigoplus_{u \in U} e_u R_E, \tag{3.2}$$

where  $e_u R_E = \text{Hom}_C(E, E(u))$  is viewed as an indecomposable projective right ideal of  $R_E$  and  $e_u$  is the primitive idempotent of  $R_E$  defined by the summand  $E(u)$  of  $E$ . Since the set  $U$  is finite,  $\sum_{u \in U} e_u$  is the identity of  $R_E$ . It is easy to see that the Jacobson radical  $J(R_E)$  of  $R_E$  has the form  $J(R_E) = \{h \in \text{End}_C E; h(\text{soc } E) = 0\}$ . It follows that the algebra  $R_E$  is semiperfect and pseudocompact with respect to the  $K$ -linear topology defined by the left ideals  $\mathfrak{a}_\beta = \text{Hom}_C(E/V_\beta, E) \subseteq R_E$ , where  $\{V_\beta\}_\beta$  is the directed set of all finite-dimensional subcomodules of  $E$ . Since  $E = \bigcup_\beta V_\beta$ , there are isomorphisms

$$R_E = \text{End}_C E \cong \varprojlim_{\beta} \text{Hom}_C(V_\beta, E) \cong \varprojlim_{\beta} R_E/\mathfrak{a}_\beta. \tag{3.3}$$

Following [3, 7, 28], we consider the homomorphism category  $\text{Map}_1(E)$  whose objects are the triples  $(E_0, E_1, \psi)$  with  $E_0, E_1$  comodules in  $\text{add}(E)$  and  $\psi: E_0 \rightarrow E_1$  a homomorphism of  $C$ -comodules such that  $\psi(\text{soc } E_0) = 0$ ; and whose morphisms are the pairs  $(f_0, f_1)$ , where  $f_0: E_0 \rightarrow E'_0, f_1: E_1 \rightarrow E'_1$  and  $\psi' \circ f_0 = f_1 \circ \psi$ . Denote by  $\text{Map}_2(E)$  the full subcategory of  $\text{Map}_1(E)$  whose objects are the triples  $(E_0, E_1, \psi)$  such that  $\text{soc } \text{Im } \psi = \text{soc } E_1$ . or equivalently,  $\psi: E_0 \rightarrow E_1$  has no nonzero direct summand of the form  $0 \rightarrow E''$ . We define the *coordinate vector* of  $(E_0, E_1, \psi)$  to be the bipartite vector

$$\mathbf{cdn}(E_0, E_1, \psi) = (\mathbf{lgth}(\text{soc } E_0) | \mathbf{lgth}(\text{soc } E_1)) \in \mathbb{Z}^U \times \mathbb{Z}^U = K_0(R_E) \times K_0(R_E). \tag{3.4}$$

Following [7], [3] (Sec.6) and [28], we denote by  $\mathcal{P}_1(R_E^{op})$  the category whose objects are the triples  $(P_1, P_0, \phi)$  with  $P_0, P_1$  finitely generated projective left  $R_E$ -modules and  $\phi: P_1 \rightarrow \text{rad}(P_0) = P_0 J(R_E)$  a homomorphism of left  $R_E$ -modules; and whose morphisms are the pairs  $(g_1, g_0)$ , where  $g_0: P_0 \rightarrow P'_0, g_1: P_1 \rightarrow P'_1$  and  $\phi' \circ g_1 = g_0 \circ \phi$ . Denote by  $\mathcal{P}_2(R_E^{op})$  the full subcategory of  $\mathcal{P}_1(R_E^{op})$  whose objects are the triples  $(P_1, P_0, \phi)$  with  $\text{Ker } \phi \subseteq \text{rad}(P_1)$ . or equivalently,  $\phi: P_1 \rightarrow P_0$  has no nonzero direct summand of the form  $P \rightarrow 0$ . We define the *coordinate vector* of  $(P_1, P_0, \phi)$  to be the bipartite vector

$$\mathbf{cdn}(P_1, P_0, \phi) = (\mathbf{lgth}(\text{top } P_1) | \mathbf{lgth}(\text{top } P_0)) \in \mathbb{Z}^U \times \mathbb{Z}^U = K_0(R_E^{op}) \times K_0(R_E^{op}).$$

We call  $\mathbf{cdn}(\text{Coker } \phi) = \mathbf{cdn}(P_1, P_0, \phi)$  the *coordinate vector* of the  $R_E$ -module  $\text{Coker } \phi$ .

We start with the following important result (here we freely use the terminology and notation introduced in [3] (Sec. 6), [7], and [28]):

**Theorem 3.1.** *Let  $K$  be an algebraically closed field,  $C$  a basic  $K$ -coalgebra with the decomposition (1.1),  $E$  a socle-finite injective direct summand (3.1) of  ${}_C C$ , and assume that the  $K$ -algebra  $R_E = \text{End}_C E$  (3.2) is finite-dimensional. Let  $\mathbf{B}_E = (A, {}_A V_A)$  be the additive Roiter bocses associated with the  $K$ -algebra  $R_E^{op}$  in [3] (Proposition 6.1). Then there is a commutative diagram*

$$\begin{array}{ccccc} \text{Map}_1(E) & \xrightarrow[\simeq]{H_E} & \mathcal{P}_1(R_E^{op}) & \xleftarrow[\simeq]{G} & \text{rep}_K(\mathbf{B}_E) \\ \text{ker}_E \downarrow & & \text{cok}_E \downarrow & & \\ C\text{-Comod}_{fc}^E & \xrightarrow[\simeq]{h_E^\bullet} & \text{mod}(R_E^{op}), & & \end{array} \tag{3.5}$$

where  $H_E$  and  $h_E^\bullet = \text{Hom}_C(\bullet, E)$  are  $K$ -linear contravariant equivalences of categories,  $G$  is a covariant  $K$ -linear equivalence of categories,  $h_E^\bullet$  is an exact functor,  $\text{ker}_E(E_0, E_1, \psi) = \text{Ker } \psi, \text{cok}_E(P_1, P_0, \phi) = \text{Coker } \phi$ , and the following conditions are satisfied:



- (a) The functors  $\mathbf{cok}_E$  and  $\mathbf{ker}_E$  are full dense and restrict to the representation equivalences  $\mathbf{ker}_E: \mathcal{M}ap_2(E) \rightarrow C\text{-Comod}_{f_c}^E$  and  $\mathbf{cok}_E: \mathcal{P}_2(R_E^{op}) \rightarrow \mathbf{mod}(R_E^{op})$ . The right-hand part in the diagram is defined as in [7] (Sec. 5) and [3, p. 476, 478], with  $R_E^{op}$ ,  $G$ ,  $\mathbf{cok}_E$  and  $\Lambda, \Xi, \mathbf{cok}$  interchanged.
- (b) If  $N$  is an indecomposable comodule in  $C\text{-Comod}_{f_c}^E$  then there exists a unique, up to isomorphism, indecomposable object  $(E_0, E_1, \psi)$  in  $\mathcal{M}ap_1(E)$  such that  $\mathbf{ker}_E(E_0, E_1, \psi) \cong N$ . In this case  $(E_0, E_1, \psi)$  lies in  $\mathcal{M}ap_2(E)$  and

$$\mathbf{cdn}(N) = \mathbf{cdn}(E_0, E_1, \psi) = \sigma(\mathbf{cdn}H_E(E_0, E_1, \psi)) = \underline{\dim}G^{-1}H_E(E_0, E_1, \psi),$$

where we set  $\sigma(v'|v'') = (v''|v')$ .

- (c) If the category  $C\text{-Comod}_{f_c}^E$  is not of  $K$ -wild representation type (shortly,  $K$ -wild) then the additive category  $\mathbf{rep}_K(\mathbf{B}_E)$  of the  $K$ -linear representations of  $\mathbf{B}_E$  is not wild and, given a nonnegative vector

$$v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \cong K_0(C) \times K_0(C),$$

there exist minimal bocses  $\mathbf{B}_1, \dots, \mathbf{B}_n$ , with  $\mathbf{B}_i = (B_i, W_i)$ , finitely  $E$ -copresented  $C$ - $B_i$ -bicomodules  $T_i$  and full functors  $F_i: \mathbf{rep}_K(\mathbf{B}_i) \rightarrow C\text{-Comod}_{f_c}^E$  which reflect isomorphisms such that

- (c<sub>1</sub>)  $F_i(X) = T_i \otimes_{B_i} X$ , for all representations  $X$  in  $\mathbf{rep}_K(\mathbf{B}_i)$ ,
- (c<sub>2</sub>) every indecomposable comodule  $N$  in  $C\text{-Comod}_{f_c}^E$ , with  $\mathbf{cdn}(N) = v$ , is isomorphic to  $F_i(X)$ , for some  $i$  and some representation  $X$  in  $\mathbf{rep}_K(\mathbf{B}_i)$ ,
- (c<sub>3</sub>) the functors  $F_i$  induce group homomorphisms  $K_0(\mathbf{B}_i) \rightarrow \mathbb{Z}^U \subseteq \mathbb{Z}^{(I_C)} \cong K_0(C)$  taking the dimension vector  $\underline{\dim}(X)$  of  $X$  to  $\mathbf{cdn}F_i(X)$ .

**Proof.** By our assumption, the injective comodule  $E = E_U$  is socle-finite and the  $K$ -algebra  $R_E = \text{End}_C E$  is finite-dimensional. Let  $D: \mathbf{mod}R_E^{op} \rightarrow \mathbf{mod}R_E$  be the standard duality given by  $L \mapsto D(L) = \text{Hom}_K(L, K)$ , for any  $L$  in  $\mathbf{mod}R_E^{op}$ . We define the contravariant functor  $h_E^\bullet$  by setting  $h_E^{(-)} = \text{Hom}_C(-, E)$ . Since  $E$  is injective, the functor  $h_E^\bullet$  is exact and, by [26] (Proposition 2.13),  $h_E^\bullet$  is an equivalence of categories such that  $(\mathbf{lgth}N)_u = (\underline{\dim}h_E^N)_u = \dim_K(h_E^N e_u)$ , for any comodule  $N$  in  $C\text{-Comod}_{f_c}^E$  and all  $u \in U$ , where  $\underline{\dim}N'$  is the dimension vector of a left  $R_U$ -module  $N'$ . This means that  $\text{res}_U(\mathbf{lgth}N) = \underline{\dim}h_E^N$ , for any comodule  $N$  in  $C\text{-Comod}_{f_c}^E$ , where  $\text{res}_U: \mathbb{Z}^{(I_E)} \rightarrow \mathbb{Z}^U$  is the restriction homomorphism.

We define the functor  $H_E$  on objects by setting  $H_E(E_0, E_1, \psi) = (h_E^{E_1}, h_E^{E_0}, h_E^\psi)$ , and on morphisms by setting  $H_E(f_0, f_1) = (h_E^{f_1}, h_E^{f_0})$ . A direct calculation shows that  $(h_E^{E_1}, h_E^{E_0}, h_E^\psi)$  belongs to  $\mathcal{P}_1(R_E^{op})$  if  $(E_0, E_1, \psi) \in \mathcal{M}ap_1(E)$  and that  $H_E$  is well defined.

For a purpose of next steps of the proof (and in order to see a nature of  $\mathcal{M}ap_1(E)$  as the bimodule problem in the sense of Drozd [5], see also [4, 17]), we give a different detailed proof of the above fact.

Let  $\mathbb{K} = \text{add}(E)$  be the full additive subcategory of  $C\text{-Comod}$  formed by finite direct sums of the injective  $C$ -comodules  $E(u)$ , with  $u \in U$ , and let  $\mathbb{H} = \mathbb{H}^E$  be the  $\mathbb{K}$ - $\mathbb{K}$ -bimodule  $\mathbb{H}(-, \cdot) = \mathbb{H}^E(-, \cdot): \mathbb{K}^{op} \times \mathbb{K} \rightarrow \mathbf{mod}K$  defined by the formula

$$\mathbb{H}(E', E'') = \{g \in \text{Hom}_C(E', E''); \psi(\text{soc } E') = 0\} \subseteq \text{Hom}_C(E', E''),$$

with  $E', E'' \in \mathbb{K}$ . Note that  $\mathbb{H}(E, E) = \{\psi \in \text{End}_C E; \psi(\text{soc } E) = 0\} = J(R_E)$  is the Jacobson radical of the algebra  $R_E$ .

We construct  $H_E$  as the composite functor

$$\mathcal{M}ap_1(E) \xrightarrow[\simeq]{H'} \mathbf{Mat}(\mathbb{K}\mathbb{H}_{\mathbb{K}}^E) \xrightarrow[\simeq]{H'} \mathcal{P}_1(R_E^{op}), \quad (3.6)$$

where  $\mathbf{Mat}(\mathbb{K}\mathbb{H}_{\mathbb{K}}^E)$  is the additive  $K$ -category of  $\mathbb{K}\mathbb{H}_{\mathbb{K}}^E$ -matrices in the sense of Drozd [5], see also [4], [10], [18] (Chap. 17), [20] (Sec. 2) for details. Recall that the objects of  $\mathbf{Mat}(\mathbb{K}\mathbb{H}_{\mathbb{K}}^E)$  are the triples  $(E', E'', \psi)$ , where  $E', E'' \in \text{ob}\mathbb{K}$  and  $\psi \in \mathbb{H}(E', E'')$ , and morphisms are defined in a natural way.

The functor  $H'$  is defined by attaching to any object  $(E_0, E_1, \psi)$  of  $\mathcal{M}ap_1(E)$ , with  $\psi \in \text{Hom}_C(E_0, E_1) = \mathbb{H}(E_0, E_1)$  and  $E_0, E_1 \in \mathbb{K}$ , the triple  $H'(E_0, E_1, \psi) = (E_0, E_1, \psi)$ , viewed as an object of  $\mathbf{Mat}(\mathbb{K}\mathbb{H}_{\mathbb{K}}^E)$ . Given a morphism  $(f_0, f_1) : (E_0, E_1, \psi) \rightarrow (E'_0, E'_1, \psi')$ , we set  $H'(f_0, f_1) = (f_0, f_1)$ . It is easy to see that  $H'$  is a  $K$ -linear equivalence of categories.

Now we construct the functor  $H''$ . In the notation of [20] (Sec. 2), we denote by  $R_E\text{-pr}$  the category of finitely generated projective left  $R_E$ -modules and we define the Nakayama equivalence  $\omega : \mathbb{K} \xrightarrow{\simeq} (R_E\text{-pr})^{op}$  that associates with any object  $x$  of  $\mathbb{K}$  the finitely generated projective left  $R_E$ -module  $\omega(x) = h_E^x = \text{Hom}_C(x, E)$ . Hence, by applying the formula (2.9) in [20] to  $\mathbf{K} = \mathbf{L} = \mathbb{K} = \text{add}(E)$  and the bimodule  $\mathbf{M} = \mathbb{H}$ , we conclude that, for any pair  $x = E', y = E''$  of objects in  $\mathbb{K}$ , the (contravariant!) functor  $\omega$  induces the natural isomorphisms

$$\begin{aligned} \mathbb{H}(E', E'') &= \mathbb{H}(x, y) \cong \text{Hom}_{R_E}(h_E^y, \mathbb{H}(x, E)) \\ &\cong \text{Hom}_{R_E}(h_E^y, \mathbb{H}(E, E) \otimes_{R_E} h_E^x) \\ &\cong \text{Hom}_{R_E}(h_E^y, J(R_E) \otimes_{R_E} h_E^x) \\ &\cong \text{Hom}_{R_E}(h_E^y, \text{rad}h_E^x) = \text{Hom}_{R_E}(h_E^{E''}, \text{rad}h_E^{E'}) \\ &\cong \text{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^y, h_E^x) \\ &\cong \text{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_E^{E''}, h_E^{E'}), \end{aligned} \quad (3.7)$$

where  $J(R_E)^+ = \text{Hom}_{R_E}(J(R_E), R_E)$  is viewed as an  $R_E$ - $R_E$ -bimodule.

Hence, if  $(E_0, E_1, \psi)$  is an object of  $\mathcal{M}ap_1(E)$  (or of  $\mathbf{Mat}(\mathbb{K}\mathbb{H}_{\mathbb{K}})$ ) then  $\psi \in \mathbb{H}(E', E'')$  and its image  $\widehat{\psi} : h_E^{E''} \rightarrow \text{rad}h_E^{E'}$  under the composite isomorphism (3.7) is such that  $h_E^\psi = u \cdot \widehat{\psi}$ , where  $u : \text{rad}h_E^{E'} \hookrightarrow h_E^{E'}$  is the imbedding. It follows that  $(h_E^{E''}, h_E^{E'}, h_E^\psi)$  lies in  $\mathcal{P}_1(R_E)$  if and only if  $(E_0, E_1, \psi)$  lies in  $\mathcal{M}ap_1(E)$ . We define  $H''$  (and  $H_E$ ) on objects  $(E_0, E_1, \psi)$  by setting

$$H''(E_0, E_1, \psi) = H_E(E_0, E_1, \psi) = (h_E^{E''}, h_E^{E'}, h_E^\psi),$$

and on morphisms  $(f_0, f_1)$  by  $H''(f_0, f_1) = H^E(f_0, f_1) = (h_E^{f_1}, h_E^{f_0})$ . Obviously,  $H = H'' \circ H'$ . Since, up to isomorphism, all objects of  $\mathcal{P}_1(R_E)$  are of the form  $(h_E^{E''}, h_E^{E'}, h_E^\psi)$ , with  $(E_0, E_1, \psi) \in \mathcal{M}ap_1(E)$ , the functors  $H''$  and  $H_E$  are equivalences of categories making the square in (3.5) commutative.

(a) The fact that the functors **ker** and **cok** are full and dense follows immediately from the definitions. It is easy to see that  $(P_1, P_0, \phi)$  is an object of  $\mathcal{P}_1(R_E)$  if and only if  $P_1 \xrightarrow{\phi} P_0 \rightarrow \text{Coker}\phi \rightarrow 0$  is a minimal

projective presentation of  $\text{Coker}\psi$  in  $\text{mod}(R_E^{op})$ . Analogously,  $(E_0, E_1, \psi)$  is an object of  $\mathcal{M}ap_1(E)$  if and only if  $0 \rightarrow \text{Ker}\psi \rightarrow E_0 \xrightarrow{\psi} E_1$  is a minimal injective  $E$ -copresentation of  $\text{Ker}\psi$ . Hence easily follows that the functors  $\mathbf{cok}_E$  and  $\mathbf{ker}_E$  restrict to the representation equivalences  $\mathbf{ker}_E: \mathcal{M}ap_2(E) \rightarrow C\text{-Comod}_{f_c}^E$  and  $\mathbf{cok}_E: \mathcal{P}_2(R_E^{op}) \rightarrow \text{mod}(R_E^{op})$ . The remaining statements in (a) follow from the definitions and [3] (Sec. 6).

(b) Let  $N$  be an indecomposable comodule in  $C\text{-Comod}_{f_c}^E$ . Then  $N$  admits a minimal injective  $E$ -copresentation  $0 \rightarrow N \rightarrow E_0 \xrightarrow{\psi} E_1$  in  $C\text{-Comod}$ , with  $E_0, E_1 \in \text{add}(E)$  and, therefore,  $(E_0, E_1, \psi)$  is an object of  $\mathcal{M}ap_1(E)$ . It follows that

$$H_E(E_0, E_1, \psi) = (h_{E_1}^{E_1}, h_{E_0}^{E_0}, h_{E_1}^{\psi}) \in \mathcal{P}_2(R_E)$$

and, hence,  $h_{E_1}^{E_1} \xrightarrow{h_{E_0}^{\psi}} h_{E_0}^{E_0} \rightarrow h_{E_1}^N \rightarrow 0$  is a minimal projective presentation of  $h_{E_1}^N$  in  $\text{mod}R_E^{op}$ . Hence the equalities  $\mathbf{cdn}(N) = \mathbf{cdn}(E_0, E_1, \psi) = \sigma(\mathbf{cdn}H_E(E_0, E_1, \psi))$  easily follow. The equality  $\sigma(\mathbf{cdn}H_E(E_0, E_1, \psi)) = \underline{\dim}G^{-1}H_E(E_0, E_1, \psi)$  is proved in [7] (Sec. 5) and [3] (Sec. 6).

(c) First we show that the functor  $G$  in (3.5) is the composite functor

$$\mathcal{P}_1(R_E^{op}) \xleftarrow[\simeq]{G'} \widehat{R}_E\text{-mod}_{pr}^{pr} \xleftarrow[\simeq]{G''} \text{rep}_K(\mathbf{B}_E), \tag{3.8}$$

where  $\widehat{R}_E\text{-mod}_{pr}^{pr}$  is the additive  $K$ -category of finite-dimensional propartite left modules over the finite-dimensional bipartite  $K$ -algebra

$$\widehat{R}_E = \begin{bmatrix} R_E & J(R_E)^+ \\ 0 & R_E \end{bmatrix} \tag{3.9}$$

in the sense of [20], with  $J(R_E)^+ = \text{Hom}_{R_E}(J(R_E), R_E)$ . First we note that if  $X = (X', X'', \xi: J(R_E)^+ \otimes_{R_E} X' \rightarrow X'')$ , is a propartite left  $\widehat{R}_E$ -module then, up to isomorphism, the projective left  $R_E$ -modules  $X', X''$  have the forms  $X' = h_{E'}^{E''}$ ,  $X'' = h_{E'}^{E'}$ , where  $E', E'' \in \text{add}(E)$ . Then, in view of the isomorphisms

$$\text{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_{E'}^{E''}, h_{E'}^{E'}) \cong \text{Hom}_{R_E}(h_{E'}^{E''}, J(R_E) \otimes_{R_E} h_{E'}^{E'}) \cong \text{Hom}_{R_E}(h_{E'}^{E''}, \text{rad}h_{E'}^{E'})$$

given in (3.7), we can view  $X$  as the triple  $X = (X', X'', \tilde{\xi})$ , where  $\tilde{\xi} = u \circ \bar{\xi}$  is the composition  $h_{E'}^{E''} \xrightarrow{\bar{\xi}} \text{rad}h_{E'}^{E'} \xrightarrow{u} h_{E'}^{E'}$  of the image  $\bar{\xi}$  of  $\xi \in \text{Hom}_{R_E}(J(R_E)^+ \otimes_{R_E} h_{E'}^{E''}, h_{E'}^{E'})$ , under the composite isomorphism, with the canonical imbedding  $u$ . In other words, the triple  $G'(X) = (X', X'', \phi) = (h_{E'}^{E''}, h_{E'}^{E'}, \phi)$  is an object of  $\mathcal{P}_1(R_E^{op})$ . This defines the equivalence  $G'$ , and we set  $G'' = G \circ (G')^{-1}$ . It is clear that the functor  $T_K = (G'')^{-1}$  is the equivalence  $T_K: \widehat{R}_E\text{-mod}_{pr}^{pr} \xrightarrow{\simeq} \text{rep}_K(\mathbf{B}_E)$  defined in [20] ((4.11)).

Following an observation of Drozd [7] (see also [3] and [20, p. 44, 45]), given a finitely generated  $K$ -algebra  $S$ , the category  $\text{rep}(\mathbf{B}_E, S)$  of right  $S$ -module representations of the boc  $\mathbf{B}_E = (A, {}_A V_A)$  has as objects the  $A$ - $S$ -bimodules  ${}_A X_S$  in  $\text{mod}_{f_p}(A \otimes S^{op})$  (the category of finitely presented left  $(A \otimes S^{op})$ -modules), which are finitely generated projective, when viewed as right  $S$ -modules, see [7], [3] and [20, p. 44, 45] for details. We set  $\text{rep}_K(\mathbf{B}_E) = \text{rep}(\mathbf{B}_E, K)$ .

By [20] (Proposition 4.9), there is an equivalence of categories

$$T_S: (\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr} \xrightarrow{\simeq} \text{rep}(\mathbf{B}_E, S), \tag{3.10}$$

for any finitely generated  $K$ -algebra  $S$ , where

$$(\widehat{R}_E \otimes S^{op}) = \begin{bmatrix} R_E \otimes S^{op} & J(R_E)^+ \otimes S^{op} \\ 0 & R_E \otimes S^{op} \end{bmatrix}.$$

The objects of  $(\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr}$  are  $\widehat{R}_E$ - $S$ -bimodules that are  $(R_E \otimes S^{op})$ - $(R_E \otimes S^{op})$ -propartite and finitely generated projective as left  $S$ -modules.

Following the above construction of the functor  $G'$ , we can construct equivalences of categories

$$\mathcal{P}_1((R_E \otimes S^{op})^{op}) \xleftarrow{G'_{E,S}} (\widehat{R}_E \otimes S^{op})\text{-mod}_{pr}^{pr} \xleftarrow{G''_{E,S}} \text{rep}(\mathbf{B}_E, S), \tag{3.11}$$

and we extend the diagram (3.5) to the following commutative diagram:

$$\begin{array}{ccc} \text{Map}_1(E \otimes S^{op}) & \xrightarrow[\cong]{H_{E,S}} & \mathcal{P}_1((R_E \otimes S^{op})^{op}) \xrightarrow[\cong]{G_{E,S}} \text{rep}(\mathbf{B}_E, S) \\ \ker \downarrow & & \text{cok} \downarrow \\ (C \otimes S^{op})\text{-Comod}_{f_c}^{E \otimes S^{op}} & \xrightarrow[\cong]{h_S^\bullet} & \text{mod}((R_E \otimes S^{op})^{op}), \end{array} \tag{3.12}$$

where  $G_{E,S} = G'_{E,S} \circ G''_{E,S}$  and  $T_S^{-1} = G''_{E,S}$ . We set  $\widehat{C} = C \otimes S^{op}$  and view it as an  $S^{op}$ -coalgebra with the comultiplication  $\widehat{\Delta} = \Delta \otimes S^{op}$  and the counit  $\widehat{\varepsilon} = \varepsilon \otimes S^{op}$ . Then  $\widehat{E} = E \otimes S^{op}$  is an injective object in the category  $\widehat{C}\text{-Comod}$  of left  $\widehat{C}$ -comodules, which is projective, when viewed as a right  $S$ -module.

We define  $\widehat{C}\text{-Comod}_{f_c}^{\widehat{E}} = (C \otimes S^{op})\text{-Comod}_{f_c}^{E \otimes S^{op}}$  to be the full subcategory  $\widehat{C}\text{-Comod}$  whose objects are the finitely  $\widehat{E}$ -copresented  $\widehat{C}$ -comodules, i.e., finitely  $E \otimes S^{op}$ -copresented  $\widehat{C}$ -bicomodules. The categories  $\text{Map}_1(E \otimes S^{op})$ ,  $\mathcal{P}_1(R_E \otimes S^{op})$ , and the functors  $\mathbf{ker} = \mathbf{ker}_{E \otimes S^{op}}$ ,  $\mathbf{cok} = \mathbf{cok}_{R_E \otimes S^{op}}$  are defined in an obvious way.

We only prove that the functor  $h_S^\bullet: (C \otimes S^{op})\text{-Comod}_{f_c}^{E \otimes S^{op}} \rightarrow \text{mod}((R_E \otimes S^{op})^{op})$  in (3.12) defined by  $Z \mapsto h_S^Z = \text{Hom}_{\widehat{C}}(Z, \widehat{E})$ , is an equivalence of categories. The fact that  $H_{E,S}$  is an equivalence of categories can be proved by applying the properties of  $h_S^\bullet$  and the isomorphism

$$\chi_{E',E''}: \text{Hom}_C(E', E'') \otimes S^{op} \rightarrow \text{Hom}_{\widehat{C}}(E' \otimes S^{op}, E'' \otimes S^{op}), \tag{3.13}$$

with  $E', E'' \in \text{add}(E)$ , given by  $g \otimes s \mapsto [(g \otimes \text{id}) \cdot s: E' \otimes S^{op} \rightarrow E'' \otimes S^{op}]$ , because the bimodule problem arguments used above extend almost verbatim to our situation. The homomorphism  $\chi_{E',E''}$  is an isomorphism of  $S$ -modules for each pair  $E', E''$  of comodules in  $\text{add}(E)$ , because it is functorial with respect to homomorphisms  $E' \rightarrow E'_1$  and  $E'' \rightarrow E''_1$  of  $C$ -comodules and it is proved in [28] ((2.10)) that  $\chi_{E',E''}$  is bijective for  $E' = E'' = E$  if the algebra  $R_E$  is finite-dimensional.

Hence easily follows that a left  $\widehat{C}$ -comodule  $Z$  lies in  $(C \otimes S^{op})\text{-Comod}_{f_c}^{E \otimes S^{op}}$  if and only if there is an exact sequence  $0 \rightarrow Z \rightarrow E_0 \otimes S^{op} \rightarrow E_1 \otimes S^{op}$ , with  $E_0, E_1 \in \text{add}(E)$ . By applying  $\text{Hom}_{\widehat{C}}(-, E \otimes S^{op})$  and the isomorphism  $\chi_{E',E''}$ , we get the exact sequence

$$h_E^{E'_1} \otimes S^{op} \rightarrow h_E^{E_0} \otimes S^{op} \rightarrow h_S^Z \rightarrow 0$$

of left  $(R_E \otimes S^{op})$ -modules that is a projective presentation of  $h_S^Z = \text{Hom}_{\widehat{C}}(Z, E \otimes S^{op})$ . Hence, we conclude that the functor  $h_S^\bullet$  in (3.12) is an equivalence of categories. It follows that the functor  $H_{E,S}$  in (3.12) is an equivalence of categories making the diagram (3.12) commutative.

Note that, by [20] (Proposition 4.9(b)) and the definition of the functors  $G_{E,S}$ ,  $H_{E,S}$  in (3.12) and the functors  $G_E$  and  $H_E$  in (3.5), for every module  $L$  in the category  $\text{fin}(S^{op})$  of finite-dimensional left  $S$ -modules and every  $\widehat{R}_E$ - $S$ -bimodule  $\widehat{R}_E X_S$  in the category  $\mathcal{R}_1((R_E \otimes S^{op})^{op})$  there exist isomorphisms

$$G_E^{-1}(\widehat{R}_E X \otimes_S L) \cong G_{E,S}^{-1}(\widehat{R}_E X_S) \otimes_S L,$$

and

$$H_E^{-1}(\widehat{R}_E X \otimes_S L) \cong H_{E,S}^{-1}(\widehat{R}_E X_S) \otimes_S L$$

that are functorial with respect to the  $S$ -module homomorphisms  $L \rightarrow L'$  and  $\widehat{R}_E$ - $S$ -bimodule homomorphisms  $\widehat{R}_E X_S \rightarrow \widehat{R}_E X'_S$ .

By applying the diagram (3.12), we reduce the proof of (c) to [7] (Propositions 11 and 13), and to [3] (Theorem B). Here we follow closely the notation and the proof of [3] (Theorem B). We recall that our functor  $G_E$  in (3.5) is just the functor  $\Xi: \text{rep}_K(\mathbf{B}_E) \rightarrow \mathcal{P}_1(R_E)$  in [3, p. 476], where  $\text{rep}_K(\mathbf{B}_E) = \text{rep}(\mathbf{B}_E, K)$ .

Assume that the category  $C\text{-Comod}_{fc}^E$  is not  $K$ -wild. Then the category  $C\text{-Comod}_{fc}^E$  is not  $K$ -wild and, by [28] (Proposition 2.8 (a)), the finite-dimensional  $K$ -algebras  $R_E$  and  $R_E^{op}$  are not wild. Hence, according to [3] (Theorem B) and its proof, the category  $\text{rep}_K(\mathbf{B}_E)$  is not wild and there exist minimal bocses  $\mathbf{B}_1, \dots, \mathbf{B}_n$ , with  $\mathbf{B}_i = (B_i, W_i)$ , finitely generated  $R_E$ - $B_i^{op}$ -bimodules  $T'_i$  and full functors  $F'_i: \text{rep}_K(\mathbf{B}_i) \rightarrow R_E\text{-mod}$  which reflect isomorphisms such that the conditions (c1), (c2) and (c3) stated in (c) are satisfied with  $C\text{-Comod}_{fc}^E$ ,  $F'_i: \text{rep}_K(\mathbf{B}_i) \rightarrow C\text{-Comod}_{fc}^E$  and  $R_E\text{-mod}$ ,  $F'_i: \text{rep}_K(\mathbf{B}_i) \rightarrow R_E\text{-mod}$  interchanged. Moreover, it is shown in the proof of [3] (Theorem B) that, for each  $i = 1, \dots, n$ , the  $R_E$ - $B_i^{op}$ -bimodules  $T'_i$  are of the form  $T'_i = \text{cok}_{B_i}(\widehat{T}'_i)$ , where  $\widehat{T}'_i \in \mathcal{P}_1(R_E \otimes B_i^{op})$ , and  $\widehat{F}'_i(X) = \widehat{T}'_i \otimes_{B_i} X$ , for all representations  $X$  of the boc  $\mathbf{B}_i$ .

Let  $\widehat{T}_i = H_{E,B_i}^{-1}(T'_i) \in \text{Map}_1(E \otimes B_i^{op})$  be the preimage of  $T'_i$  under the functor  $H_{E,S}$  in (3.12), with  $S = B_i$ . Finally, let  $T_i = \ker(\widehat{T}_i) \in \widehat{C}\text{-Comod}_{fc}^E$  be the image of  $\widehat{T}_i$  under the functor  $\ker$  in (3.12), applied to  $S = B_i$ . Then  $T_i$  is a finitely  $E$ -copresented  $C$ - $B_i$ -bicomodule and we set  $F_i(-) = T_i \otimes_{B_i} (-)$ .

In view of (a), (b) and the properties of the functors  $F'_i: \text{rep}_K(\mathbf{B}_i) \rightarrow R_E\text{-mod}$  listed above, the conditions (c<sub>1</sub>)–(c<sub>3</sub>) are satisfied, because the arguments given in the proof of [3] (Theorem B) extends almost verbatim. The details are left to the reader.

**Corollary 3.1.** *Under the assumption made in Theorem 3.1, for a given socle-finite injective direct summand  $E$  of  ${}_C C$  such that  $\dim_K \text{End}_C E < \infty$ , the following conditions are equivalent:*

- (a) *The category  $C\text{-Comod}_{fc}^E$  is  $K$ -wild.*
- (b)  *$C\text{-Comod}_{fc}^E$  is properly  $fc$ -wild (or smooth) [20] (Sec. 6), i.e., for every finitely generated  $K$ -algebra  $\Lambda$  (equivalently, for  $\Lambda = K\langle t_1, t_2 \rangle$ , or  $\Lambda = \Gamma_3(K)$ ) there exists a finitely  $E$ -copresented  $C$ - $\Lambda$ -bicomodule  ${}_C N_\Lambda$  that induces a representation imbedding  ${}_C N \otimes_\Lambda (-): \text{fin}(\Lambda^{op}) \rightarrow C\text{-Comod}_{fc}^E$ .*
- (c) *The finite-dimensional  $K$ -algebras  $R_E^{op}$  and  $R_E$  are wild.*
- (d) *The additive  $K$ -category  $\text{rep}_K(\mathbf{B}_E)$  is wild, where  $\mathbf{B}_E$  is the Roiter boc of  $R_E^{op}$ , see (3.5).*
- (e) *The additive  $K$ -category  $\widehat{R}_E\text{-mod}_{pr}^{pr}$  is wild, where  $\widehat{R}_E$  is the bipartite algebra (3.9).*

**Proof.** Since the functor  $h_E^\bullet: C\text{-Comod}_{f_c}^E \rightarrow R_E\text{-mod}$  in (3.5) is an exact equivalence of categories, condition (a) implies (c). The inverse implication (c)  $\Rightarrow$  (a) and the equivalence of (a) and (b) follows from [28] (Corollary 2.12). The implication (d)  $\Rightarrow$  (a) follows from Theorem 3.1 (c). The equivalence (d)  $\Leftrightarrow$  (e) follows from [20] (Proposition 4.9). Since (c)  $\Leftrightarrow$  (d) follows from [7] (Sec. 5) and [3], the proof is complete.

In the proof of the  $f_c$ -tame-wild dichotomy we use the following lemma:

**Lemma 3.2.** *Under the assumption made in Theorem 3.1, for a given socle-finite injective direct summand  $E = E_U$  of  ${}_C C$  such that  $R_E = \text{End}_C E$  is of finite dimension,*

- (a) *the  $f_c$ -tameness of the category  $C\text{-Comod}_{f_c}^E$  implies the tameness of the additive  $K$ -categories  $\text{Map}_1(E) \cong \text{rep}_K(\mathbf{B}_E) \cong \widehat{R}_E\text{-mod}_{pr}^{op}$  and the tameness of the algebras  $R_E$  and  $R^{op}$ , where  $\widehat{R}_E$  is the bipartite algebra (3.9) and  $\mathbf{B}_E$  is the Roiter boc of  $R_E^{op}$ , see (3.5),*
- (b) *given a proper bipartite vector  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$  we have*

$$\widehat{\mu}_C^1(v) = \widehat{\mu}_{\widehat{R}_E}^1(\sigma(v)) = \widehat{\mu}_{R_E^{op}}^1(\sigma(v)),$$

where  $\widehat{\mu}_{\widehat{R}_E}^1(\sigma(v))$  and  $\widehat{\mu}_{R_E^{op}}^1(\sigma(v))$  is the minimal cardinality of an almost parametrizing family for  $\text{ind}_{\sigma(v)}(\widehat{R}_E\text{-mod}_{pr}^{op})$  and  $\text{ind}_{\sigma(v)}(\text{mod}(R_E^{op}))$ , respectively.

**Proof.** Assume that the category  $C\text{-Comod}_{f_c}^E$  is  $f_c$ -tame, i.e., for any proper nonnegative bipartite vector  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$ , there exist a nonzero polynomial  $h \in K[t]$ ,  $C\text{-}K[t]_h$ -bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  that are finitely  $E\text{-}K[t]_h$ -copresented and form an almost parametrizing family for the family  $\text{ind}_v(C\text{-Comod}_{f_c}^E)$  of all indecomposable  $C$ -comodules  $M$  with  $\text{cdn}M = v$ . It follows that all  $L^{(j)}$  lie in  $C\text{-Comod}_{f_c}^{E \otimes K[t]_h}$ . Then, for each  $j \in \{1, \dots, r_v\}$ , there is an exact sequence

$$0 \rightarrow {}_C L_{K[t]_h}^{(j)} \rightarrow E_0^{(j)} \otimes K[t]_h \xrightarrow{\psi^{(j)}} E_1^{(j)} \otimes K[t]_h$$

in  $C\text{-Comod}_{f_c}^{E \otimes K[t]_h}$ , with  $E_0^{(j)}, E_1^{(j)}$  in  $\text{add}(E)$ , such that

$$\widehat{L}^{(j)} = (E_0^{(j)} \otimes K[t]_h, E_1^{(j)} \otimes K[t]_h, \psi^{(j)})$$

is an object of  $\text{Map}_1(E \otimes K[t]_h)$ , see (3.12). By applying Theorem 3.1, one can show that the objects  $\widehat{L}^{(1)}, \dots, \widehat{L}^{(r_v)}$  form a finitely  $E$ -copresented almost parametrizing family for  $\text{ind}_v(\text{Map}_1(E))$ , i.e., all but finitely many indecomposable objects  $(E', E'', g)$  in  $\text{Map}_1(E)$ , with  $\text{cdn}(E', E'', g) = v$ , are of the form

$$(E', E'', g) \cong \widehat{L}^{(s)} \otimes K[t]_h := (E_0^{(s)} \otimes K[t]_h, E_1^{(s)} \otimes K[t]_h, \psi^{(j)} \otimes K_\lambda^1),$$

where  $s \leq r_v$ ,  $K_\lambda^1 = K[t]/(t - \lambda)$  and  $\lambda \in K$ . This shows that the category  $\text{Map}_1(E)$  is tame. The functor  $G_S^{-1} \circ H_{E,S}$  in the diagram (3.12), with  $S = K[t]_h$ , carries each of the objects  $\widehat{L}^{(s)}$  to some object  $U^{(s)} \in \text{rep}(\mathbf{B}_E, K[t]_h)$  such that all but finitely many indecomposable objects  $X$  in  $\text{rep}_K(\mathbf{B}_E)$ , with  $\underline{\dim}(X) = \sigma(v)$ , are of the form  $X \cong U^{(j)} \otimes K_\lambda^1$ , where  $s \leq r_v$ . This shows that the category  $\text{rep}_K(\mathbf{B}_E)$  is tame and, by [3] (Sec. 6) and [7], the algebra  $R_E^{op}$  and  $R_E$  are tame. Since, by Proposition 4.9 (b) and Theorem 6.5 in [20], the category  $\text{rep}_K(\mathbf{B}_E)$  is tame if and only if  $\widehat{R}_E\text{-mod}_{pr}^{op}$  is tame, the proof of (a) is complete.

Moreover, it follows that, given a proper vector  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$ , any almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{f_c}^E)$  consisting of finitely  $E$ -copresented bicomodules  $L^{(1)}, \dots, L^{(r_v)}$  leads to an almost parametrizing family  $\widehat{L}^{(1)}, \dots, \widehat{L}^{(r_v)} \in \text{Map}_1(E \otimes S)$ , with  $S = K[t]_h$ , for  $\text{ind}_v(\text{Map}_1(E))$ . By applying the functor  $H_{E,S}$  in (3.12) and then the functor  $(G'_{E,S})^{-1}$  in (3.11), to  $\widehat{L}^{(1)}, \dots, \widehat{L}^{(r_v)}$ , we get an almost parametrizing family  $\widehat{\widehat{L}}^{(1)}, \dots, \widehat{\widehat{L}}^{(r_v)} \in (\widehat{R}_E \otimes S)\text{-mod}_{pr}^{pr}$ , for  $\text{ind}_v(\widehat{R}_E\text{-mod}_{pr}^{pr})$ . Since the vector  $v = (v'|v'')$  is proper, up to a localization of  $S = K[t]_h$ , by applying the functor  $\text{cok}$  in (3.12) we get an almost parametrizing family  $\text{cok}(\widehat{\widehat{L}}^{(1)}), \dots, \text{cok}(\widehat{\widehat{L}}^{(r_v)})$  for  $\text{ind}_{\sigma(v)}(\text{mod}(R_E^{op}))$ .

By Lemma 3.1, any finitely copresented family for  $\text{ind}_v(C\text{-Comod}_{f_c})$  can be corrected to a finitely  $E$ -copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{f_c}) = \text{ind}_v(C\text{-Comod}_{f_c}^E)$ , for any  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U$ . Hence (b) follows and the proof is complete.

We are now able to give an alternative proof of the  $f_c$ -tame-wild dichotomy for computable coalgebras established in [28].

**Theorem 3.2.** *Assume that  $C$  is a basic coalgebra over an algebraically closed field  $K$  such that  $\dim_K \text{Hom}_K(E', E'')$  is finite for each pair  $E', E''$  of indecomposable direct summands of  ${}_C C$ . Then  $C$  is either of tame  $f_c$ -comodule type or of wild  $f_c$ -comodule type, and these two types are mutually exclusive.*

**Proof.** Since  $C$  is basic,  ${}_C C$  has a decomposition (1.1). Assume that  $C$  is not of  $f_c$ -wild comodule type. To show that  $C$  is of  $f_c$ -tame comodule type, fix a nonnegative bipartite vector  $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} \cong K_0(C) \times K_0(C)$ . Since the support  $U_v = \text{supp}(v)$  of  $v$  is a finite subset of  $I_C$ , the injective  $C$ -comodule  $E = E_{U_v} = \bigoplus_{j \in U_v} E(j)$  is socle-finite and, according to our assumption the algebra  $R_E = \text{End}_C E$  is finite-dimensional. Moreover, every left  $C$ -comodule  $N$ , with  $\text{cdn}(N) = v$  lies in the subcategory  $C\text{-Comod}_{f_c}^E$  of  $C\text{-Comod}_{f_c}$ . Then  $\text{ind}_v(C\text{-Comod}_{f_c}) = \text{ind}_v(C\text{-Comod}_{f_c}^E)$  and, by our assumption, the category  $C\text{-Comod}_{f_c}^E$  is not of  $K$ -wild comodule type. Then, by Theorem 3.1, there exist minimal bocses  $\mathbf{B}_1, \dots, \mathbf{B}_n$ , with  $\mathbf{B}_i = (B_i, W_i)$ , finitely  $E \otimes R_i$ -copresented  $C\text{-}B_i$ -bicomodules  $T_i$  and full functors  $F_i(-) = T_i \otimes_{B_i} (-) : \text{rep}_K(\mathbf{B}_i) \rightarrow C\text{-Comod}_{f_c}^E$  which reflect isomorphisms such that the conditions (c<sub>1</sub>)–(c<sub>3</sub>) in Theorem 3.1 are satisfied. In particular, every indecomposable comodule  $N$  in  $C\text{-Comod}_{f_c}^E$  with  $\text{cdn}(N) = v$  is isomorphic to  $F_i(X)$ , for some  $i$  and some representation  $X$  in  $\text{rep}_K(\mathbf{B}_i)$ . Hence we conclude, as in the proof of [3] (Corollary C), that there is a finite set of pairs  $(R_i, L^{(i)})$ , where each  $R_i = K[t]_h$  is a localization of  $K[t]$  and  $L^{(i)}$  is a finitely  $E$ -copresented  $C\text{-}R_i$ -bicomodule such that

$$L^{(i)} \in (C \otimes R_i^{op})\text{-Comod}_{f_c}^{E \otimes R_i^{op}} \tag{3.14}$$

and all but finitely many indecomposable left  $C$ -comodules  $N$  in  $C\text{-Comod}_{f_c}$ , with  $\text{cdn}(N) = v$ , are of the form  $N \cong L^{(s)} \otimes Y$ , for some  $i$  and some indecomposable  $R_i$ -module  $Y$ . Hence we conclude, as in the proof of Theorem 14.18 in [18, p. 297], that there exist finitely  $E$ -copresented  $C\text{-}K[t]_h$ -bicomodules  $\widehat{L}^{(1)}, \dots, \widehat{L}^{(r_v)}$  such that all but finitely many indecomposable left  $C$ -comodules  $N$  in  $C\text{-Comod}_{f_c}$ , with  $\text{cdn}(N) = v$ , are of the form  $N \cong \widehat{L}^{(s)} \otimes K_\lambda^1$ , where  $s \leq r_v$ ,  $K_\lambda^1 = K[t]/(t - \lambda)$  and  $\lambda \in K$ . Consequently, the coalgebra is of  $f_c$ -tame comodule type.

It remains to prove that the coalgebra  $C$  cannot be both of  $f_c$ -tame and of  $f_c$ -wild comodule type. Assume to the contrary that  $C$  is of  $f_c$ -tame and of  $f_c$ -wild comodule type. Let  $T : \text{mod}\Gamma_3(K) \rightarrow C\text{-Comod}_{f_c}$  be an exact  $K$ -linear representation imbedding, where  $\Gamma_3(K) = \begin{bmatrix} K & K^3 \\ 0 & K \end{bmatrix}$ . Let  $S_1$  be the unique simple injective right  $\Gamma_3(K)$ -module, and let  $S_2$  be the unique simple projective right  $\Gamma_3(K)$ -module, up to isomorphism. Since  $T(S_1)$

and  $T(S_2)$  lie in  $C\text{-Comod}_{fc}$ , there are exact sequences  $0 \rightarrow T(S_1) \rightarrow E_0^{(1)} \rightarrow E_1^{(1)}$  and  $0 \rightarrow T(S_2) \rightarrow E_0^{(2)} \rightarrow E_1^{(2)}$ , where  $E_0^{(1)}, E_1^{(1)}, E_0^{(2)}, E_1^{(2)}$  are socle-finite injective  $C$ -modules.

Let  $E$  be a socle-finite direct summand of  $C$  such that the comodules  $E_0^{(1)}, E_1^{(1)}, E_0^{(2)}, E_1^{(2)}$  lies in  $\text{add}(E)$ . We show that  $\text{Im} T \subseteq C\text{-Comod}_{fc}^E$ . Indeed, if  $N = T(X)$  lies in  $\text{Im} F$ , where  $X$  is a module in  $\text{mod}\Gamma_3(K)$ , then there is an exact sequence  $0 \rightarrow S_2^n \rightarrow X \rightarrow S_1^m \rightarrow 0$ , with  $n, m \geq 0$ . Since  $T$  is exact, we get the exact sequence  $0 \rightarrow T(S_2)^n \rightarrow N \rightarrow T(S_1)^m \rightarrow 0$  in  $C\text{-Comod}$ . The comodules  $T(S_1)^m$  and  $T(S_2)^n$  obviously lie in  $C\text{-Comod}_{fc}^E$  and, hence, also  $N$  lies in  $C\text{-Comod}_{fc}^E$ . This shows that  $\text{Im} T \subseteq C\text{-Comod}_{fc}^E$  and, hence, the category  $C\text{-Comod}_{fc}^E$  is  $fc$ -wild and, according to Corollary 3.1, the finite-dimensional algebra  $R_E$  is wild.

On the other hand, in view of the  $fc$ -parametrization correction lemma (Lemma 3.1), the assumption that  $C$  is of  $fc$ -tame comodule type implies that  $C\text{-Comod}_{fc}^E$  is  $fc$ -tame. Hence, by Lemma 3.2, the finite-dimensional algebra  $R_E$  is tame and we get a contradiction with the tame-wild dichotomy [7] for finite-dimensional  $K$ -algebras.

We can now complete [28] (Proposition 2.8 (a)) as follows:

**Corollary 3.2.** *Under the assumption made in Theorem 3.1, for a given socle-finite injective direct summand  $E = E_U$  of  ${}_C C$  such that the algebra  $R_E = \text{End}_C E$  is of finite dimension, the following conditions are equivalent:*

- (a) *The category  $C\text{-Comod}_{fc}^E$  is  $fc$ -tame.*
- (b) *The finite-dimensional  $K$ -algebra  $R_E$  is tame.*
- (c) *The additive  $K$ -categories  $\text{Map}_1(E) \cong \text{rep}_K(\mathbf{B}_E)$  are tame, where  $\mathbf{B}_E$  is the additive Roiter boc of  $R_E^{\text{op}}$ , see (3.5).*
- (d) *The additive  $K$ -category  $\widehat{R}_E\text{-mod}_{pr}^{\text{pr}}$  is tame, where  $\widehat{R}_E$  is the bipartite algebra (3.9).*

Moreover, if  $C\text{-Comod}_{fc}^E$  is  $fc$ -tame then, given a proper bipartite vector  $v = (v'|v'') \in \mathbb{Z}^U \times \mathbb{Z}^U \subseteq K_0(C) \times K_0(C)$ , we have  $\widehat{\mu}_C^1(v) = \widehat{\mu}_{R_E}^1(\sigma(v)) = \widehat{\mu}_{R_E^{\text{op}}}^1(\sigma(v))$ . In particular,  $C\text{-Comod}_{fc}^E$  is of polynomial growth if and only if  $\widehat{R}_E\text{-mod}_{pr}^{\text{pr}}$  is of polynomial growth.

**Proof.** The equivalence (b)  $\Leftrightarrow$  (c) follows from the theorem of Drozd [7] (see also [3], [28] (Proposition 2.8) and from the proof of Theorem 3.1. The equivalence (c)  $\Leftrightarrow$  (d) follows from [20] (Theorem 6.5) (or from the proof of Theorem 3.1). To prove (c)  $\Rightarrow$  (a), note that, according to [7], if  $\text{rep}_K(\mathbf{B}_E)$  is tame, it is not wild. Then, by Theorem 3.2 and its proof, the category  $C\text{-Comod}_{fc}^E$  is  $fc$ -tame. Since (a)  $\Rightarrow$  (c) follows from Lemma 3.2 (a), the conditions (a)–(d) are equivalent. The remaining statement follows from Lemma 3.2 (b).

**Corollary 3.3.** *Let  $C$  be a basic coalgebra over an algebraically closed field  $K$  such that  $\dim_K \text{Hom}_K(E', E'')$  is finite for each pair  $E', E''$  of indecomposable direct summands of  ${}_C C$ . The following conditions are equivalent:*

- (a) *The coalgebra  $C$  is of tame  $fc$ -comodule type.*
- (b) *For any proper bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , there is a finitely  $E_{U_v}$ -copresented almost parametrizing family for  $\text{ind}_v(C\text{-Comod}_{fc}) = \text{ind}_v(C\text{-Comod}_{fc}^{E_{U_v}})$ , where  $U_v = \text{supp}(v) \subseteq \mathbb{Z}^{(I_C)}$  is the support of  $v$  and  $E_{U_v} = \bigoplus_{j \in U_v} E(j)$ .*
- (c) *For any socle-finite direct summand  $E$  of  ${}_C C$ ,  $C\text{-Comod}_{fc}^E$  is  $fc$ -tame.*
- (d) *For any socle-finite direct summand  $E$  of  ${}_C C$ ,  $C\text{-Comod}_{fc}^E$  is not  $fc$ -wild.*



- (e) For any socle-finite direct summand  $E$  of  ${}_C C$ , the finite-dimensional  $K$ -algebra  $R_E = \text{End}_C E$  is tame.
- (f) For any socle-finite direct summand  $E$  of  ${}_C C$ , the category  $\widehat{R}_E\text{-mod}_{pr}^{pr}$  is tame, where  $\widehat{R}_E$  is the bipartite algebra (3.9).

The coalgebra  $C$  is of  $f$ -discrete comodule type if and only if, for any proper bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , the family  $\text{ind}_v(C\text{-Comod}_{f_C}^{E_{U_v}})$ , with  $U_v = \text{supp}(v) \subseteq \mathbb{Z}^{(I_C)}$ , is finite up to isomorphism, or equivalently, the family  $\text{ind}_v(\widehat{R}_{E_{U_v}}\text{-mod}_{pr}^{pr})$  is finite up to isomorphism.

**Proof.** The implication (b)  $\Rightarrow$  (a) is obvious. The implication (c)  $\Rightarrow$  (b) and the equivalence of the statements (c)–(f) is an immediate consequence of previous results.

To prove (a)  $\Leftrightarrow$  (b), we fix a proper bipartite vector  $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$  and set  $U_v = \text{supp}(v)$ ,  $E_{U_v} = \bigoplus_{j \in U_v} E(j)$ . It is clear that  $\text{ind}_v(C\text{-Comod}_{f_C}) = \text{ind}_v(C\text{-Comod}_{f_C}^{E_{U_v}})$ . Since  $C$  is  $f$ -tame, there are finitely copresented  $C\text{-}K[t]_h\text{-bimodules } L^{(1)}, \dots, L^{(r_v)}$  forming an almost parametrizing family of for  $\text{ind}_v(C\text{-Comod}_{f_C})$ . By Lemma 3.1, the family corrects to an almost parametrizing family  $\widetilde{L}^{(1)}, \dots, \widetilde{L}^{(r_v)}$  for  $\text{ind}_v(C\text{-Comod}_{f_C}) = \text{ind}_v(C\text{-Comod}_{f_C}^{E_{U_v}})$  consisting of finitely  $E_{U_v}$ -copresented bicomodules. Hence (b) follows and the conditions (a)–(f) are equivalent. Since the remaining statement of corollary is a consequence of Lemma 3.2 (b), the proof is complete.

#### 4. A Geometry Context for Computable Coalgebras

Throughout we assume that  $K$  is an algebraically closed field and  $C$  a basic computable  $K$ -coalgebra with a fixed decomposition  ${}_C C = \bigoplus_{j \in I_C} E(j)$  (1.1). Following [7, 17, 19, 20], we introduce in Definitions 4.1 and 4.2 a geometry context for a coalgebra  $C$ , compare with [15]. We use it in the study of comodules over a  $K$ -coalgebra  $C$  by applying the geometry of orbits. In particular, we give a geometric characterization of  $f$ -tame coalgebras.

**Definition 4.1.** Given a computable  $K$ -coalgebra  $C$  (1.1) and a bipartite nonnegative vector  $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$ , we define an action

$$* : \mathbf{G}_v^C \times \mathbf{Map}_v^C \longrightarrow \mathbf{Map}_v^C \tag{4.1}$$

of an algebraic (parabolic) group  $\mathbf{G}_v^C$  on an affine  $K$ -variety  $\mathbf{Map}_v^C$  as follows:

- (a)  $\mathbf{G}_v^C = \text{Aut}_C \mathbf{E}(v') \times \text{Aut}_C \mathbf{E}(v'')$  viewed as an algebraic group with respect to Zariski topology, where  $\mathbf{E}(v') = \bigoplus_{i \in I_C} E(j)^{v'_i}$  and  $\mathbf{E}(v'') = \bigoplus_{j \in I_C} E(j)^{v''_j}$  are the standard injective  $C$ -comodules (2.2) with  $\text{lgth} \mathbf{E}(v') = (v'|0)$  and  $\text{lgth} \mathbf{E}(v'') = (v''|0)$ .
- (b)  $\mathbf{Map}_v^C = \{\psi \in \text{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v'')); \psi(\text{soc } \mathbf{E}(v')) = 0\} \subseteq \text{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v''))$  is viewed as an affine  $K$ -variety (Zariski closed subset of the affine space  $\text{Hom}_C(\mathbf{E}(v'), \mathbf{E}(v''))$ ) of finite  $K$ -dimension.
- (c) The algebraic group (left) action (4.1) of  $\mathbf{G}_v^C$  on  $\mathbf{Map}_v^C$  is defined by the conjugation  $(f', f'') * \psi = f'' \circ \psi \circ (f')^{-1}$ , where  $\psi \in \mathbf{Map}_v^C$ ,  $f' \in \text{Aut}_C \mathbf{E}(v')$  and  $f'' \in \text{Aut}_C \mathbf{E}(v'')$ .

**Definition 4.2.** Given a computable  $K$ -coalgebra  $C$  and a bipartite nonnegative vector  $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} = K_0(C) \times K_0(C)$ , the open subset

$$\mathbf{Comod}_v^C = \{\psi \in \mathbf{Map}_v^C; \text{soc } \mathbf{E}(v'') \subseteq \text{Im } \psi\} \tag{4.2}$$

of the variety  $\mathbf{Map}_v^C$  is called a variety of  $C$ -comodules  $N$  with  $\text{cdn}(N) = v$ .

We start with the following useful facts:

**Lemma 4.1.** *Let  $C$  be a computable  $K$ -coalgebra and  $v = (v'|v'') \in \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)} = K_0(C) \times K_0(C)$  a nonnegative bipartite vector.*

- (a)  $\mathbf{Comod}_v^C$  is a  $\mathbf{G}_v^C$ -invariant and Zariski open subset of the affine variety  $\mathbf{Map}_v^C$ .
- (b) The map  $\psi \mapsto \mathbf{Ker} \psi$  defines a bijection between the  $\mathbf{G}_v^C$ -orbits of  $\mathbf{Comod}_v^C$  and the isomorphism classes of comodules  $N$  in  $C\text{-Comod}_{f_C}$  such that  $\mathbf{cdn}(N) = v$ .

**Proof.** (a) To see that  $\mathbf{Comod}_v^C$  is a Zariski open subset of  $\mathbf{Map}_v^C$ , note that, given  $a \in \mathbf{supp}(v'') \subseteq I_C$ , the subset  $\mathfrak{D}_a$  of  $\mathbf{Map}_v^C$  consisting of all  $\psi \in \mathbf{Map}_v^C$  such that  $\psi: \mathbf{E}(v') \rightarrow \mathbf{E}(v'')$  has a factorization through the subcomodule  $\mathbf{E}(v'')_a = \bigoplus_{j \neq a} E(j)^{v''_j}$  of  $E(v'')$  is Zariski closed. Since the set  $\mathbf{supp}(v'')$  is finite,  $\mathfrak{D} = \bigcup_{a \in \mathbf{supp}(v'')} \mathfrak{D}_a$  is closed and therefore  $\mathbf{Comod}_v^C = \mathbf{Map}_v^C \setminus \mathfrak{D}$  is open. The fact that  $\mathbf{Comod}_v^C$  is a  $\mathbf{G}_v^C$ -invariant subset of  $\mathbf{Map}_v^C$  follows by applying the definitions.

(b) Note that a  $C$ -comodule homomorphism  $\psi: \mathbf{E}(v') \rightarrow \mathbf{E}(v'')$  is an element of  $\mathbf{Comod}_v^C$  if and only if  $0 \rightarrow \mathbf{Ker} \psi \rightarrow \mathbf{E}(v') \xrightarrow{\psi} \mathbf{E}(v'')$  is a minimal injective copresentation of  $\mathbf{Ker} \psi$  in  $C\text{-Comod}_{f_C}$ . Hence every comodule  $N$  in  $C\text{-Comod}_{f_C}$ , with  $\mathbf{cdn}(N) = v$ , is isomorphic to  $\mathbf{Ker} \psi$ , for some  $\psi: \mathbf{E}(v') \rightarrow \mathbf{E}(v'')$  in  $\mathbf{Comod}_v^C$ . Obviously, two elements  $\psi: \mathbf{E}(v') \rightarrow \mathbf{E}(v'')$  and  $\psi': \mathbf{E}(v') \rightarrow \mathbf{E}(v'')$  of  $\mathbf{Comod}_v^C$  lie in the same  $\mathbf{G}_v^C$ -orbits if and only if the comodules  $\mathbf{Ker} \psi$  and  $\mathbf{Ker} \psi'$  are isomorphic. Hence (b) follows.

The lemma is proved.

Now we characterize computable  $K$ -coalgebras of  $f_C$ -discrete comodule type in terms of the  $\mathbf{G}_v^C$ -orbits of  $\mathbf{Comod}_v^C$  as follows:

**Proposition 4.1.** *Let  $K$  be an algebraically closed field and  $C$  a computable  $K$ -coalgebra. The following four conditions are equivalent:*

- (a) The coalgebra  $C$  is  $f_C$ -tame of discrete comodule type.
- (b) For every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , there is only a finite number of indecomposable objects  $(E_0, E_1, \psi)$  in  $\mathcal{Map}_1(E_{U_v})$  with  $\mathbf{cdn}(E_0, E_1, \psi) = v$ , up to isomorphism, where  $U_v = \mathbf{supp}(v)$ .
- (c) The number of  $\mathbf{G}_v^C$ -orbits in  $\mathbf{Comod}_v^C$  is finite for every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ .
- (d) The number of  $\mathbf{G}_v^C$ -orbits in  $\mathbf{Map}_v^C$  is finite for every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ .

**Proof.** (a)  $\Rightarrow$  (b) Assume that  $C$  is  $f_C$ -tame of discrete comodule type. Let  $v = (v'|v'')$  be a bipartite vector in  $K_0(C) \times K_0(C)$  and let  $(E_0, E_1, \psi)$  be an indecomposable object of  $\mathcal{Map}_1(E_U)$  such that  $\mathbf{cdn}(E_0, E_1, \psi) = (v'|v'')$ , where we set  $U = U_v = \mathbf{supp}(v)$ .

If  $v' = 0$  then  $E_0 = 0$ ,  $E_1 \cong E(a)$ , with  $a \in U$ , and therefore the number of the indecomposable objects  $(E_0, E_1, \psi)$  of  $\mathcal{Map}_1(E_U)$  with  $\mathbf{cdn}(E_0, E_1, \psi) = (0|v'')$  equals the cardinality of the finite subset  $U = \mathbf{supp}(v)$  of  $I_C$ .

Assume that  $v' \neq 0$ , i.e., the vector  $v$  is proper. Since  $(E_0, E_1, \psi)$  is indecomposable, it lies in  $\mathcal{Map}_2(E_U)$ , because it has no nonzero direct summand of the form  $(0, Z, 0)$ , By Proposition 4.1 (a), with

$E$  and  $E_U$  interchanged, the functor  $\mathbf{ker}_{E_U}$  in the diagram (3.5) restrict to the representation equivalence  $\mathbf{ker}_{E_U}: \mathcal{M}ap_2(E_U) \rightarrow C\text{-Comod}_{fc}^{E_U}$ . Then  $\mathbf{Ker} \psi = \mathbf{ker}_{E_U}(E_0, E_1, \psi)$  is an indecomposable comodule in  $C\text{-Comod}_{fc}^{E_U}$  such that  $\mathbf{cdn}(\mathbf{Ker} \psi) = \mathbf{cdn}(E_0, E_1, \psi) = v$ , see Proposition 4.1 (b). Since  $C$  is  $fc$ -tame of discrete comodule type, the number of the isomorphism classes of such comodules is finite and, hence, the number of the isomorphism classes of indecomposable objects  $(E_0, E_1, \psi)$  in  $\mathcal{M}ap_1(E_U)$  with  $\mathbf{cdn}(E_0, E_1, \psi) = v$  is also finite.

(b)  $\Rightarrow$  (d) Let  $v = (v'|v'') \in K_0(C) \times K_0(C)$  be a vector with nonnegative coordinates and let  $(E_0, E_1, \psi)$  be an object in  $\mathcal{M}ap_1(E_U)$ . Since the coalgebra  $C$  is assumed to be computable, the endomorphism ring  $\mathbf{End}(\psi)$  of  $(E_0, E_1, \psi)$  is a finite dimension  $K$ -algebra, and  $\mathbf{End}(\psi)$  is a local algebra if  $(E_0, E_1, \psi)$  is indecomposable. It follows that  $\mathcal{M}ap_1(E_U)$ , with  $U = \mathbf{supp}(v) \subseteq I_C$ , is a Krull–Schmidt category such that each of its objects is a finite direct sum of indecomposable objects, and every such a decomposition is unique up to isomorphism and a permutation of the indecomposables.

By our assumption, there is only a finite number of indecomposable objects  $(E'_0, E'_1, \psi')$  in  $\mathcal{M}ap_1(E_{U_v})$  with  $\mathbf{cdn}(E'_0, E'_1, \psi') \leq v$ , up to isomorphism. Let  $\mathbb{E}_1, \dots, \mathbb{E}_{s_v}$  be a complete set of such indecomposable objects. Then, up to isomorphism, any object  $(E_0, E_1, \psi)$  in  $\mathcal{M}ap_1(E_{U_v})$ , with  $\mathbf{cdn}(E_0, E_1, \psi) = v$ , has the form

$$(\mathbf{E}(v'), \mathbf{E}(v''), \psi) \cong \mathbb{E}_1^{\ell_1} \oplus \dots \oplus \mathbb{E}_{s_v}^{\ell_{s_v}},$$

where  $\ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi) = (\ell_1, \dots, \ell_{s_v}) \in \mathbb{N}^{s_v}$  is a vector with nonnegative coordinates such that

$$\ell_1 \cdot \mathbf{cdn}(\mathbb{E}_1) + \dots + \ell_{s_v} \cdot \mathbf{cdn}(\mathbb{E}_{s_v}) = v.$$

Obviously, the number of such vectors  $(\ell_1, \dots, \ell_{s_v})$  is finite. The unique decomposition property in  $\mathcal{M}ap_1(E_{U_v})$  yields

$$\ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi) = \ell(\mathbf{E}(v'), \mathbf{E}(v''), \psi') \quad \text{if and only if} \quad (\mathbf{E}(v'), \mathbf{E}(v''), \psi) \cong (\mathbf{E}(v'), \mathbf{E}(v''), \psi'),$$

or equivalently, if and only if the elements  $\psi$  and  $\psi'$  of  $\mathbf{Map}_v^C$  lie in the same  $\mathbf{G}_v^C$ -orbit. Hence the number of  $\mathbf{G}_v^C$ -orbits in  $\mathbf{Map}_v^C$  is finite and (d) follows.

Since the implication (d)  $\Rightarrow$  (c) is obvious and the implication (c)  $\Rightarrow$  (a) follows from Lemma 4.1 (b), the proof is complete.

Now we present a characterization of computable  $fc$ -tame coalgebras in terms of geometry of the  $\mathbf{G}_v^C$ -orbits of  $\mathbf{Comod}_v^C$ .

**Theorem 4.1.** *Let  $K$  be an algebraically closed field and  $C$  a computable  $K$ -coalgebra.*

- (a)  $C$  is  $fc$ -tame.
- (b) For every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , the category  $\mathcal{M}ap_1(E_{U_v})$ , with  $U_v = \mathbf{supp}(v)$ , is tame.
- (c) For every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , the subset  $\mathbf{indComod}_v^C$  of  $\mathbf{Comod}_v^C$  defined by the indecomposable  $C$ -comodules is constructible and there exists a constructible subset  $\mathcal{C}(v)$  of  $\mathbf{indComod}_v^C$  such that

$$\mathbf{G}_v^C * \mathcal{C}(v) = \mathbf{indComod}_v^C \quad \text{and} \quad \dim \mathcal{C}(v) \leq 1.$$

(d) For every bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$ , the subset  $\mathbf{indMap}_v^C$  of  $\mathbf{Map}_v^C$  defined by the indecomposable  $C$ -comodules is constructible and there exists a constructible subset  $\widehat{\mathcal{C}}(v)$  of  $\mathbf{indMap}_v^C$  such that

$$\mathbf{G}_v^C * \widehat{\mathcal{C}}(v) = \mathbf{indMap}_v^C \quad \text{and} \quad \dim \widehat{\mathcal{C}}(v) \leq 1.$$

**Proof.** (a)  $\Rightarrow$  (b) Apply Lemma 3.2 (a) to  $E = E_U = \bigoplus_{j \in U} E(j)$ , where  $U = \mathbf{supp}(v) \subseteq I_C$ .

(b)  $\Rightarrow$  (a) Apply Corollary 3.3.

We prove the equivalence of (b), (c) and (d) by applying the arguments used by Drozd [7], see also [3], [18] (Sec. 15.2) and [20] (Theorem 6.5).

(b)  $\Rightarrow$  (d) Fix a bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$  and assume that the category  $\mathcal{M}ap_1(E_{U_v})$ , with  $U_v = \mathbf{supp}(v)$ , is tame. Then there is a parametrizing family of functors

$$\widehat{L}^{(1)}, \dots, \widehat{L}^{(r)}: \mathbf{ind}_1(K[t]_h) \longrightarrow \mathcal{M}ap_1(E_{U_v})$$

for the family  $\mathbf{ind}_v(\mathcal{M}ap_1(E_{U_v}))$ , where  $h \in K[t]$  and  $U_v = \mathbf{supp}(v)$ . Here  $\mathbf{ind}_1(K[t]_h)$  is the category of one-dimensional  $K[t]_h$ -modules. Hence we conclude, as in [18] (Lemma 14.30, Remark 14.27) that the functors  $\widehat{L}^{(1)}, \dots, \widehat{L}^{(r)}$  induce regular maps

$$\ell_1, \dots, \ell_r: \mathbf{mod}^{K[t]_h}(1) \longrightarrow \mathbf{Map}_v^C$$

such that every point of  $\mathbf{indMap}_v^C$  belongs to an  $\mathbf{G}_v^C$ -orbit of the set

$$\widehat{\mathcal{C}}(v) = \text{Im } \ell_1 \cup \dots \cup \text{Im } \ell_r.$$

Here  $\mathbf{mod}^{K[t]_h}(1)$  is the variety of one-dimensional  $K[t]_h$ -modules. Since  $\dim \mathbf{mod}^{K[t]_h}(1) = 1$ , according to the Chevalley Theorem the subsets  $\text{Im } \ell_1, \dots, \text{Im } \ell_r$  of  $\mathbf{indMap}_v^C$  are constructible and therefore  $\widehat{\mathcal{C}}(v)$  is a constructible subset of  $\mathbf{indMap}_v^C$ . Moreover, it follows that  $\dim(\text{Im } \ell_j) \leq 1$ , for  $j = 1, \dots, r$ , and therefore  $\dim \widehat{\mathcal{C}}(v) \leq 1$ , compare with [15] and [18, p. 317].

The equivalence (d)  $\Leftrightarrow$  (c) easily follows from the fact that  $\mathbf{indMap}_v^C \setminus \mathbf{indComod}_v^C$  is a finite set and  $\mathbf{Comod}_v^C$  is an open subset of  $\mathbf{Map}_v^C$ , by Lemma 4.1.

(d)  $\Rightarrow$  (b) Assume to the contrary that there is a bipartite vector  $v = (v'|v'') \in K_0(C) \times K_0(C)$  such that the category  $\mathcal{M}ap_1(E_{U_v})$ , with  $U_v = \mathbf{supp}(v)$ , is not tame. By Corollary 3.2, the finite-dimensional algebra  $R_{U_v}$  is not tame. Then  $R_{U_v}$  is wild [7] and therefore the category  $\mathcal{M}ap_1(E_{U_v})$  is wild, by [3] (Sec. 6) and the proof of Theorem 3.1.

Let  $\mathcal{W} = K\langle t_1, t_2 \rangle$  be the free polynomial  $K$ -algebra in two noncommuting indeterminates  $t_1$  and  $t_2$ . Since the category  $\mathcal{M}ap_1(E_{U_v})$  is wild, there exists an object  ${}_C N_{\mathcal{W}} = (E' \otimes \mathcal{W}, E'' \otimes \mathcal{W}, \psi)$  in  $\mathcal{M}ap_1(E_{U_v} \otimes \mathcal{W})$ , with  $E', E''$  in  $\mathbf{add}(E_{U_v})$ , such that the functor

$$\widehat{N} = {}_C N \otimes_{\mathcal{W}} (-): \mathbf{fin}(\mathcal{W}) \longrightarrow \mathcal{M}ap_1(E_{U_v})$$

preserves the indecomposability and respects the isomorphism classes.

Let  $w = (w'|w'')$ , where  $w' = \text{lgth}(\text{soc } E')$  and  $w'' = \text{lgth}(\text{soc } E'')$ . It is well known that  $\text{indMap}_w^C$  is a constructible subset of  $\text{Map}_w^C$ , compare with [18] (Lemma 14.32).

Note that  $U_w = \text{supp}(w) \subseteq U_v$ ,  $\text{cdn}(\widehat{N}(X)) = w$ , and  $\widehat{N}(X) \cong (\mathbf{E}(w'), \mathbf{E}(w''), \psi)$ , for some  $\psi \in \text{indMap}_w^C \subseteq \text{Map}_w^C$  if  $X \in \text{fin}(\mathcal{W})$  and  $\dim_K X = 1$ . It follows that the restriction  $\widehat{N}: \text{ind}_1(\mathcal{W}) \rightarrow \text{Map}_1(E_{U_v})$  of  $\widehat{N}$  to  $\text{ind}_1(\mathcal{W})$  induces a regular map (see [18], Lemma 14.30)

$$\ell_N: \text{mod}^{\mathcal{W}}(1) \longrightarrow \text{indMap}_w^C \subseteq \text{Map}_w^C.$$

Since  $\text{mod}^{\mathcal{W}}(1) \cong K^2$ , the map  $\ell_N$  is injective, and according to the Chevalley Theorem the set  $\text{Im } \ell_N$  is constructible, the variety dimension  $\dim(\text{Im } \ell_N)$  of  $\text{Im } \ell_N$  equals two. Hence, in view of (d) with  $v$  and  $w$  interchanged, we get the contradiction  $2 = \dim(\text{Im } \ell_N) \leq \dim \mathcal{C}(v) \leq 1$  (apply [12] (Lemma 3.16) or [18] (Lemma 15.15)). This completes the proof.

### 5. On $f_c$ -Tameness for Arbitrary Coalgebras

The  $f_c$ -tame-wild dichotomy for an arbitrary basic coalgebra  $C$  over an algebraically closed field  $K$  remains an open problem. Some suggestions for the proof in case  $C$  is not computable is given in the following proposition, which collects important consequences of the technique described in Sec. 3. In particular, it shows that the coalgebra  $C$  is  $f_c$ -tame if and only if every socle-finite colocalization  $C_E \cong R_E^\circ$  of  $C$  (in the sense of [11, 25]) is  $f_c$ -tame.

**Proposition 5.1.** *Assume that  $K$  is an algebraically closed field and  $C$  is an arbitrary basic coalgebra with a decomposition  ${}_C C = \bigoplus_{j \in I_C} E(j)$  (1.1).*

- (a) *Given a socle-finite injective direct summand  $E = E_U = \bigoplus_{u \in U} E(u)$  (3.1) of  ${}_C C$ , with a finite subset  $U$  of  $I_C$ , the  $K$ -algebra  $R_E = \text{End}_C E$  is semi-perfect and pseudocompact with respect to the topology defined by (5.2) below. There is a commutative diagram*

$$\begin{CD} \text{Map}_1(E) @>H_E>> \mathcal{P}_1(R_E^{op}) @<G'<< \widehat{R}_E\text{-mod}_{pr}^{pr} \\ @V\ker_E VV @VV\text{cok}_E V \\ C_E\text{-Comod}_{f_c} \cong C\text{-Comod}_{f_c}^E @>h_E^\bullet>> \text{mod}_{fp}(R_E^{op}), \end{CD} \tag{5.1}$$

where  $C_E \cong R_E^\circ$  is the colocalization of  $C$  at  $E$  in the sense of [11, 25],  $\text{mod}_{fp}(R_E^{op})$  is the category of finitely presented left  $R_E$ -modules,  $\widehat{R}_E\text{-mod}_{pr}^{pr}$  is the category of finitely generated pro-partite left modules over the bipartite  $K$ -algebra  $\widehat{R}_E$  (3.9),  $H_E$  and  $h_E^\bullet = \text{Hom}_C(\bullet, E)$  are  $K$ -linear contravariant equivalences of categories defined as in (3.5),  $G'$  is the covariant  $K$ -linear equivalence of categories defined in (3.9),  $h_E^\bullet$  is an exact functor,  $\ker_E(E_0, E_1, \psi) = \text{Ker } \psi$ ,  $\text{cok}_E(P_1, P_0, \phi) = \text{Coker } \phi$ .

- (b) *For any socle-finite comodule  $E = E_U$  as in (a), the  $f_c$ -tameness of the coalgebra  $C$  implies that the category  $C\text{-Comod}_{f_c}^{E_U}$  is  $f_c$ -tame, i.e., the coalgebra  $C_{E_U}$  is  $f_c$ -tame.*
- (c) *Conversely, if the category  $C_{E_U}\text{-Comod}_{f_c} \cong C\text{-Comod}_{f_c}^{E_U}$  is  $f_c$ -tame for all socle-finite injective direct summands  $E = E_U$ , then the coalgebra  $C$  is  $f_c$ -tame.*

**Proof.** (a) Let  $E = E_U$  be a socle-finite direct summand of  $C$  as in (a). The  $K$ -algebra  $R_E = \text{End}_C E$  has the decomposition  $R_E = \bigoplus_{u \in U} e_u R_E$ , where  $e_u R_E = \text{Hom}_C(E, E(u))$  is an indecomposable projective right ideal of  $R_E$  and  $e_u$  is the primitive idempotent of  $R_E$  defined by the summand  $E(u)$  of  $E$ . Since the set  $U$  is finite,  $\sum_{u \in U} e_u$  is the identity of  $R_E$ , see [25, 26, 28]. It is easy to see that the Jacobson radical  $J(R_E)$  of  $R_E$  has the form  $J(R_E) = \{h \in \text{End}_C E; h(\text{soc } E) = 0\}$ . It follows that the algebra  $R_E$  is semiperfect and pseudocompact with respect to the  $K$ -linear topology defined by the left ideals  $\mathfrak{a}_\beta = \text{Hom}_C(E/V_\beta, E) \subseteq R_E$ , where  $\{V_\beta\}_\beta$  is the directed set of all finite-dimensional subcomodules of  $E$ . Since  $E = \bigcup_\beta V_\beta$ , there are isomorphisms

$$R_E = \text{End}_C E \cong \varprojlim_{\beta} \text{Hom}_C(V_\beta, E) \cong \varprojlim_{\beta} R_E/\mathfrak{a}_\beta. \tag{5.2}$$

The remaining statements in (a) follow from the proof of Theorem 3.1.

For the proof of (b) and (c), apply Lemma 3.1 and the arguments used in the proof of Theorem 3.1.

It follows from [28] (Corollaries 2.12 and 2.13) and the results of Sec. 3 that the  $f_c$ -tameness and  $f_c$ -wildness of a computable coalgebra  $C$  is equivalent, respectively, to the  $K$ -tameness and the  $K$ -wildness of the finite-dimensional algebra  $R_E$ , for every socle-finite direct summand of  $C$ . Proposition 5.1 shows that the  $f_c$ -tameness and  $f_c$ -wildness of a basic coalgebra  $C$  (that is not necessarily computable) can be studied by means of the tameness and wildness of the categories  $\widehat{R}_E\text{-mod}_{pr}^{pp}$  and  $\text{mod}_{fp}(R_E^{op})$  over the semiperfect algebras  $\widehat{R}_E$  and  $R_E$  that are not finite-dimensional, in general.

We recall from [26] (Corollary 2.10) that a socle-finite coalgebra  $C$  is computable if and only if  $\dim_K C$  is finite. Hence, if  $C$  is a cocommutative noncomputable coalgebra with simple socle then  $C$  is infinite-dimensional and, in view of Proposition 5.1, we have the following consequence of Drozd [6]:

**Corollary 5.1.** *Assume that  $K$  is an algebraically closed field and  $C$  is a basic infinite-dimensional cocommutative  $K$ -coalgebra with a unique simple subcoalgebra  $S$ . If  $S$  is finitely copresented and  $C$  is not  $f_c$ -wild then*

- (i)  $C$  is a subcoalgebra of the path  $K$ -coalgebra  $K^\square(\mathcal{L}_2, \Omega)$  (see [21] (Example 6.18), [22], [24]), where  $\mathcal{L}_2$  is the two loop quiver



and  $\Omega \subseteq K\mathcal{L}_2$  is the ideal of the path algebra  $K\mathcal{L}_2$  generated by the two zero-relations  $\beta_1\beta_2$  and  $\beta_2\beta_1$ , and

- (ii)  $K^\square(\mathcal{L}_2, \Omega)$  is a string coalgebra in the sense of [22] (Sec. 6),
- (iii) the coalgebras  $K^\square(\mathcal{L}_2, \Omega)$  and  $C$  are of tame comodule type, and  $K^\square(\mathcal{L}_2, \Omega)$  is of nonpolynomial growth.

**Proof.** By our assumption,  $C$  has a simple socle  $S$  and  $C = E(S)$  is the injective envelope of  $S$ , i.e., the set  $I_C$  in the decomposition (1.1) has one element and Proposition 5.1 applies to  $E = E(S) = C$ . It follows that the  $K$ -algebra  $R_E$  is pseudocompact, infinite-dimensional, commutative, local, and complete. Since  $C$  is not  $f_c$ -wild, the category  $\text{mod}_{fp}(R_E)$  is not  $K$ -wild, by Proposition 5.1. Since  $S$  is finitely copresented,  $C\text{-comod} \subseteq C\text{-Comod}_{f_c}$  and therefore  $\text{fin}(R_E) \subseteq \text{mod}_{fp}(R_E)$ . It follows that the category  $\text{fin}(R_E)$  is not  $K$ -wild. Hence, by [6], the unique maximal ideal  $J(R_E)$  of  $R_E$  is generated by at most two elements and  $R_E$  is isomorphic to a quotient of the  $K$ -algebra  $K[[t_1, t_2]]/(t_1 t_2)$ , where  $K[[t_1, t_2]]$  is the power series  $K$ -algebra in two commuting indeterminates  $t_1, t_2$  and  $(t_1 t_2)$  is the ideal of  $K[[t_1, t_2]]$  generated by  $t_1 t_2$ .

It is easy to see that the path coalgebra  $K^\square(\mathcal{L}_2, \Omega) = \Omega^\perp \subseteq K^\square \mathcal{L}_2$  is isomorphic with the coalgebra

$$K[t_1, t_2]^\diamond = K \oplus \bigoplus_{n=1}^{\infty} K\bar{t}_1^n \oplus \bigoplus_{m=1}^{\infty} K\bar{t}_2^m,$$

where the comultiplication  $\Delta: K[t_1, t_2]^\diamond \longrightarrow K[t_1, t_2]^\diamond \otimes K[t_1, t_2]^\diamond$  and the counity  $\varepsilon: K[t_1, t_2]^\diamond \longrightarrow K$  are defined by the formulae  $\Delta(\bar{t}_j^m) = \sum_{r+s=m} \bar{t}_j^r \otimes \bar{t}_j^s$  for  $j = 1, 2$ ,  $\varepsilon(1) = 1$  and  $\varepsilon(\bar{t}_j^s) = 0$  for  $s \geq 1$  and  $j = 1, 2$ , see [21] (Example 6.18).

Moreover, it follows from [24] that  $C$  is isomorphic to a subcoalgebra of  $K^\square(\mathcal{L}_2, \Omega)$ . Since  $K^\square(\mathcal{L}_2, \Omega)$  is a string coalgebra, according to [21] (Example 6.18) and [22] (Theorem 6.2)  $K^\square(\mathcal{L}_2, \Omega) \cong K[t_1, t_2]^\diamond$  is of tame comodule type and, hence, the coalgebra  $C$  is of tame comodule type, too. It is shown in [21] (Example 6.18) that  $K^\square(\mathcal{L}_2, \Omega) \cong K[t_1, t_2]^\diamond$  is tame of nonpolynomial growth.

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## REFERENCES

1. I. Assem, D. Simson, and A. Skowroński, “Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory,” in: *London Math. Soc. Student Texts 65*, Cambridge University Press, Cambridge–New York (2006).
2. W. Chin, “A brief introduction to coalgebra representation theory,” *Lect. Notes Pure Appl. Math.*, **237**, 109–131 (2004).
3. W. W. Crawley-Boevey, “On tame algebras and bocses,” *Proc. London Math. Soc.*, **56**, 451–483 (1988).
4. W. W. Crawley-Boevey, “Matrix problems and Drozd’s theorem,” in: S. Balcerzyk, T. Józefiak, J. Krempa, D. Simson, and W. Vogel (editors), *Topics Algebra, Pt. I: Rings and Represent. Algebras*, PWN, Warszawa (1990), pp. 199–222.
5. Ju. A. Drozd, “Matrix problems and categories of matrices,” *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, **28**, 144–153 (1972).
6. Ju. A. Drozd, “Representations of commutative algebras,” *Funkts. Anal. Prilozhen.*, **6**, 41–43 (1972).
7. Ju. A. Drozd, “Tame and wild matrix problems,” in: *Represent. and Quadr. Forms* [in Russian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (1979), pp. 39–74.
8. Ju. A. Drozd and V. V. Kirichenko, *Finite-Dimensional Algebras*, Springer, Berlin (1994).
9. Ju. A. Drozd, S. A. Ovsienko, and B. Ju. Furchin, “Categorical constructions in representation theory,” in: *Algebr. Structures and Appl.* [in Russian], UMK VO, Kiev (1988), pp. 43–73.
10. P. Gabriel and A. V. Roiter, “Representations of finite-dimensional algebras,” *Algebra VIII: Encyclopedia Math. Sci.*, **73** (1992).
11. P. Jara, L. Merino, and G. Navarro, “Localization in tame and wild coalgebras,” *J. Pure Appl. Algebra*, **211**, 342–359 (2007).
12. S. Kasjan and D. Simson, “Varieties of poset representations and minimal posets of wild prinjective type,” *Represent. Algebras: Can. Math. Soc. Conf. Proc.*, **14**, 245–284 (1993).
13. M. Kleiner and A. V. Roiter, “Representations of differential graded categories,” in: *Matrix Problems* [in Russian], Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1977), pp. 5–70.
14. M. Kleiner and A. V. Roiter, “Abelian categories, almost split sequences and comodules,” *Trans. Amer. Math. Soc.*, **357**, 3201–3214 (2005).
15. H. Kraft and K. Riedtmann, “Geometry of representations of quivers,” *Represent. Algebras: London Math. Soc. Lect. Notes*, **116**, 109–145 (1986).
16. S. Montgomery, “Hopf algebras and their actions on rings,” *CMBS*, No. 82 (1993).
17. J. A. de la Peña and D. Simson, “Prinjective modules, reflection functors, quadratic forms and Auslander–Reiten sequences,” *Trans. Amer. Math. Soc.*, **329**, 733–753 (1992).
18. D. Simson, “Linear representations of partially ordered sets and vector space categories,” *Algebra, Logic Appl.*, **4** (1992).
19. D. Simson, “Representation embedding problems, categories of extensions and prinjective modules,” in: R. Bautista, R. Martinez-Villa, and J. A. de la Peña (editors), *Represent. Theory Algebras*, Can. Math. Soc. Conf. Proc., **18**, (1996), pp. 601–639.
20. D. Simson, “Prinjective modules, propartite modules, representations of bocses and lattices over orders,” *J. Math. Soc. Jpn.*, **49**, 1–68 (1997).
21. D. Simson, “Coalgebras, comodules, pseudocompact algebras and tame comodule type,” *Colloq. Math.*, **90**, 101–150 (2001).
22. D. Simson, “Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras,” *Lect. Notes Pure Appl. Math.*, **236**, 465–492 (2004).
23. D. Simson, “On Corner type Endo–Wild algebras,” *J. Pure Appl. Algebra*, **202**, 118–132 (2005).

24. D. Simson, "Path coalgebras of profinite bound quivers, cotensor coalgebras of bound species and locally nilpotent representations," *Colloq. Math.*, **109**, 307–343 (2007).
25. D. Simson, "Localising embeddings of comodule categories with applications to tame and Euler coalgebras," *J. Algebra*, **312**, 455–494 (2007).
26. D. Simson, "Hom-computable coalgebras, a composition factors matrix and the Euler bilinear form of an Euler coalgebra," *J. Algebra*, **315**, 42–75 (2007).
27. D. Simson, "Representation-directed incidence coalgebras of intervally finite posets and the tame-wild dichotomy," *Commun. Algebra*, **36**, 2764–2784 (2008).
28. D. Simson, "Tame-wild dichotomy for coalgebras," *J. London Math. Soc.*, **78**, 783–797 (2008).
29. Woodcock D., "Some categorical remarks on the representation theory of coalgebras," *Commun. Algebra*, **25**, 775–2794 (1997).